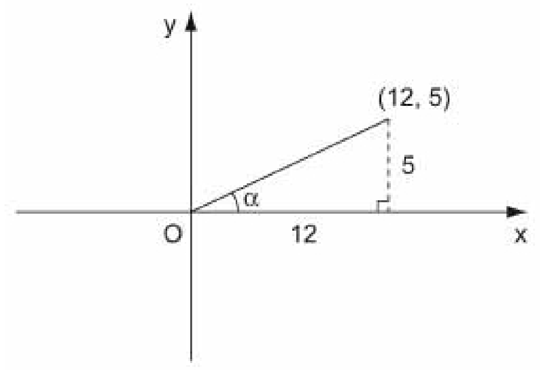
Argand diagrams 2B

1 a $z = 12 + 5\text{i}$

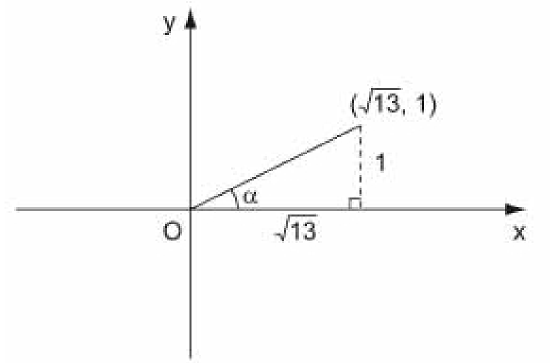


$|z| = \sqrt{(12^2 + 5^2)} = \sqrt{169} = 13$

$\tan\alpha = \dfrac{5}{12} \quad \alpha = 0.39 \text{ rad.}$

$\arg z = 0.39$ radians (2 d.p.)

b $z = \sqrt{3} + \text{i}$

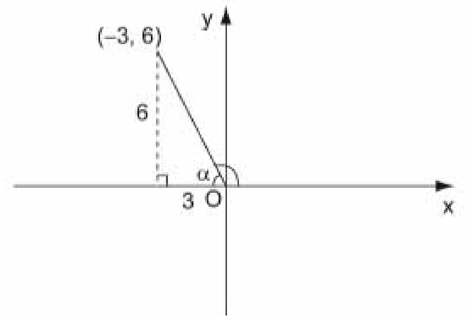


$|z| = \sqrt{\left((\sqrt{3})^2 + 1^2\right)} = \sqrt{4} = 2$

$\tan\alpha = \dfrac{1}{\sqrt{3}} \quad \alpha = \dfrac{\pi}{6}$

$\arg z = \dfrac{\pi}{6}$

c $z = -3 + 6\text{i}$

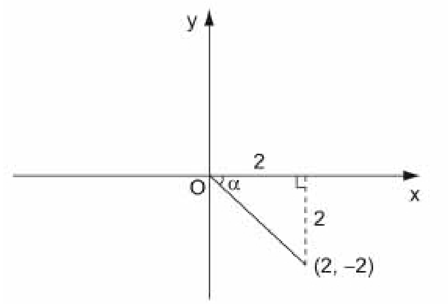


$|z| = \sqrt{\left((-3)^2 + 6^2\right)} = \sqrt{45} = 3\sqrt{5}$

$\tan\alpha = \dfrac{6}{3} \quad \alpha = 1.107$

$\arg z = \pi - \alpha = 2.03$ radians (2 d.p.)

d $z = 2 - 2\text{i}$

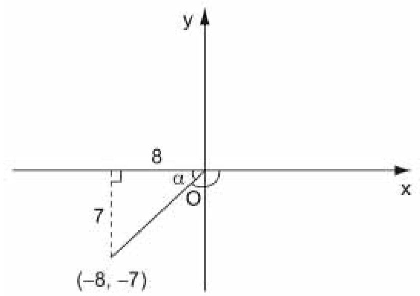


$|z| = \sqrt{\left(2^2 + (-2)^2\right)} = \sqrt{8} = 2\sqrt{2}$

$\tan\alpha = \dfrac{2}{2} \quad \alpha = \dfrac{\pi}{4}$

$\arg z = -\alpha = -\dfrac{\pi}{4}$

e $z = -8 - 7\text{i}$

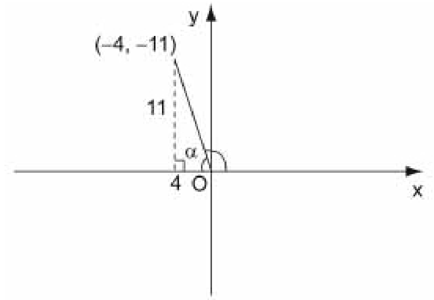


$|z| = \sqrt{\left((-8)^2 + (-7)^2\right)} = \sqrt{113}$

$\tan\alpha = \dfrac{7}{8} \quad \alpha = 0.7188$

$\arg z = -(\pi - \alpha) = -2.42$ radians (2 d.p.)

