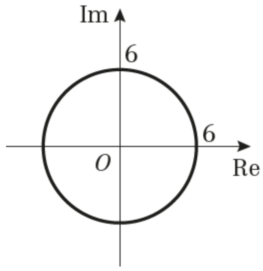
Argand diagrams 2E

1 a $|z| = 6$

circle centre (0, 0), radius 6

equation: $x^2 + y^2 = 6^2$

$x^2 + y^2 = 36$

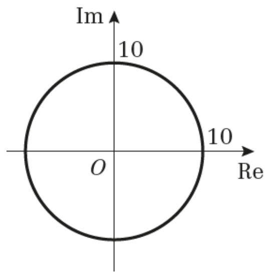


b $|z| = 10$

circle centre (0, 0), radius 10

equation: $x^2 + y^2 = 10^2$

$x^2 + y^2 = 100$

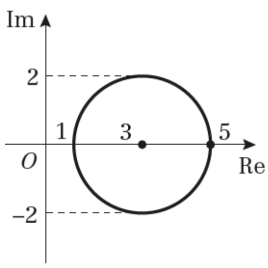


c $|z - 3| = 2$

circle centre (3, 0), radius 2

equation: $(x-3)^2 + y^2 = 2^2$

$(x-3)^2 + y^2 = 4$

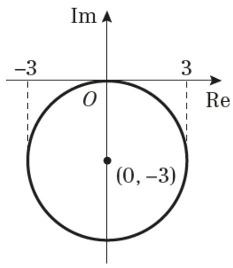


d $|z + 3\text{i}| = 3 \Rightarrow |z - (-3\text{i})| = 3$

circle centre (0, –3), radius 3

equation: $x^2 + (y+3)^2 = 3^2$

$x^2 + (y+3)^2 = 9$

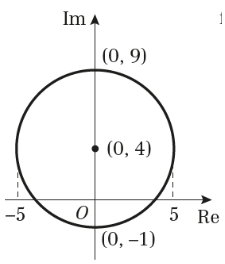


e $|z - 4\text{i}| = 5$

circle centre (0, 4), radius 5

equation: $x^2 + (y-4)^2 = 5^2$

$x^2 + (y-4)^2 = 25$

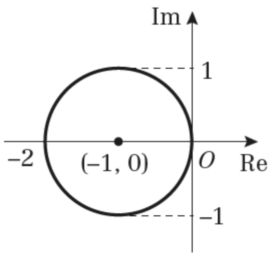


f $|z + 1| = 1 \Rightarrow |z - (-1)| = 1$

circle centre (–1, 0), radius 1

equation: $(x+1)^2 + y^2 = 1^2$

$(x+1)^2 + y^2 = 1$

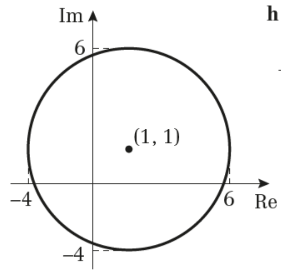


1 g $|z-1-\mathrm{i}|=5 \Rightarrow |z-(1+\mathrm{i})|=5$

circle centre (1, 1), radius 5

equation: $(x-1)^2+(y-1)^2=5^2$

$(x-1)^2+(y-1)^2=25$

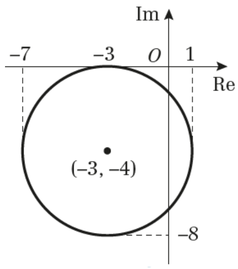


h $|z+3+4\mathrm{i}|=4 \Rightarrow |z-(-3-4\mathrm{i})|=4$

circle centre (–3, –4), radius 4

equation: $(x+3)^2+(y+4)^2=4^2$

$(x+3)^2+(y+4)^2=16$

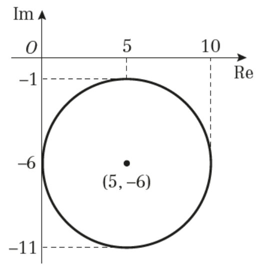


i $|z-5+6\mathrm{i}|=5 \Rightarrow |z-(5-6\mathrm{i})|=4$

circle centre (5, –6), radius 5

equation: $(x-5)^2+(y+6)^2=5^2$

$(x-5)^2+(y+6)^2=25$

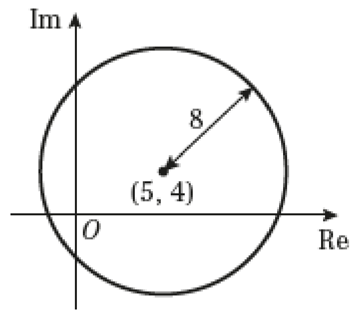


2 a $|z-5-4\mathrm{i}|=8$

$|z-(5+4\mathrm{i})|=8$

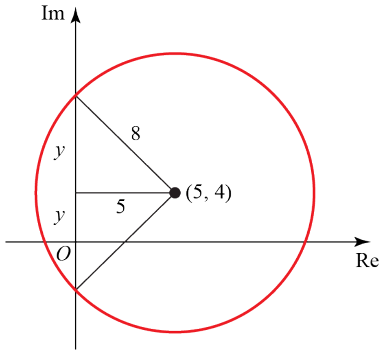
The locus of z is a circle centre $(5,4)$ and radius 8.

a



2 b i As $\text{Re}(z)=0$, this implies that the points lie on the Im axis.

Let y be the vertical distance between the centre of the circle and the points where the circle crosses the Im axis, as shown in the diagram.



Using Pythagoras' Theorem,

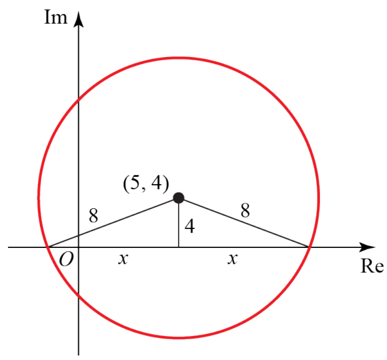
$$5^2+y^2=8^2$$
$$25+y^2=64$$
$$y^2=39$$
$$y=\pm\sqrt{39}$$

So $z=(4+y)\text{i}$ or $z=(4-y)\text{i}$

So $z=\left(4+\sqrt{39}\right)\text{i}$ or $z=\left(4-\sqrt{39}\right)\text{i}$

b ii As $\text{Im}(z)=0$, this implies that the points lie on the Re axis.

