Argand diagrams 2D

1 a i $|z_1z_2| = |z_1||z_2|$

$|z_1| = 5, |z_2| = 6$

$|z_1z_2| = 5\times 6 = 30$

ii $\arg(z_1z_2) = \arg z_1 + \arg z_2$

$\arg z_1 = \frac{3\pi}{8}, \arg z_2 = \frac{7\pi}{8}$

$\arg(z_1z_2) = \frac{3\pi}{8} + \frac{7\pi}{8}$

$= \frac{10\pi}{8}$

$= \frac{5\pi}{4}$

iii $z_1z_2 = r(\cos\theta + \mathrm{i}\sin\theta)$

$= 30\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right)$

b i $|z_1z_2| = |z_1||z_2|$

$|z_1| = \sqrt{2}, |z_2| = 4\sqrt{2}$

$|z_1z_2| = \sqrt{2}\times 4\sqrt{2} = 4\times 2 = 8$

ii $\arg(z_1z_2) = \arg z_1 + \arg z_2$

$\arg z_1 = \frac{\pi}{3}, \arg z_2 = \frac{3\pi}{4}$

$\arg(z_1z_2) = \frac{\pi}{3} + \frac{3\pi}{4}$

$= \frac{13\pi}{12}$

iii $z_1z_2 = r(\cos\theta + \mathrm{i}\sin\theta)$

$= 8\left(\cos\frac{13\pi}{12} + \mathrm{i}\sin\frac{13\pi}{12}\right)$

2 a $|z_1z_2| = 8\times 4 = 32$

$\arg(z_1z_2) = \frac{8\pi}{5} + \frac{2\pi}{3}$

$= \frac{24\pi}{15} + \frac{10\pi}{15}$

$= \frac{34\pi}{15}$

$\frac{34\pi}{15}$ is not in the range $-\pi < \theta < \pi$

$\frac{34\pi}{15} - 2\pi = \frac{34\pi}{15} - \frac{30\pi}{15} = \frac{4\pi}{15}$

So $\arg(z_1z_2) = \frac{4\pi}{15}$

b $\left|\frac{z_1}{z_2}\right| = \frac{8}{4} = 2$

$\arg\left(\frac{z_1}{z_2}\right) = \frac{8\pi}{5} - \frac{2\pi}{3}$

$= \frac{24\pi}{15} - \frac{10\pi}{15}$

$= \frac{14\pi}{15}$

c $|z_1^2| = |z_1z_1| = 8\times 8 = 64$

$\arg(z_1^2) = \arg(z_1z_1)$

$= \frac{8\pi}{5} + \frac{8\pi}{5}$

$= \frac{16\pi}{5}$

$\frac{16\pi}{5}$ is not in the range $-\pi < \theta < \pi$

$\frac{16\pi}{5} - 2\pi = \frac{16\pi}{5} - \frac{10\pi}{5} = \frac{6\pi}{5}$

$\frac{6\pi}{5}$ is still not in the range $-\pi < \theta < \pi$, so we need to subtract 2π again:

$\frac{6\pi}{5} - 2\pi = \frac{6\pi}{5} - \frac{10\pi}{5} = -\frac{4\pi}{5}$

So $\arg(z_1^2) = -\frac{4\pi}{5}$

3 a $(\cos 2\theta + i\sin 2\theta)(\cos 3\theta + i\sin 3\theta)$

$= \cos(2\theta + 3\theta) + i\sin(2\theta + 3\theta)$

$= \cos 5\theta + i\sin 5\theta$

b $\left(\cos\frac{3\pi}{11} + i\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + i\sin\frac{8\pi}{11}\right)$

$= \cos\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right) + i\sin\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right)$

$= \cos\pi + i\sin\pi$

$= -1 + i(0)$

$= -1$

c $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= (3 \times 2)\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$= 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 3 + 3\sqrt{3}i$

d $\sqrt{6}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

The first factor is not in modulus-argument form. Use

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

to write

$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3}$ and $\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}$