1 f $z=-4+11\mathrm{i}$

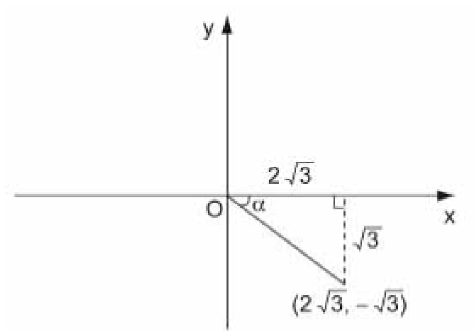


$|z|=\sqrt{\left((-4)^2+11^2\right)}=\sqrt{137}$

$\tan\ \alpha=\frac{11}{4}\quad \alpha=1.222$

$\arg\ z=\pi-\alpha=1.92$ radians (2 d.p.)

g $z=2\sqrt{3}-\mathrm{i}\sqrt{3}$

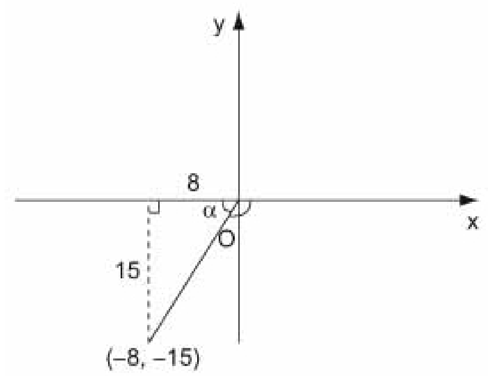


$|z|=\sqrt{\left((2\sqrt{3})^2+(-\sqrt{3})^2\right)}=\sqrt{15}$

$\tan\ \alpha=\frac{\sqrt{3}}{2\sqrt{3}}=\frac{1}{2}\quad \alpha=0.4636$

$\arg\ z=-0.46$ radians (2 d.p.)

h $z=-8-15\mathrm{i}$



$|z|=\sqrt{\left((-8)^2+(-15)^2\right)}=\sqrt{289}=17$

$\tan\ \alpha=\frac{15}{8}\quad \alpha=1.0808$

$\arg\ z=-(\pi-\alpha)=-2.06$ radians (2 d.p.)

2 a i $|2+2\mathrm{i}|=\sqrt{(2)^2+(2)^2}$
$=\sqrt{8}$
$=2\sqrt{2}$

ii $\tan\alpha=\frac{2}{2}$, so $\alpha=\frac{\pi}{4}$

$\arg z=\frac{\pi}{4}$

b i $|5+5\mathrm{i}|=\sqrt{(5)^2+(5)^2}$
$=\sqrt{50}$
$=5\sqrt{2}$

ii $\tan\alpha=\frac{5}{5}$, so $\alpha=\frac{\pi}{4}$

$\arg z=\frac{\pi}{4}$

c i $|-6+6\mathrm{i}|=\sqrt{(-6)^2+(6)^2}$
$=\sqrt{72}$
$=6\sqrt{2}$

ii $\tan\alpha=\frac{6}{6}$, so $\alpha=\frac{\pi}{4}$

Here z is in the second quadrant, so

$\arg z=(\pi-\alpha)$

$\arg z=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

d i $|-a-a\mathrm{i}|=\sqrt{(-a)^2+(-a)^2}$
$=\sqrt{2a^2}$
$=a\sqrt{2}$

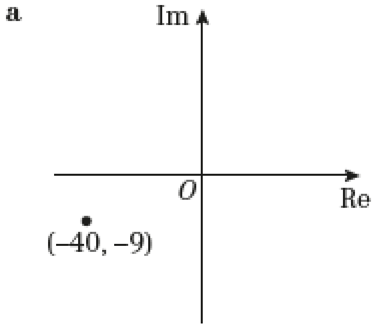
ii $\tan\alpha=\frac{a}{a}$, so $\alpha=\frac{\pi}{4}$.