Let x be the horizontal distance between the centre of the circle and the points where the circle crosses the Re axis, as shown in the diagram.



Using Pythagoras' Theorem,

$$4^2+x^2=8^2$$
$$16+x^2=64$$
$$x^2=48$$
$$x=\pm4\sqrt{3}$$

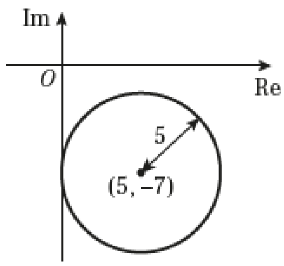
So $z=5+x$ or $z=5-x$

So $z=5+4\sqrt{3}$ or $z=5-4\sqrt{3}$

3 a $|z-5+7\text{i}|=5$

$|z-(5-7\text{i})|=5$

The locus of z is a circle with centre $(5,-7)$ and radius 5.



b Let $z=x+\text{i}y$

$$|z-5+7\text{i}|=5$$
$$|x+\text{i}y-5+7\text{i}|=5$$
$$|(x-5)+\text{i}(y+7)|=5$$

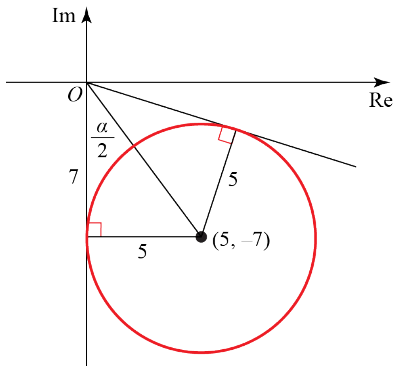
So, $(x-5)^2+(y+7)^2=5^2$

or $(x-5)^2+(y+7)^2=25$

3 c $\tan\frac{\alpha}{2} = \frac{5}{7}$

$\frac{\alpha}{2} = \arctan\left(\frac{5}{7}\right)$

$\alpha = 2\arctan\left(\frac{5}{7}\right)$



The maximum value of $\arg z$ is:

$\arg z = -\frac{\pi}{2} + 2\arctan\left(\frac{5}{7}\right)$

$\arg z = -\left(\frac{\pi}{2} - 2\arctan\left(\frac{5}{7}\right)\right)$

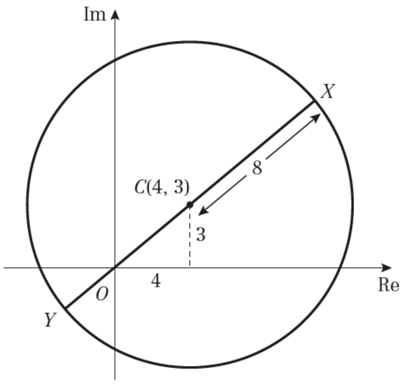
$\arg z = -0.330$ radians (3 s.f.)

4 a $|z-4-3\mathrm{i}| = 8 \Rightarrow |z-(4+3\mathrm{i})| = 8$

This is a circle centre (4, 3), radius 8

Hence the Cartesian equation of the locus of P is $(x-4)^2 + (y-3)^2 = 64$

b



c $|z|$ is the distance from (0, 0) to the locus of points.

$|z|_{max}$ is the distance OX

$|z|_{min}$ is the distance OY

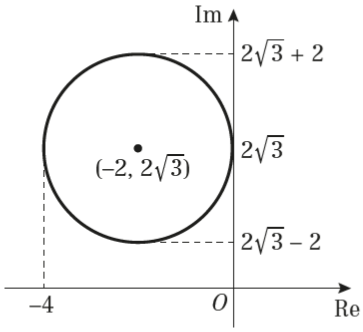
Now radius = $CY = CX = 8$ and $OC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

From the diagram, $|z|_{max} = OC + CX = 5 + 8 = 13$

$|z|_{min} = CY - OC = 8 - 5 = 3$

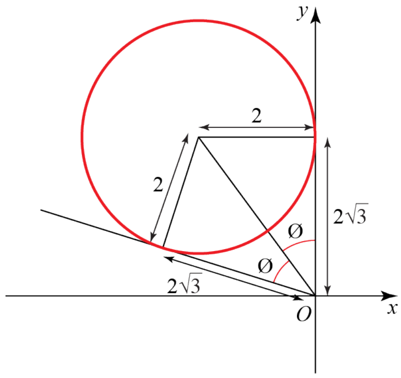
The maximum value of $|z|$ is 13 and the minimum value of $|z|$ is 3.

5 **a** $\left|z+2-2\sqrt{3}\mathrm{i}\right|=2$ is a circle centre $(-2,\ 2\sqrt{3})$, radius 2



b From the diagram, the minimum value of arg(z) is $\dfrac{\pi}{2}$.

c From the diagram, the maximum value of arg z is $\dfrac{\pi}{2}+2\phi$



Now

$$\tan\phi=\frac{2}{2\sqrt{3}}$$

$$\tan\phi=\frac{1}{\sqrt{3}}$$

$$\phi=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Hence $\arg(z)_{\max}=\dfrac{\pi}{2}+2\left(\dfrac{\pi}{6}\right)=\dfrac{5\pi}{6}$.

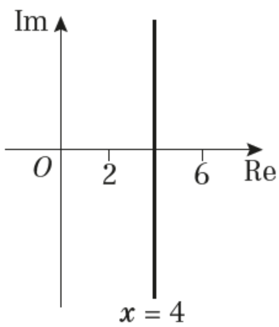
The maximum value of $\arg(z)$ is $\dfrac{5\pi}{6}$.

