Complex numbers 1E

1 a $z^2+2z+26=0$

Using the quadratic formula gives

$$z=\frac{-2\pm\sqrt{4-4(1)(26)}}{2}$$
$$=\frac{-2\pm10\text{i}}{2}$$

$\alpha=-1+5\text{i},\ \beta=-1-5\text{i}$ (or vice versa)

b $\alpha+\beta=(-1+5\text{i})+(-1-5\text{i})=-2$

c $\alpha\beta=(-1+5\text{i})(-1-5\text{i})$
$=-1(-1-5\text{i})+5\text{i}(-1-5\text{i})$
$=1+5\text{i}-5\text{i}-25\text{i}^2$
$=26$

2 a $z^2-8z+25=0$

$$z=\frac{8\pm\sqrt{(-8)^2-4(1)(25)}}{2}$$
$$=\frac{8\pm6\text{i}}{2}$$

$\alpha=4+3\text{i},\ \beta=4-3\text{i}$ (or vice versa)

b $\alpha+\beta=(4+3\text{i})+(4-3\text{i})=8$

c $\alpha\beta=(4+3\text{i})(4-3\text{i})$
$=4(4-3\text{i})+3\text{i}(4-3\text{i})$
$=16-12\text{i}+12\text{i}-9\text{i}^2$
$=25$

3 a The other root is the complex conjugate $2-3\text{i}$.

b $(z-(2+3\text{i}))(z-(2-3\text{i}))$
$=z^2-(2-3\text{i})z-(2+3\text{i})z+(2+3\text{i})(2-3\text{i})$
$=z^2-2z+3\text{i}z-2z-3\text{i}z+4-6\text{i}+6\text{i}-9\text{i}^2$
$=z^2-4z+4+9$
$=z^2-4z+13$

Equation is $z^2-4z+13=0$

where $b=-4,\ c=13$

4 a The other root is the complex conjugate $5+\text{i}$.

b $(z-(5-\text{i}))(z-(5+\text{i}))$
$=z^2-(5+\text{i})z-(5-\text{i})z+(5-\text{i})(5+\text{i})$
$=z^2-5z-\text{i}z-5z+\text{i}z+25+5\text{i}-5\text{i}-\text{i}^2$
$=z^2-10z+25+1$
$=z^2-10z+26$

So, $p=-10$ and $q=26$

5 The second root is the complex conjugate of $z_1=-5+4i$

i.e. $z_2=-5-4i$

$(z-(-5+4\text{i}))(z-(-5-4\text{i}))$
$=z^2-(-5-4\text{i})z-(-5+4\text{i})z+(-5+4\text{i})(-5-4\text{i})$
$=z^2+5z+4\text{i}z+5z-4\text{i}z+25+20\text{i}-20\text{i}-16\text{i}^2$
$=z^2+10z+25+16$
$=z^2+10z+41$

So, $b=10$ and $c=41$

6 The other root is 1 – 2i.

If the roots are α and β, the equation is $(z-\alpha)(z-\beta)=z^2-(\alpha+\beta)z+\alpha\beta=0$

$\alpha+\beta=(1+2\text{i})+(1-2\text{i})=2$
$\alpha\beta=(1+2\text{i})(1-2\text{i})$
$=1(1-2\text{i})+2\text{i}(1-2\text{i})$
$=1-2\text{i}+2\text{i}-4\text{i}^2=5$

Equation is $z^2-2z+5=0$

7 The other root is 3 + 5i.

If the roots are α and β, the equation is
$(z-\alpha)(z-\beta)=z^2-(\alpha+\beta)z+\alpha\beta=0$

$$\begin{aligned}\alpha+\beta&=(3-5\mathrm{i})+(3+5\mathrm{i})=6\\ \alpha\beta&=(3-5\mathrm{i})(3+5\mathrm{i})\\ &=3(3+5\mathrm{i})-5\mathrm{i}(3+5\mathrm{i})\\ &=9+15\mathrm{i}-15\mathrm{i}-25\mathrm{i}^2=34\end{aligned}$$

Equation is $z^2-6z+34=0$

8 a $$\begin{aligned}z&=\frac{5}{3-\mathrm{i}}\\ &=\frac{5}{(3-\mathrm{i})}\times\frac{(3+\mathrm{i})}{(3+\mathrm{i})}\\ &=\frac{15+5\mathrm{i}}{9+3\mathrm{i}-3\mathrm{i}-\mathrm{i}^2}\\ &=\frac{15}{10}+\frac{5}{10}\mathrm{i}\\ &=\frac{3}{2}+\frac{1}{2}\mathrm{i}\end{aligned}$$

b If $z=\frac{3}{2}+\frac{1}{2}\mathrm{i}$ is a root, then $z=\frac{3}{2}-\frac{1}{2}\mathrm{i}$ is also a root.

$$\begin{aligned}0&=\left(z-\left(\frac{3}{2}+\frac{1}{2}\mathrm{i}\right)\right)\left(z-\left(\frac{3}{2}-\frac{1}{2}\mathrm{i}\right)\right)\\ &=z^2-\left(\frac{3}{2}-\frac{1}{2}\mathrm{i}\right)z-\left(\frac{3}{2}+\frac{1}{2}\mathrm{i}\right)z\\ &\qquad+\left(\frac{3}{2}+\frac{1}{2}\mathrm{i}\right)\left(\frac{3}{2}-\frac{1}{2}\mathrm{i}\right)\\ &=z^2-\frac{3}{2}z+\frac{1}{2}\mathrm{i}z-\frac{3}{2}z-\frac{1}{2}\mathrm{i}z\\ &\qquad+\frac{9}{4}-\frac{3}{4}\mathrm{i}+\frac{3}{4}\mathrm{i}-\frac{1}{4}\mathrm{i}^2\\ &=z^2-3z+\frac{10}{4}\\ &=z^2-3z+\frac{5}{2}\end{aligned}$$

So $p=-3$ and $q=\frac{5}{2}$

9 $z=5+q\mathrm{i}$
The second root is the complex conjugate, $z=5-q\mathrm{i}$

$$\begin{aligned}0&=\left(z-\left(5+q\mathrm{i}\right)\right)\left(z-\left(5-q\mathrm{i}\right)\right)\\ &=z^2-\left(5-q\mathrm{i}\right)z-\left(5+q\mathrm{i}\right)z\\ &\qquad+\left(5+q\mathrm{i}\right)\left(5-q\mathrm{i}\right)\\ &=z^2-5z+q\mathrm{i}z-5z-q\mathrm{i}z\\ &\qquad+25-5q\mathrm{i}+5q\mathrm{i}-q^2\mathrm{i}^2\\ &=z^2-10z+25+q^2\end{aligned}$$

So $z^2-10z+25+q^2\equiv z^2-4pz+34$
Equating z terms:
$4p=10\Rightarrow p=\frac{5}{2}$
Equating constant terms:
$$\begin{aligned}25+q^2=34&\Rightarrow q^2=9\\ &\Rightarrow q=\pm3\end{aligned}$$
But $q>0$, and hence $q=3$