Here z is in the third quadrant, so

$\arg z=-(\pi-\alpha)$

$\arg z=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$

3 a



b $\tan\alpha = \dfrac{9}{40}$

$\alpha = \arctan\left(\dfrac{9}{40}\right) = 0.2213...$

Here z is in the third quadrant,
so $\arg z = -(\pi - \alpha)$

$\arg z = -(\pi - 0.2213...) = -2.92$

4 a $z = 3 + 4\text{i}$

$z^2 = (3+4\text{i})^2 = 9 + 12\text{i} + 12\text{i} + 16\text{i}^2$
$= -7 + 24\text{i}$

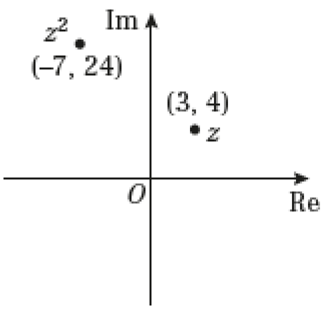
b $|z^2| = \sqrt{(-7)^2 + (24)^2} = \sqrt{625} = 25$

c $\tan\alpha = \dfrac{24}{7}$

$\alpha = \arctan\left(\dfrac{24}{7}\right) = 1.2870...$

z^2 is in the second quadrant, so
$\arg z^2 = \pi - \alpha$
$\arg z^2 = \pi - 1.2870... = 1.85$

d



5 a $\dfrac{z_1}{z_2} = \dfrac{4+6\text{i}}{1+\text{i}}$

Multiply by the complex conjugate

$\dfrac{4+6\text{i}}{1+\text{i}} = \dfrac{(4+6\text{i})}{(1+\text{i})} \times \dfrac{(1-\text{i})}{(1-\text{i})}$

$= \dfrac{4 - 4\text{i} + 6\text{i} - 6\text{i}^2}{1 - \text{i} + \text{i} - \text{i}^2}$

$= \dfrac{10}{2} + \dfrac{2}{2}\text{i}$

$= 5 + \text{i}$

b $\left|\dfrac{z_1}{z_2}\right| = \sqrt{(5)^2 + (1)^2}$

$= \sqrt{26}$

c $\tan\alpha = \dfrac{1}{5}$

$\alpha = \arctan\dfrac{1}{5} = 0.20$

As z is in the first quadrant
$\arg z = \alpha = 0.20$ radians (2 d.p.)

6 a $\dfrac{z_1}{z_2} = 1 - \text{i} \Rightarrow z_2 = \dfrac{z_1}{1-\text{i}}$

$z_2 = \dfrac{3+2p\text{i}}{1-\text{i}}$

Multiply by the complex conjugate

$z_2 = \dfrac{(3+2p\text{i})}{(1-\text{i})} \times \dfrac{(1+\text{i})}{(1+\text{i})}$

$= \dfrac{3 + 3\text{i} + 2p\text{i} + 2p\text{i}^2}{1 + \text{i} - \text{i} - \text{i}^2}$

$= \left(\dfrac{3-2p}{2}\right) + \left(\dfrac{3+2p}{2}\right)\text{i}$

b $\arg z_2 = \arctan\left(\dfrac{\frac{3+2p}{2}}{\frac{3-2p}{2}}\right)$

$= \arctan\left(\dfrac{3+2p}{3-2p}\right)$

Since we are told that $\arg z_2 = \arctan 5$ it follows that:

$\dfrac{3+2p}{3-2p} = 5$

$3 + 2p = 15 - 10p$

$-12 = -12p$

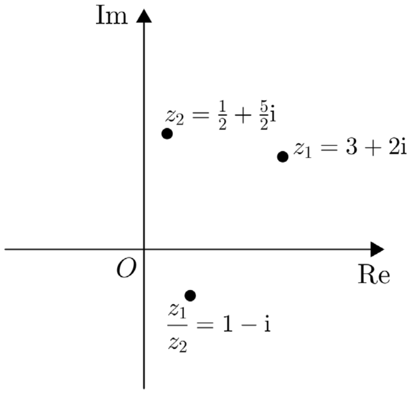
$p = 1$

6 c $z_2 = \frac{3-2p}{2} + \frac{3+2p}{2}\text{i}$

When $p = 1,\ z_2 = \frac{1}{2} + \frac{5}{2}\text{i}$

$|z_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{26}{4}} = \frac{\sqrt{26}}{2}$