6 All parts can be solved using either a geometrical approach, or by substitution of $z=x+\mathrm{i}y$. Here, we have used a mixture of the two techniques.

a $|z-6|=|z-2|$

Locus is the perpendicular bisector of the line joining (6, 0) and (2, 0).
The midpoint of the line joining (6, 0) and (2, 0) is (4, 0)

Hece, the locus has equation: $x=4$

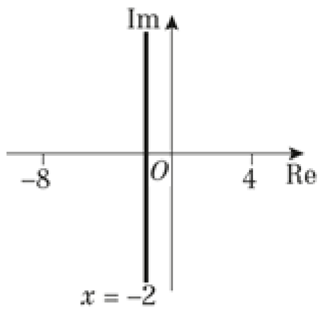


6 b $|z+8|=|z-4|$

$\Rightarrow |z-(-8)|=|z-4|$

Locus is the perpendicular bisector of the line joining (–8, 0) and (4, 0)
The midpoint of the line joining (–8, 0) and (4, 0) is (–2, 0)

Hence, the locus has equation: $x = -2$

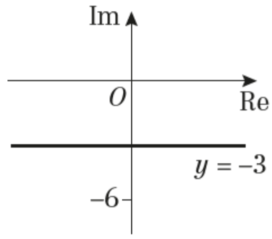


c $|z|=|z+6\text{i}|$

$\Rightarrow |z|=|z-(-6\text{i})|$

Locus is the perpendicular bisector of the line joining (0, 0) to (0, –6).
The midpoint of the line joining (0, 0) to (0, –6) is (0, –3)

Hence, the locus has equation: $y = -3$



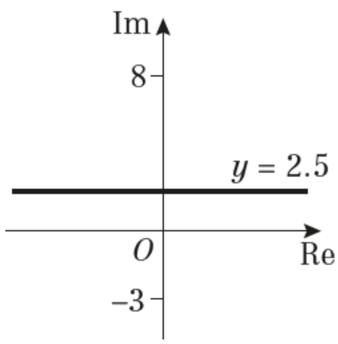
d $|z+3\text{i}|=|z-8\text{i}|$

The locus is the perpendicular bisector of the points $(0,-3)$ and $(0,8)$
As both these points lie on the Im axis, the perpendicular bisector will be a horizontal line.

The midpoint of $(0,-3)$ and $(0,8)$ is $\left(0,\frac{5}{2}\right)$

The equation of the perpendicular bisector is the horizontal line through $\left(0,\frac{5}{2}\right)$.

So the Cartesian equation of the locus is $y=\frac{5}{2}$



e $|z-2-2\text{i}|=|z+2+2\text{i}|$

Substitute $z = x+\text{i}y$

$|x+\text{i}y-2-2\text{i}|=|x+\text{i}y+2+2\text{i}|$

$\Rightarrow |(x-2)+\text{i}(y-2)|=|(x+2)+\text{i}(y+2)|$

$\Rightarrow (x-2)^2+(y-2)^2=(x+2)^2+(y+2)^2$

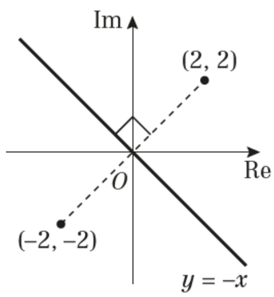
$\Rightarrow x^2-4x+4+y^2-4y+4=x^2+4x+4+y^2+4y+4$

$\Rightarrow -4x-4y^2+8=4x+4y+8$

$\Rightarrow 0=8x+8y$

$\Rightarrow -8x=8y$

$\Rightarrow y=-x$

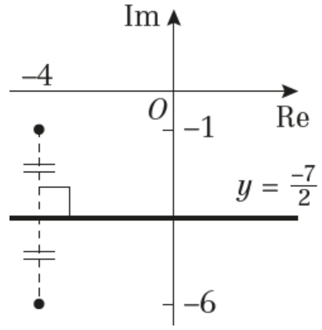


6 f $|z+4+\mathrm{i}| = |z+4+6\mathrm{i}|$

$\Rightarrow |z-(-4-\mathrm{i})| = |z+(-4-6\mathrm{i})|$

perpendicular bisector of the line joining (–4, –1) to (–4, –6).

Equation: $y = -\dfrac{7}{2}$

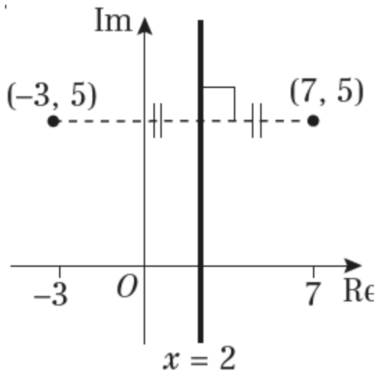


g $|z+3-5\mathrm{i}| = |z-7-5\mathrm{i}|$

$\Rightarrow |z-(-3+5\mathrm{i})| = |z-(7+5\mathrm{i})|$

perpendicular bisector of the line joining (–3, 5) to (7, 5).

Equation: $x = 2$



h

$|z+4-2\mathrm{i}| = |z-8+2\mathrm{i}|$

Substitute $z = x + \mathrm{i}y$

$|x+\mathrm{i}y+4-2\mathrm{i}| = |x+\mathrm{i}y-8+2\mathrm{i}|$

$\Rightarrow |(x+4)+\mathrm{i}(y-2)| = |(x-8)+\mathrm{i}(y+2)|$

$\Rightarrow (x+4)^2+(y-2)^2 = (x-8)^2+(y+2)^2$

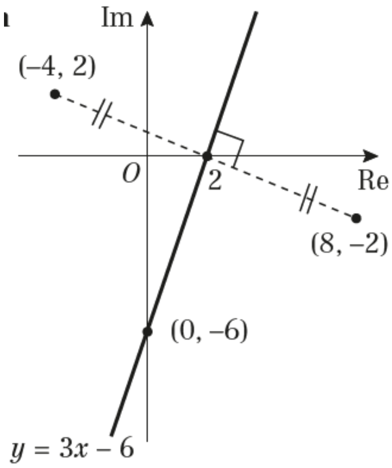
$\Rightarrow \cancel{x^2}+8x+16+\cancel{y^2}-4y+4 = \cancel{x^2}-16x+64+\cancel{y^2}+4y+4$

