

Complex numbers 1D

1 a $z^* = 8 - 2i$

b $z^* = 6 + 5i$

c $z^* = \frac{2}{3} + \frac{1}{2}i$

d $z^* = \sqrt{5} - i\sqrt{10}$

2 a $z + z^* = (6 - 3i) + (6 + 3i)$
 $= 12$

$$\begin{aligned} zz^* &= (6 - 3i)(6 + 3i) \\ &= 6(6 + 3i) - 3i(6 + 3i) \\ &= 36 + 18i - 18i - 9i^2 \\ &= (36 + 9) + i(18 - 18) \\ &= 45 \end{aligned}$$

b $z + z^* = (10 + 5i) + (10 - 5i)$
 $= 20$

$$\begin{aligned} zz^* &= (10 + 5i)(10 - 5i) \\ &= 10(10 - 5i) + 5i(10 - 5i) \\ &= 100 - 50i + 50i - 25i^2 \\ &= (100 + 25) + i(-50 + 50) \\ &= 125 \end{aligned}$$

c $z + z^* = \left(\frac{3}{4} + \frac{1}{4}i\right) + \left(\frac{3}{4} - \frac{1}{4}i\right)$
 $= \frac{3}{2}$

$$\begin{aligned} zz^* &= \left(\frac{3}{4} + \frac{1}{4}i\right)\left(\frac{3}{4} - \frac{1}{4}i\right) \\ &= \frac{3}{4}\left(\frac{3}{4} - \frac{1}{4}i\right) + \frac{1}{4}i\left(\frac{3}{4} - \frac{1}{4}i\right) \\ &= \frac{9}{16} - \frac{3}{16}i + \frac{3}{16}i - \frac{1}{16}i^2 \\ &= \frac{10}{16} = \frac{5}{8} \end{aligned}$$

2 d $z + z^* = (\sqrt{5} - 3i\sqrt{5}) + (\sqrt{5} + 3i\sqrt{5})$
 $= 2\sqrt{5}$

$$\begin{aligned} zz^* &= (\sqrt{5} - 3i\sqrt{5})(\sqrt{5} + 3i\sqrt{5}) \\ &= \sqrt{5}(\sqrt{5} + 3i\sqrt{5}) - 3i\sqrt{5}(\sqrt{5} + 3i\sqrt{5}) \\ &= 5 + 15i - 15i - 45i^2 \\ &= (5 + 45) + i(15 - 15) \\ &= 50 \end{aligned}$$

3 a $\frac{3 - 5i}{1 + 3i} = \frac{(3 - 5i)(1 - 3i)}{(1 + 3i)(1 - 3i)}$
 $(3 - 5i)(1 - 3i) = 3(1 - 3i) - 5i(1 - 3i)$
 $= 3 - 9i - 5i + 15i^2$
 $= -12 - 14i$
 $(1 + 3i)(1 - 3i) = 1(1 - 3i) + 3i(1 - 3i)$
 $= 1 - 3i + 3i - 9i^2$
 $= 10$
 $\frac{3 - 5i}{1 + 3i} = \frac{-12 - 14i}{10} = -\frac{6}{5} - \frac{7}{5}i$

b $\frac{3 + 5i}{6 - 8i} = \frac{(3 + 5i)(6 + 8i)}{(6 - 8i)(6 + 8i)}$
 $(3 + 5i)(6 + 8i) = 3(6 + 8i) + 5i(6 + 8i)$
 $= 18 + 24i + 30i + 40i^2$
 $= -22 + 54i$
 $(6 - 8i)(6 + 8i) = 6(6 + 8i) - 8i(6 + 8i)$
 $= 36 + 48i - 48i - 64i^2$
 $= 100$
 $\frac{3 + 5i}{6 - 8i} = \frac{-22 + 54i}{100} = -\frac{11}{50} + \frac{27}{50}i$

$$\begin{aligned}
 3 \quad c \quad \frac{28-3i}{1-i} &= \frac{(28-3i)(1+i)}{(1-i)(1+i)} \\
 (28-3i)(1+i) &= 28(1+i) - 3i(1+i) \\
 &= 28 + 28i - 3i - 3i^2 \\
 &= 31 + 25i \\
 (1-i)(1+i) &= 1(1+i) - i(1+i) \\
 &= 1 + i - i - i^2 \\
 &= 2 \\
 \frac{28-3i}{1-i} &= \frac{31+25i}{2} = \frac{31}{2} + \frac{25}{2}i
 \end{aligned}$$