d



7 a $z = \frac{26}{2-3\text{i}}$

Multiply by the complex conjugate

$z = \frac{26}{(2-3\text{i})} \times \frac{(2+3\text{i})}{(2+3\text{i})}$

$= \frac{52+78\text{i}}{4+6\text{i}-6\text{i}-9\text{i}^2}$

$= \frac{52}{13} + \frac{78}{13}\text{i}$

$= 4 + 6\text{i}$

b $z^2 = (4+6\text{i})^2 = 16 + 24\text{i} + 24\text{i} + 36\text{i}^2$

So $z^2 = -20 + 48\text{i}$

c $|z| = \sqrt{(4)^2 + (6)^2} = \sqrt{52} = 2\sqrt{13}$

d $\tan\alpha = \frac{48}{20}$

$\alpha = \tan^{-1}\left(\frac{48}{20}\right) = 1.1760...$

Here z^2 is in the second quadrant

so $\arg(z^2) = \pi - \alpha$

$\arg(z^2) = \pi - 1.1760... = 1.97$

radians (2 d.p.)

8 a $z_1 + z_2 = (4+2\text{i}) + (2+4\text{i}) = 6 + 6\text{i}$

$|z_1 + z_2| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$

b $w = \frac{z_1 z_3}{z_2} = \frac{(4+2\text{i})(a+b\text{i})}{2+4\text{i}}$

$z_1 z_3 = (4+2\text{i})(a+b\text{i})$

$= 4a + 4b\text{i} + 2a\text{i} + 2b\text{i}^2$

$= (4a-2b) + (4b+2a)\text{i}$

$w = \frac{(4a-2b)+(4b+2a)\text{i}}{2+4\text{i}}$

Multiply by the complex conjugate

$w = \frac{((4a-2b)+(4b+2a)\text{i})}{(2+4\text{i})} \times \frac{(2-4\text{i})}{(2-4\text{i})}$

$= \frac{8a-4b-(16a-8b)\text{i}+(8b+4a)\text{i}-(16b+8a)\text{i}^2}{4-8\text{i}+8\text{i}-16\text{i}^2}$

$= \frac{(16a+12b)+(-12a+16b)\text{i}}{20}$

$= \frac{16a+12b}{20} + \frac{-12a+16b}{20}\text{i}$

$= \frac{4a+3b}{5} + \frac{-3a+4b}{5}\text{i}$

c $w = \frac{21}{5} - \frac{22}{5}\text{i} = \frac{4a+3b}{5} + \frac{-3a+4b}{5}\text{i}$

Equate real coefficients:

$4a + 3b = 21$ **(1)**

Equate imaginary coefficients:

$-3a + 4b = -22$ **(2)**

$3 \times$ **(1)**: $12a + 9b = 63$ **(3)**

$4 \times$ **(2)**: $-12a + 16b = -88$ **(4)**

(3) + **(4)** $\Rightarrow 25b = -25$

$b = -1$

Substituting $b = -1$ into (1): $4a - 3 = 21$

$a = 6$

d $\tan\alpha = \frac{\frac{22}{5}}{\frac{21}{5}} = \frac{22}{21}$

$\alpha = \arctan\left(\frac{22}{21}\right) = 0.8086...$

Here z is in the fourth quadrant, so $\arg z = -\alpha$

$\arg z = -0.81$ radians (2 d.p.)

9 a $|w| = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 3\sqrt{5}$

b $\tan\alpha = \frac{3}{6} = \frac{1}{2}$

$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636...$

Here w is in the first quadrant

so $\arg z = \alpha$

$\arg z = 0.46$ radians (2 d.p.)

c $\arg(\lambda + 5\mathrm{i} + w)$

$= \arg(\lambda + 5\mathrm{i} + 6 + 3\mathrm{i})$

$= \arg((\lambda + 6) + 8\mathrm{i})$

$\tan\alpha = \frac{8}{\lambda + 6}$

As $\tan\frac{\pi}{4} = 1$, it follows that

$\frac{8}{\lambda + 6} = 1$, so $\lambda = 2$.

10 a $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

b $z = -1 - \mathrm{i}\sqrt{3}$, so $z^* = -1 + \mathrm{i}\sqrt{3}$

$\frac{z}{z^*} = \frac{-1 - \mathrm{i}\sqrt{3}}{-1 + \mathrm{i}\sqrt{3}}$

Multiply by the complex conjugate

$\frac{z}{z^*} = \frac{(-1 - \mathrm{i}\sqrt{3})}{(-1 + \mathrm{i}\sqrt{3})} \times \frac{(-1 - \mathrm{i}\sqrt{3})}{(-1 - \mathrm{i}\sqrt{3})}$

$= \frac{1 + \mathrm{i}\sqrt{3} + \mathrm{i}\sqrt{3} + \mathrm{i}^2(\sqrt{3})^2}{1 + \mathrm{i}\sqrt{3} - \mathrm{i}\sqrt{3} - \mathrm{i}^2(\sqrt{3})^2}$