$\Rightarrow 8x-4y+20 = -16x+4y+68$

$\Rightarrow 0 = -24x+8y+48$

$\Rightarrow 0 = -3x+y+6$

$\Rightarrow 3x-6 = y$



6 i

$\left|\dfrac{z+3}{z-6\mathrm{i}}\right|=1$

$\Rightarrow |z+3|=|z-6\mathrm{i}|$

Substitute $z=x+\mathrm{i}y$

$|x+\mathrm{i}y+3|=|x+\mathrm{i}y-6\mathrm{i}|$

$\Rightarrow |(x+3)+\mathrm{i}y|=|x+\mathrm{i}(y-6)|$

$\Rightarrow (x+3)^2+y^2=x^2+(y-6)^2$

$\Rightarrow x^2+6x+9+y^2=x^2+y^2-12y+36$

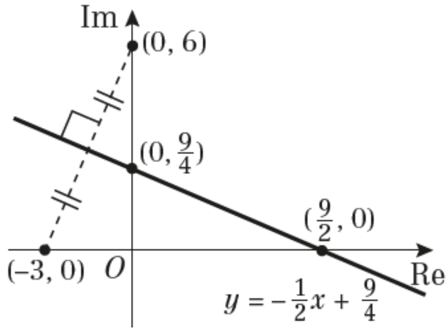
$\Rightarrow 6x+12y=36-9$

$\Rightarrow 6x+12y=27$

$\Rightarrow 2x+4y=9$

$\Rightarrow 4y=9-2x$

$\Rightarrow y=-\dfrac{1}{2}x+\dfrac{9}{4}$



j

$\dfrac{|z+6-\mathrm{i}|}{|z-10-5\mathrm{i}|}=1$

$|z+6-\mathrm{i}|=|z-10-5\mathrm{i}|$

$|z-(-6+\mathrm{i})|=|z-(10+5\mathrm{i})|$

The locus of z is the perpendicular bisector of the line segment joining the points $(-6,1)$ and $(10,5)$.

The gradient of the line joining $(-6,1)$ and $(10,5)$ is $\dfrac{1}{4}$.

So, the gradient of the perpendicular bisector is -4.

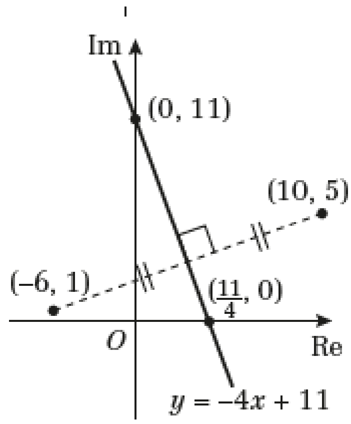
The midpoint of $(-6,1)$ and $(10,5)$ is $(2,3)$

The equation of the perpendicular bisector is found by using $y-y_1=m(x-x_1)$ with $m=-4$ and $(x_1,\ y_1)=(2,\ 3)$

$y-3=-4(x-2)$

$y-3=-4x+8$

$y=-4x+11$



So the Cartesian equation of the locus is $y=-4x+11$.

7 a $|z-3|=|z-6\text{i}|$

The locus of z is the perpendicular bisector of the line segment joining the points $(3,0)$ and $(0,6)$.

The gradient of the line joining $(3,0)$ and $(0,6)=-2$.

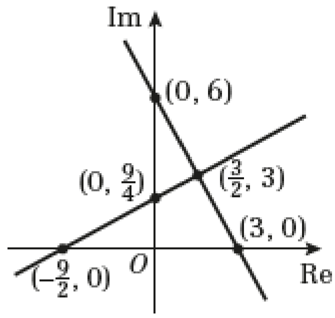
So, the gradient of the perpendicular bisector is $\frac{1}{2}$.

The midpoint of $(3,0)$ and $(0,6)$ is $\left(\frac{3}{2},3\right)$.

The equation of the perpendicular bisector is found by using

$y-y_1=m(x-x_1)$ with $m=\frac{1}{2}$ and $(x_1, y_1)=\left(\frac{3}{2}, 3\right)$

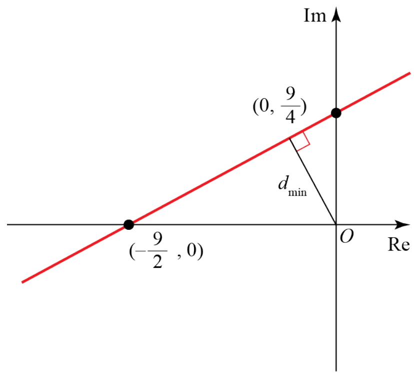
$$y-3=\frac{1}{2}\left(x-\frac{3}{2}\right)$$
$$y-3=\frac{1}{2}x-\frac{3}{4}$$
$$y=\frac{1}{2}x+\frac{9}{4}$$



b $|z|_{\min}$ occurs at the point on the locus where a line from the origin meets the locus of z at right angles (see diagram).

The line labelled $d_{\min}$ is perpendicular to the locus, so has gradient –2

It also passes through the origin, so has equation $y=-2x$



The lines intersect where:

$$-2x=\frac{1}{2}x+\frac{9}{4}$$
$$-\frac{5}{2}x=\frac{9}{4}$$
$$x=-\frac{18}{20}=-\frac{9}{10}$$

If $x=-\frac{9}{10}$, then $y=-2\left(-\frac{9}{10}\right)=\frac{9}{5}$

$$d_{\min}=\sqrt{\left(-\frac{9}{10}\right)^2+\left(\frac{9}{5}\right)^2}$$
$$=\sqrt{\frac{81}{100}+\frac{81}{25}}$$
$$=\sqrt{\frac{405}{100}}$$
$$=\frac{9\sqrt{5}}{10}$$

8 a&b $|z+3+3\mathrm{i}|=|z-9-5\mathrm{i}|$

$|z-(-3-3\mathrm{i})|=|z-(9+5\mathrm{i})|$

The locus of z is the equation of the perpendicular bisector joining $(-3,-3)$ and $(9,\ 5)$.

The gradient of the line joining $(-3,-3)$ and $(9,5)$ is $\frac{8}{12}=\frac{2}{3}$.

So the gradient of the perpendicular bisector is $-\frac{3}{2}$.