$$\begin{aligned}
 d \quad \frac{2+i}{1+4i} &= \frac{(2+i)(1-4i)}{(1+4i)(1-4i)} \\
 (2+i)(1-4i) &= 2(1-4i) + i(1-4i) \\
 &= 2 - 8i + i - 4i^2 \\
 &= 6 - 7i \\
 (1+4i)(1-4i) &= 1(1-4i) + 4i(1-4i) \\
 &= 1 - 4i + 4i - 16i^2 \\
 &= 17 \\
 \frac{2+i}{1+4i} &= \frac{6-7i}{17} = \frac{6}{17} - \frac{7}{17}i
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \frac{(3-4i)^2}{1+i} &= \frac{(9-24i+16i^2)}{(1+i)} \times \frac{(1-i)}{(1-i)} \\
 &= \frac{9-24i+16i^2-9i+24i^2-16i^3}{1-i+i-i^2} \\
 &= \frac{9-24i-16-9i-24+16i}{1-(-1)} \\
 &= \frac{-31-17i}{2} \\
 &= -\frac{31}{2} - \frac{17}{2}i
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad z_1 z_2 &= (1+i)(2+i) \\
 &= 1(2+i) + i(2+i) \\
 &= 2 + i + 2i + i^2 \\
 &= 1 + 3i \\
 \frac{z_1 z_2}{z_3} &= \frac{1+3i}{3+i} = \frac{(1+3i)(3-i)}{(3+i)(3-i)} \\
 (1+3i)(3-i) &= 1(3-i) + 3i(3-i) \\
 &= 3 - i + 9i - 3i^2 \\
 &= 6 + 8i \\
 (3+i)(3-i) &= 3(3-i) + i(3-i) \\
 &= 9 - 3i + 3i - i^2 \\
 &= 10 \\
 \frac{z_1 z_2}{z_3} &= \frac{6+8i}{10} = \frac{3}{5} + \frac{4}{5}i
 \end{aligned}$$

$$\begin{aligned}
 b \quad (z_2)^2 &= (2+i)(2+i) \\
 &= 2(2+i) + i(2+i) \\
 &= 4 + 2i + 2i + i^2 \\
 &= 3 + 4i \\
 \frac{(z_2)^2}{z_1} &= \frac{3+4i}{1+i} = \frac{(3+4i)(1-i)}{(1+i)(1-i)} \\
 (3+4i)(1-i) &= 3(1-i) + 4i(1-i) \\
 &= 3 - 3i + 4i - 4i^2 \\
 &= 7 + i \\
 (1+i)(1-i) &= 1(1-i) + i(1-i) \\
 &= 1 - i + i - i^2 \\
 &= 2 \\
 \frac{(z_2)^2}{z_1} &= \frac{7+i}{2} = \frac{7}{2} + \frac{1}{2}i
 \end{aligned}$$

$$\begin{aligned}
 5 \quad c \quad 2z_1 + 5z_3 &= 2(1+i) + 5(3+i) \\
 &= 2 + 2i + 15 + 5i \\
 &= 17 + 7i \\
 \frac{2z_1 + 5z_3}{z_2} &= \frac{17 + 7i}{2 + i} = \frac{(17 + 7i)(2 - i)}{(2 + i)(2 - i)} \\
 (17 + 7i)(2 - i) &= 17(2 - i) + 7i(2 - i) \\
 &= 34 - 17i + 14i - 7i^2 \\
 &= 41 - 3i \\
 (2 + i)(2 - i) &= 2(2 - i) + i(2 - i) \\
 &= 4 - 2i + 2i - i^2 \\
 &= 5 \\
 \frac{2z_1 + 5z_3}{z_2} &= \frac{41 - 3i}{5} = \frac{41}{5} - \frac{3}{5}i
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \frac{5 + 2i}{z} &= 2 - i \\
 z &= \frac{5 + 2i}{2 - i} = \frac{(5 + 2i)(2 + i)}{(2 - i)(2 + i)} \\
 (5 + 2i)(2 + i) &= 5(2 + i) + 2i(2 + i) \\
 &= 10 + 5i + 4i + 2i^2 \\
 &= 8 + 9i \\
 (2 - i)(2 + i) &= 2(2 + i) - i(2 + i) \\
 &= 4 + 2i - 2i - i^2 \\
 &= 5 \\
 z &= \frac{8 + 9i}{5} = \frac{8}{5} + \frac{9}{5}i
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \frac{6 + 8i}{1 + i} + \frac{6 + 8i}{1 - i} &= \frac{(6 + 8i)(1 - i) + (6 + 8i)(1 + i)}{(1 + i)(1 - i)} \\
 &= \frac{6(1 - i) + 8i(1 - i) + 6(1 + i) + 8i(1 + i)}{1(1 - i) + i(1 - i)} \\
 &= \frac{6 - 6i + 8i - 8i^2 + 6 + 6i + 8i + 8i^2}{1 - i + i - i^2} \\
 &= \frac{12 + 16i}{2} = 6 + 8i
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \frac{4}{8 - i\sqrt{2}} &= \frac{4}{(8 - i\sqrt{2})} \times \frac{(8 + i\sqrt{2})}{(8 + i\sqrt{2})} \\
 &= \frac{32 + 4i\sqrt{2}}{64 + 8i\sqrt{2} - 8i\sqrt{2} - i^2(\sqrt{2})^2} \\
 &= \frac{32 + 4i\sqrt{2}}{66} \\
 &= \frac{16}{33} + \frac{2\sqrt{2}}{33}i
 \end{aligned}$$