$\sqrt{6}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= \sqrt{6} \times \sqrt{3}\left(\cos\left(-\frac{\pi}{3} + \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3} + \frac{\pi}{3}\right)\right)$

$= \sqrt{18}(\cos 0 + i\sin 0)$

$= \sqrt{18}(1 + 0i)$

$= 3\sqrt{2}$

e

$4\left(\cos\frac{5\pi}{9} - i\sin\frac{5\pi}{9}\right) \times \frac{1}{2}\left(\cos\frac{5\pi}{18} - i\sin\frac{5\pi}{18}\right)$

Here, neither factor is in modulus-argument form. Use

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

to write:

$\cos\left(-\frac{5\pi}{9}\right) = \cos\frac{5\pi}{9}$ and

$\sin\left(-\frac{5\pi}{9}\right) = -\sin\frac{5\pi}{9}$

Also write:

$\cos\left(-\frac{5\pi}{18}\right) = \cos\frac{5\pi}{18}$ and

$\sin\left(-\frac{5\pi}{18}\right) = -\sin\frac{5\pi}{18}$

$4\left(\cos\left(-\frac{5\pi}{9}\right) + i\sin\left(-\frac{5\pi}{9}\right)\right) \times$

$\frac{1}{2}\left(\cos\left(-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{18}\right)\right)$

$= 4 \times \frac{1}{2}\left(\cos\left(-\frac{5\pi}{9} - \frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{9} - \frac{5\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{15\pi}{18}\right) + i\sin\left(-\frac{15\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -\sqrt{3} - i$

f

$6\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\times \frac{1}{3}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$= \left(6 \times 5 \times \frac{1}{3}\right)\left(\cos\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right)\right)$

$= 10\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 10\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -5\sqrt{3} + 5i$

3 g First, re-write the second factor so it is in modulus-argument form:

$$(\cos 4\theta + i\sin 4\theta)(\cos\theta - i\sin\theta)$$
$$= (\cos 4\theta + i\sin 4\theta)(\cos(-\theta) + i\sin(-\theta))$$
$$= \cos\left(4\theta + (-\theta)\right) + i\sin\left(4\theta + (-\theta)\right)$$
$$= \cos 3\theta + i\sin 3\theta$$

h

$$3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$= 3\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right) \times \sqrt{2}\left(\cos\left(-\tfrac{\pi}{3}\right) + i\sin\left(-\tfrac{\pi}{3}\right)\right)$$
$$= 3(\sqrt{2})\left(\cos\left(\frac{\pi}{12} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{12} - \frac{\pi}{3}\right)\right)$$
$$= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$
$$= 3 - 3i$$

4 a

$$\frac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$$
$$= \cos(5\theta - 2\theta) + i\sin(5\theta - 2\theta)$$
$$= \cos 3\theta + i\sin 3\theta$$

b

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$$
$$= \frac{\sqrt{2}}{\left(\frac{1}{2}\right)}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$
$$= 2 + 2i$$

c

$$\frac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$$
$$= \frac{3}{4}\left(\cos\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)\right)$$
$$= \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$
$$= \frac{3}{4}(0 - i)$$
$$= -\frac{3}{4}i$$

d

$$\frac{\cos 2\theta - i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$$
$$= \frac{\cos(-2\theta) + i\sin(-2\theta)}{\cos 3\theta + i\sin 3\theta}$$
$$= \cos(-2\theta - 3\theta) + i\sin(-2\theta - 3\theta)$$
$$= \cos(-5\theta) + i\sin(-5\theta)$$
$$= \cos 5\theta - i\sin 5\theta$$

5 a $z = -9 + 3i\sqrt{3}$

$$|z| = \sqrt{(-9)^2 + \left(3\sqrt{3}\right)^2}$$
$$= \sqrt{81 + 27}$$
$$= \sqrt{108}$$
$$= 6\sqrt{3}$$
$$\tan\alpha = \frac{3\sqrt{3}}{9} = \frac{\sqrt{3}}{3}$$
$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