$= \frac{-2 + 2\mathrm{i}\sqrt{3}}{4}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

$\left|\frac{z}{z^*}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

c For $\arg z$, $\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \arctan(\sqrt{3}) = \frac{\pi}{3}$

Here z is in the third quadrant, so

$\arg z = -(\pi - \alpha)$

$\arg z = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

For $\arg z^*$, $\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \arctan(\sqrt{3}) = \frac{\pi}{3}$

Here z^* is in the second quadrant, so $\arg z^* = \pi - \alpha$

$\arg z^* = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

For $\arg\left(\frac{z}{z^*}\right)$, $\tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\alpha = \arctan(\sqrt{3}) = \frac{\pi}{3}$

Here $\frac{z}{z^*}$ is in the second quadrant, so $\arg\left(\frac{z}{z^*}\right) = \pi - \alpha$

$\arg\left(\frac{z}{z^*}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

11 $w+z=(k+\mathrm{i})+(-4+5k\mathrm{i})$

$=(k-4)+(1+5k)\mathrm{i}$

$\arg(w+z)=\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}=-\sqrt{3}$

$\frac{1+5k}{k-4}=\frac{-\sqrt{3}}{1}$

$1+5k=-k\sqrt{3}+4\sqrt{3}$

So $5k+k\sqrt{3}=4\sqrt{3}-1$

$k(5+\sqrt{3})=4\sqrt{3}-1$

$k=\frac{4\sqrt{3}-1}{5+\sqrt{3}}$

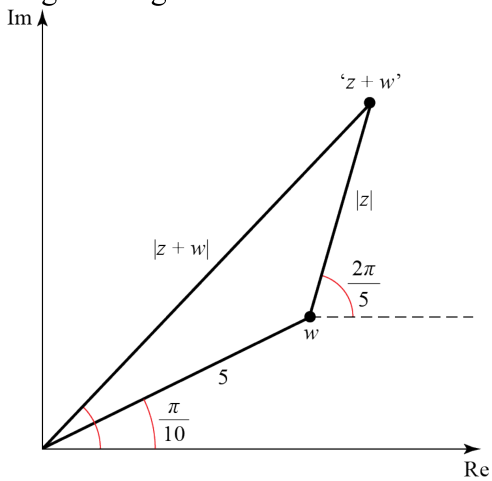
Rationalise the denominator by multiplying by the conjugate

$k=\frac{(4\sqrt{3}-1)}{(5+\sqrt{3})}\times\frac{(5-\sqrt{3})}{(5-\sqrt{3})}$

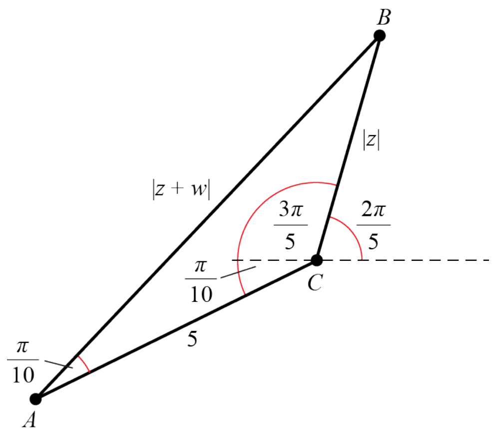
$=\frac{20\sqrt{3}-12-5+\sqrt{3}}{25-5\sqrt{3}+5\sqrt{3}-3}$

$=\frac{21\sqrt{3}-17}{22}$

12 Represent the given information on an Argand diagram:



Using this information, consider the triangle ΔABC shown below:



$\angle A=\frac{\pi}{5}-\frac{\pi}{10}=\frac{\pi}{10}$

$\angle C=\frac{3\pi}{5}+\frac{\pi}{10}=\frac{7\pi}{10}$

The angles in a triangle sum to π, so

$\angle B=\pi-\angle C-\angle A$

$=\pi-\frac{7\pi}{10}-\frac{\pi}{10}$

$=\frac{\pi}{5}$

Using the sine rule:

$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

$\frac{\sin\frac{\pi}{10}}{|z|}=\frac{\sin\frac{\pi}{5}}{5}=\frac{\sin\frac{7\pi}{10}}{|z+w|}$

$5\sin\frac{\pi}{10}=|z|\sin\frac{\pi}{5}$

$|z|=\frac{5\sin\frac{\pi}{10}}{\sin\frac{\pi}{5}}\approx 2.63$