$$\begin{aligned}
 9 \quad \frac{1}{w} &= \frac{1}{1 - 9i} \\
 &= \frac{1}{(1 - 9i)} \times \frac{(1 + 9i)}{(1 + 9i)} \\
 &= \frac{1 + 9i}{1 + 9i - 9i - 81i^2} \\
 &= \frac{1 + 9i}{82} \\
 &= \frac{1}{82} + \frac{9}{82}i
 \end{aligned}$$

$$\begin{aligned}
 10 \quad \frac{z + 4}{z - 3} &= \frac{4 - i\sqrt{2} + 4}{4 - i\sqrt{2} - 3} \\
 &= \frac{8 - i\sqrt{2}}{1 - i\sqrt{2}} \\
 &= \frac{(8 - i\sqrt{2})}{(1 - i\sqrt{2})} \times \frac{(1 + i\sqrt{2})}{(1 + i\sqrt{2})} \\
 &= \frac{8 + 8i\sqrt{2} - i\sqrt{2} - i^2(\sqrt{2})^2}{1 + i\sqrt{2} - i\sqrt{2} - i^2(\sqrt{2})^2} \\
 &= \frac{8 + 7i\sqrt{2} + 2}{1 + 2} \\
 &= \frac{10}{3} + \frac{7\sqrt{2}}{3}i
 \end{aligned}$$

$$11 \quad (4+2i)(z-2i) = 6-4i$$

$$z-2i = \frac{6-4i}{4+2i} = \frac{(6-4i)}{(4+2i)} \times \frac{(4-2i)}{(4-2i)}$$

$$z-2i = \frac{24-12i-16i+8i^2}{16-8i+8i-(2i)^2}$$

$$z-2i = \frac{16-28i}{20}$$

$$z-2i = \frac{4}{5} - \frac{7}{5}i$$

$$z = \frac{4}{5} + \frac{3}{5}i$$

Alternative Method:

Let $z = a + bi$

Substitute $z = a + bi$ into the equation

$$6-4i = (4+2i)(z-2i):$$

$$\begin{aligned} 6-4i &= (4+2i)(a+(b-2)i) \\ &= 4a+4(b-2)i+2ai+2(b-2)i^2 \\ &= 4a+4bi-8i+2ai-2(b-2) \\ &= 4a+4bi-8i+2ai-2b+4 \\ &= (4a-2b+4)+(2a+4b-8)i \end{aligned}$$

Equate real parts:

$$6 = 4a - 2b + 4 \Rightarrow 2a - b = 1 \quad (1)$$

Equate imaginary parts:

$$-4 = 2a + 4b - 8 \Rightarrow a + 2b = 2 \quad (2)$$

$$(2) + 2 \times (1) \Rightarrow 5a = 4$$

$$a = \frac{4}{5}$$

Substitute $a = \frac{4}{5}$ into (2)

$$\frac{4}{5} + 2b = 2$$

$$b = \frac{3}{5}$$

$$\text{Hence } z = \frac{4}{5} + \frac{3}{5}i$$

$$\begin{aligned} 12 \quad \frac{z_1}{z_2} &= \frac{p-7i}{2+5i} \\ &= \frac{(p-7i)}{(2+5i)} \times \frac{(2-5i)}{(2-5i)} \\ &= \frac{2p-5pi-14i+35i^2}{4-10i+10i-25i^2} \\ &= \frac{2p-35}{29} + \frac{-5p-14}{29}i \end{aligned}$$

$$\begin{aligned} 13 \quad z &= \sqrt{5} + 4i \\ z^* &= \sqrt{5} - 4i \\ \frac{z}{z^*} &= \frac{(\sqrt{5}+4i)}{(\sqrt{5}-4i)} \times \frac{(\sqrt{5}+4i)}{(\sqrt{5}+4i)} \\ &= \frac{5+4\sqrt{5}i+4\sqrt{5}i+16i^2}{5+4\sqrt{5}i-4\sqrt{5}i-16i^2} \\ &= -\frac{11}{21} + \frac{8\sqrt{5}}{21}i \end{aligned}$$

$$\begin{aligned} 14 \text{ a} \quad \frac{p+5i}{p-2i} &= \frac{(p+5i)}{(p-2i)} \times \frac{(p+2i)}{(p+2i)} \\ &= \frac{p^2+2pi+5pi+10i^2}{p^2+2pi-2pi-4i^2} \\ &= \frac{p^2-10}{p^2+4} + \frac{7p}{p^2+4}i \quad (1) \end{aligned}$$

The real part is equal to $\frac{1}{2}$, so

$$\frac{p^2-10}{p^2+4} = \frac{1}{2}$$

$$2p^2-20 = p^2+4$$

$$p^2 = 24$$

$$p = \pm 2\sqrt{6}$$

But since $p > 0$, then $p = 2\sqrt{6}$

b Substituting $p = 2\sqrt{6}$ into (1):

$$\begin{aligned} z &= \frac{(2\sqrt{6})^2-10}{(2\sqrt{6})^2+4} + \frac{7(2\sqrt{6})}{(2\sqrt{6})^2+4}i \\ &= \frac{24-10}{24+4} + \frac{14\sqrt{6}}{24+4}i \\ &= \frac{1}{2} + \frac{\sqrt{6}}{2}i \end{aligned}$$