As z is in the second quadrant, $\arg z = \pi - \alpha$

$$\arg z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$
$$z = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

b i $w = \sqrt{3}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

5 b ii $zw = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)\times \sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$

$$= 6\sqrt{3}\times\sqrt{3}\left(\cos\left(\tfrac{5\pi}{6}+\tfrac{7\pi}{12}\right) + \mathrm{i}\sin\left(\tfrac{5\pi}{6}+\tfrac{7\pi}{12}\right)\right)$$

$$= 18\left(\cos\frac{17\pi}{12} + \mathrm{i}\sin\frac{17\pi}{12}\right)$$

$\frac{17\pi}{12}$ is not in the range $-\pi < \theta < \pi$.

$$\frac{17\pi}{12} - 2\pi = \frac{17\pi}{12} - \frac{24\pi}{12} = -\frac{7\pi}{12}$$

$$zw = 18\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$= 18\left(\cos\frac{7\pi}{12} - \mathrm{i}\sin\frac{7\pi}{12}\right)$$

iii

$$\frac{z}{w} = \frac{6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}{\sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{9\pi}{12}\right)}$$

$$= \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\left(\tfrac{5\pi}{6}-\tfrac{7\pi}{12}\right) + \mathrm{i}\sin\left(\tfrac{5\pi}{6}-\tfrac{7\pi}{12}\right)\right)$$

$$= 6\left(\cos\frac{3\pi}{12} + \mathrm{i}\sin\frac{3\pi}{12}\right)$$

$$= 6\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$$

Challenge

a $z = 1 + \mathrm{i}\sqrt{3}$

$$|z| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$\tan\alpha = \frac{\sqrt{3}}{1}$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$z = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$$

$$z^7 = 2^7\left(\cos\frac{7\pi}{3} + \mathrm{i}\sin\frac{7\pi}{3}\right)$$

$\frac{7\pi}{3}$ is not in the range $-\pi < \theta < \pi$.

$$\frac{7\pi}{3} - 2\pi = \frac{7\pi}{3} - \frac{6\pi}{3} = \frac{\pi}{3}$$

$$z^7 = 128\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) \quad (*)$$

$$= 64\left[2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)\right]$$

$$= 64z$$

Alternatively, from (*):

$$z^7 = 128\left(\frac{1}{2} + \mathrm{i}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$z^7 = 64 + 64\mathrm{i}\sqrt{3}$$

So $z^7 = 64\left(1 + \mathrm{i}\sqrt{3}\right) = 64z$

Hence $k = 64$.

b $z^4 = 2^4\left(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}\right)$

$$z^4 = 2^4\left(-\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right) \quad (**)$$

$$= -2^3\left[2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)\right]$$

$$= -8z$$

Alternatively, from (**):

$$z^4 = 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)$$

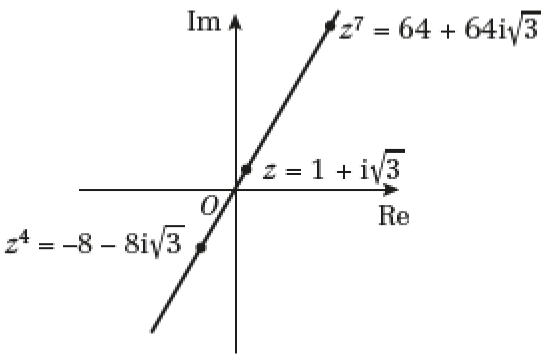
$$z^4 = -8 - 8\mathrm{i}\sqrt{3}$$

$$z^4 = -8\left(1 + \mathrm{i}\sqrt{3}\right) = -8z$$

So $p = -8$.

Challenge

c When the points z, z^7 and z^4 are plotted on an Argand diagram, they form a straight line as shown below.



[Note that you can also deduce that z, z^7 and z^4 lie in a straight line by knowing that z^7 and z^4 are scalar multiples of z by factors $k = 64$ and $p = -8$ respectively.]