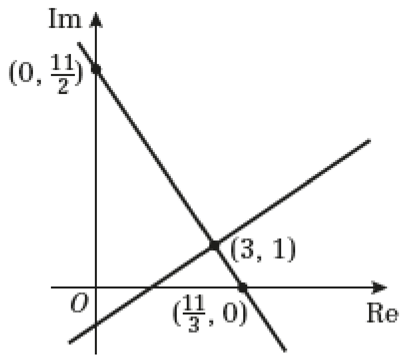
The midpoint of $(-3,-3)$ and $(9,\ 5)$ is $(3,\ 1)$

The equation of the perpendicular bisector is found by using $y-y_1=m(x-x_1)$ with $m=-\frac{3}{2}$ and $(x_1,\ y_1)=(3,\ 1)$

$$y-1=-\frac{3}{2}(x-3)$$

$$y-1=-\frac{3}{2}x+\frac{9}{2}$$

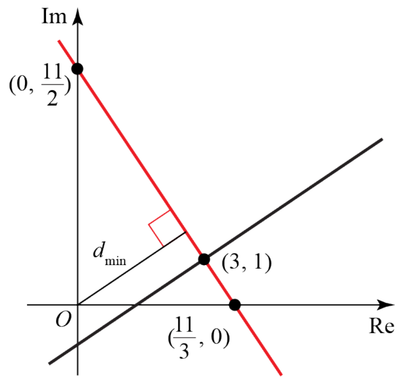
$$y=-\frac{3}{2}x+\frac{11}{2}$$



c $|z|_{\min}$ occurs at the point on the locus where a line from the origin meets the locus of z at right angles (see diagram).

The line labelled $d_{\min}$ is perpendicular to the locus, so has gradient –2

It also passes through the origin, so has equation $y=\frac{2}{3}x$



The lines intersect where:

$$\frac{2}{3}x=-\frac{3}{2}x+\frac{11}{2}$$

$$\left(\frac{4}{6}+\frac{9}{6}\right)x=\frac{11}{2}$$

$$\frac{13}{6}x=\frac{11}{2}$$

$$x=\frac{11\times 6}{2\times 13}=\frac{66}{26}=\frac{33}{13}$$

If $x=\frac{33}{13}$, $y=\frac{2}{3}\left(\frac{33}{13}\right)=\frac{22}{13}$

$$d_{\min}=\sqrt{\left(\frac{33}{13}\right)^2+\left(\frac{22}{13}\right)^2}$$

$$=\sqrt{\frac{1573}{169}}$$

$$=\frac{\sqrt{121}\sqrt{13}}{13}$$

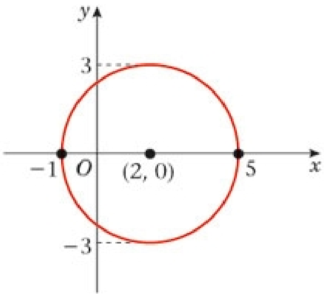
$$=\frac{11\sqrt{13}}{13}$$

9 a $|2-z|=3$

$|(-1)(z-2)|=3$

$|-1||(z-2)|=3$ ← $|-1|=1$

$|(z-2)|=3$



circle centre (2, 0), radius, 3

equation: $(x-2)^2+y^2=3^2$

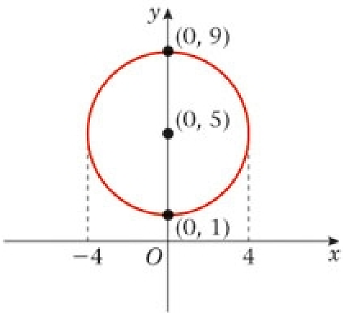
$(x-2)^2+y^2=9$

b $|5\text{i}-z|=4$

$|(-1)(z-5\text{i})|=4$

$|(-1)||(z-5\text{i})|=4$ ← $|-1|=1$

$|(z-5\text{i})|=4$



circle centre (0, 5), radius 4

equation: $x^2+(y-5)^2=4^2$

$x^2+(y-5)^2=16$

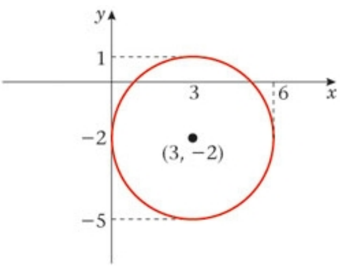
c $|3-2\text{i}-z|=3$

$|(-1)(z-3+2\text{i})|=3$

$|(-1)||(z-3+2\text{i})|=3$ ← $|-1|=1$

$|z-3+2\text{i}|=3$

$|z-(3-2\text{i})|=3$

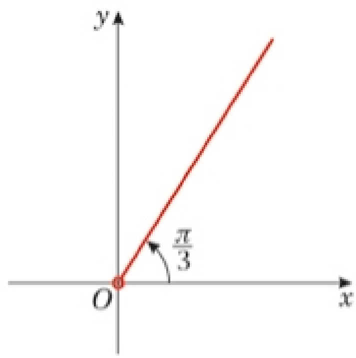


circle centre $(3,-2)$ radius 3

equation: $(x-3)^2+(y+2)^2=3^2$

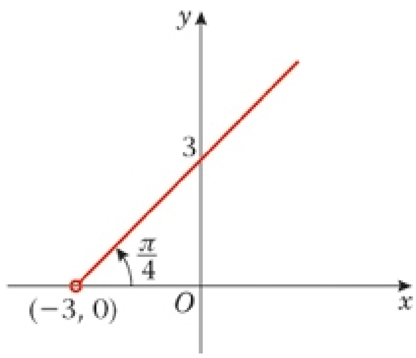
$(x-3)^2+(y+2)^2=9$

10 a $\arg z = \dfrac{\pi}{3}$

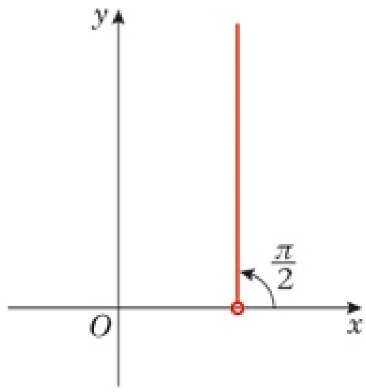


b $\arg(z+3) = \dfrac{\pi}{4}$

$\Rightarrow \arg(z-(-3)) = \dfrac{\pi}{4}$

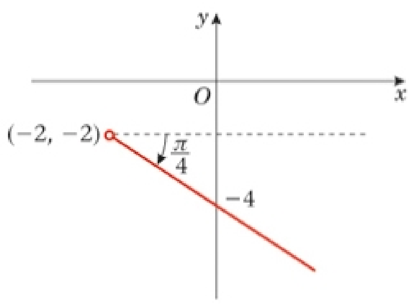


c $\arg(z+2) = \dfrac{\pi}{2}$



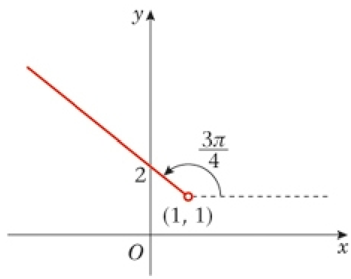
d $\arg(z+2+2\text{i}) = -\dfrac{\pi}{4}$

$\Rightarrow \arg(z-(-2-2\text{i})) = -\dfrac{\pi}{4}$



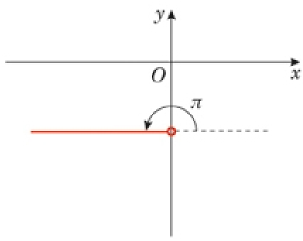
e $\arg(z-1-\text{i}) = \dfrac{3\pi}{4}$

$\Rightarrow \arg(z-(1+\text{i})) = \dfrac{3\pi}{4}$



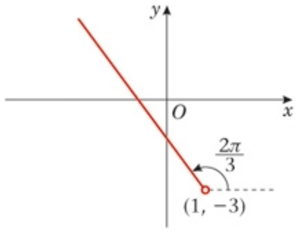
10 f $\arg(z+3\mathrm{i})=\pi$

$\Rightarrow \arg(z-(-3\mathrm{i}))=\pi$



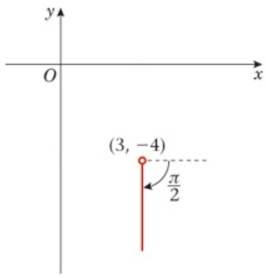
g $\arg(z-1+3\mathrm{i})=\frac{2\pi}{3}$

$\Rightarrow \arg(z-(1-3\mathrm{i}))=\frac{2\pi}{3}$

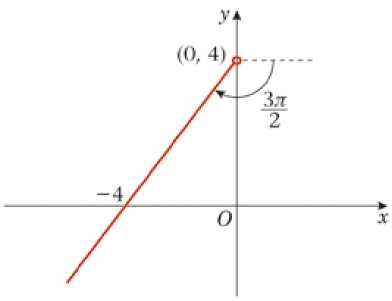


h $\arg(z-3+4\mathrm{i})=-\frac{\pi}{2}$

$\Rightarrow \arg(z-(3-4\mathrm{i}))=-\frac{\pi}{2}$

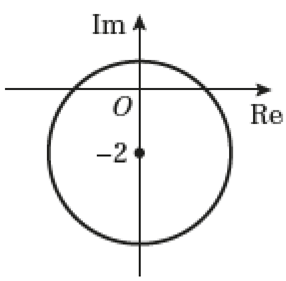


i $\arg(z-4\mathrm{i})=-\frac{3\pi}{4}$

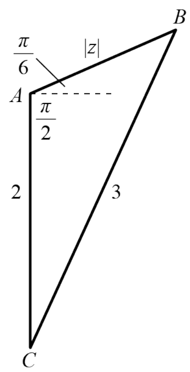


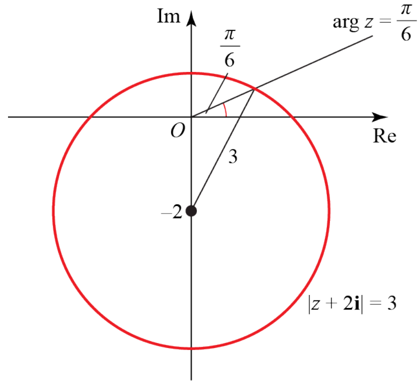
11 a $|z+2\mathrm{i}|=3$

The locus of z is a circle with centre $(0,-2)$ and radius 3.



11 b Using the information on the diagram to the right, we can create a triangle ABC as follows.





$\angle A = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$

Use the cosine rule to find $|z|$:

$a^2 = b^2 + c^2 - 2bc \times \cos A$

$3^2 = 2^2 + |z|^2 - 2(2)|z|\cos\frac{2\pi}{3}$

$9 = 4 + |z|^2 - 4|z|\left(-\frac{1}{2}\right)$

$9 = 4 + |z|^2 + 2|z|$

$0 = |z|^2 + 2|z| - 5$

Solve for $|z|$ by completing the square:

$(|z|+1)^2 - 1 - 5 = 0$

$(|z|+1)^2 = 6$

$|z| + 1 = \pm\sqrt{6}$

$|z| = -1 \pm \sqrt{6}$

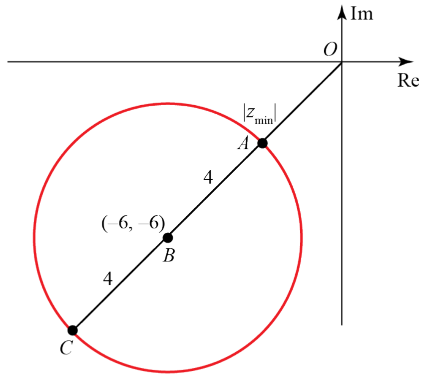
$|z| > 0$, so $|z| = -1 + \sqrt{6}$

12 a $|z+6+6\mathrm{i}|=4$

$|z-(-6-6\mathrm{i})|=4$

The locus of z is a circle with centre $(-6,-6)$ and radius 4.

This is shown in the diagram.



First find the distance from OB:

$|OB|=\sqrt{(-6)^2+(-6)^2}=\sqrt{72}=6\sqrt{2}$

Then $\quad |z_{\min}|=|OB|-|AB|$

$=6\sqrt{2}-4$

and $\quad |z_{\max}|=|OB|+|BC|$

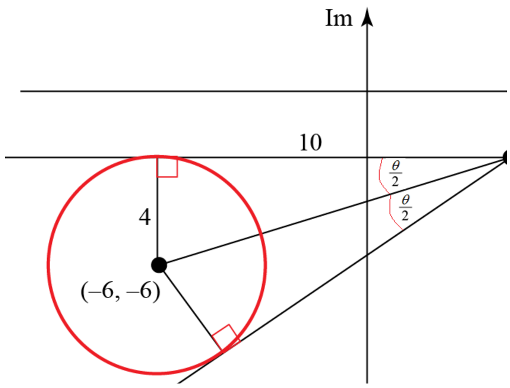
$=6\sqrt{2}+4$

b $\arg(z-4+2\mathrm{i})=\theta$

$\arg(z-(4-2\mathrm{i}))=\theta$

This is the half-line from $(4,-2)$

which makes an angle θ with the positive real axis.



Consider the diagram.

$\tan\left(\frac{\alpha}{2}\right)=\frac{4}{10}=\frac{2}{5}$

$\frac{\alpha}{2}=\arctan\left(\frac{2}{5}\right)$

$\alpha=2\arctan\left(\frac{2}{5}\right)=0.7610..$

The half-line $\arg(z-(4-2\mathrm{i}))=\theta$ intersects the circle $|z-(-6-6\mathrm{i})|=4$ when

$-\pi\leqslant\theta\leqslant-\pi+0.7610$

$\Rightarrow -\pi\leqslant\theta\leqslant-2.38...\qquad (*)$

So, there will be *no* common solutions for $\arg(z-(4-2\mathrm{i}))=\theta$ and $|z-(-6-6\mathrm{i})|=4$ for values of θ outside of the range given in (*), that is, no common solutions for $-2.38<\theta<\pi$.

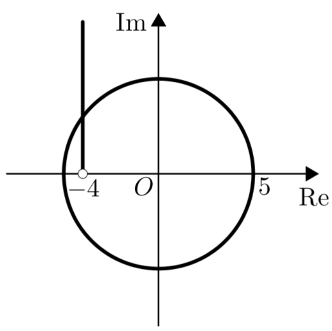
13 a Consider P:

$|z|=5\Rightarrow$ The locus of P is a circle centre $(0,0)$ and radius 5.

Consider Q:

$\arg(z+4)=\frac{\pi}{2}\Rightarrow \arg(z-(-4))=\frac{\pi}{2}$

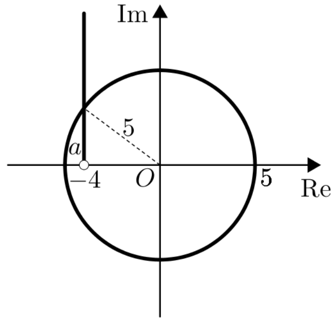
The locus of Q is a half-line from $(-4,0)$ making an angle of $\frac{\pi}{2}$ with the positive real axis.



13 b Consider the diagram shown.
Using Pythagoras, the loci of P and Q intersect when

$$(-4)^2 + a^2 = 5^2$$
$$16 + a^2 = 25$$
$$a^2 = 9$$
$$a = \pm 3$$

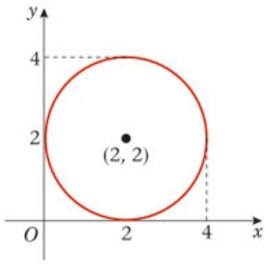
As seen in the diagram, $a > 0$, so $a = 3$.
So $z = -4 + 3\text{i}$.



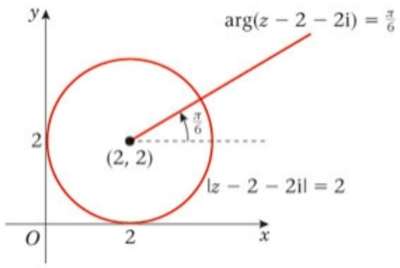
14 a $|z - 2 - 2\text{i}| = 2$

$\Rightarrow |z - (2 + 2\text{i})| = 2$

The locus of z is a circle centre (2, 2), radius 2.



b $\arg(z - 2 - 2\text{i}) = \frac{\pi}{6}$, is a half-line from (2, 2) which makes an angle $\frac{\pi}{6}$ with the positive real axis, as shown in the diagram.



$$|z - 2 - 2\text{i}| = 2 \Rightarrow (x-2)^2 + (y-2)^2 = 4 \qquad (1)$$

$$\arg(z - 2 - 2\text{i}) = \frac{\pi}{6} \Rightarrow \arg(x + \text{i}y - 2 - 2\text{i}) = \frac{\pi}{6}$$

$$\Rightarrow \arg((x-2) + \text{i}(y-2)) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y-2}{x-2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(x-2) \qquad (*)$$

$$\Rightarrow (y-2)^2 = \left[\frac{1}{\sqrt{3}}(x-2)\right]^2$$

$$\Rightarrow (y-2)^2 = \frac{1}{3}(x-2)^2 \qquad (2)$$

14 b Substituting (**2**) into (**1**) gives $(x-2)^2+\frac{1}{3}(x-2)^2=4$

$$\Rightarrow \frac{4}{3}(x-2)^2=4$$
$$\Rightarrow 4(x-2)^2=12$$
$$\Rightarrow (x-2)^2=3$$
$$\Rightarrow x-2=\pm\sqrt{3}$$
$$\Rightarrow x=2\pm\sqrt{3}$$

From the Argand diagram, $x>2$ so $x=2+\sqrt{3}$ (**3**)

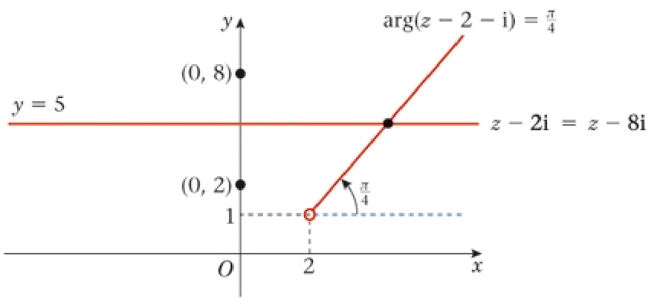
From (*), $y-2=\frac{1}{\sqrt{3}}(x-2)$

Substituting (**3**) into (*) gives $y-2=\frac{1}{\sqrt{3}}(2+\sqrt{3}-2)$

$$\Rightarrow y-2=\frac{1}{\sqrt{3}}(\sqrt{3})$$
$$\Rightarrow y-2=1$$
$$\Rightarrow y=3$$

Therefore, $z=(2+\sqrt{3})+3\text{i}$

15 a $|z-2\text{i}|=|z-8\text{i}|$
Locus is the perpendicular bisector of the line joining (0, 2) to (0, 8).
This is the line with equation $y=5$.



b $\arg(z-2-\text{i})=\frac{\pi}{4}$
Locus is the half-line from (2, 1) which makes an angle of $\frac{\pi}{4}$ with the positive real axis.

c Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:
$|z-2\text{i}|=|z-8\text{i}|$ has Cartesian equation $y=5$ (**1**)

$$\arg(z-2-\text{i})=\frac{\pi}{4}\Rightarrow \arg(x+\text{i}y-2-\text{i})=\frac{\pi}{4}$$
$$\Rightarrow \arg((x-2)+\text{i}(y-1))=\frac{\pi}{4}$$
$$\Rightarrow \frac{y-1}{x-2}=\tan\left(\frac{\pi}{4}\right)=1$$
$$\Rightarrow y-1=x-2$$
$$\Rightarrow y=x-1$$ (**2**)

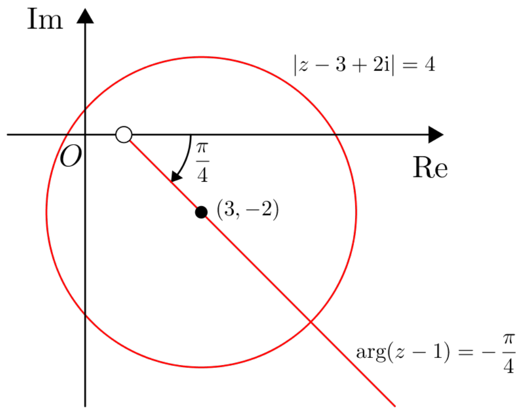
15 c Substituting (**1**) into (**2**) gives $5 = x - 1$

$\Rightarrow 6 = x$

From (**2**), $y = 6 - 1 = 5$

Therefore, $z = 6 + 5\text{i}$

16 a $|z - 3 + 2\text{i}| = 4$ is a circle centre (3, –2) radius 4.



b $\arg(z-1) = -\frac{\pi}{4}$ is a half-line from (1, 0) making an angle of $-\frac{\pi}{4}$ with the positive real axis.

c Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:

$$|z-3+2\text{i}| = 4 \Rightarrow (x-3)^2 + (y+2)^2 = 16 \quad (\mathbf{1})$$

$$\arg(z-1) = -\frac{\pi}{4} \Rightarrow \arg(x+\text{i}y-1) = -\frac{\pi}{4}$$

$$\Rightarrow \arg((x-1)+\text{i}y) = -\frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x-1} = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow = \frac{y}{x-1} = -1$$

$$\Rightarrow y = -1(x-1)$$

$$\Rightarrow y = -x + 1 \quad \text{for } x > 1, y < 0 \quad (\mathbf{2})$$

16 c **(cont.)**

Substituting **(2)** into **(1)** gives $(x-3)^2+(-x+1+2)^2=16$

$$\Rightarrow (x-3)^2+(-x+3)^2=16$$
$$\Rightarrow x^2-6x+9+x^2-6x+9=16$$
$$\Rightarrow 2x^2-12x+18=16$$
$$\Rightarrow 2x^2-12x+2=0$$
$$\Rightarrow x^2-6x+1=0$$
$$\Rightarrow x=\frac{6\pm\sqrt{36-4(1)(1)}}{2}$$
$$\Rightarrow x=\frac{6\pm\sqrt{32}}{2}$$
$$\Rightarrow x=\frac{6\pm\sqrt{16}\sqrt{2}}{2}$$
$$\Rightarrow x=\frac{6\pm4\sqrt{2}}{2}$$
$$\Rightarrow x=3\pm2\sqrt{2}$$

But $x>1$ so $x=3+2\sqrt{2}$

Substituting $x=3+2\sqrt{2}$ back into (**2**) gives

$y=-(3+2\sqrt{2})+1$

$y=-3-2\sqrt{2}+1$

$y=-2-2\sqrt{2}$

Therefore, $z=(3+2\sqrt{2})+(-2-2\sqrt{2})\text{i}$

So $a=3+2\sqrt{2},\ b=-2-2\sqrt{2}$

17 a Find the Cartesian equations of both loci, and then solve them simultaneously to find the point of intersection:

$\arg z=\frac{\pi}{3}\Rightarrow\arg(x+\text{i}y)=\frac{\pi}{3}$

$\Rightarrow\frac{y}{x}=\tan\frac{\pi}{3}$

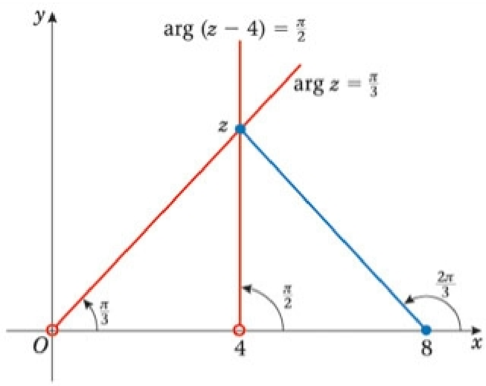
$\Rightarrow\frac{y}{x}=\sqrt{3}$

$\Rightarrow y=\sqrt{3}x\quad(\text{for } x>0, y>0)\qquad(\mathbf{1})$

$\arg\ (z-4)=\frac{\pi}{2}\Rightarrow x=4\,(\text{for } y>0)\qquad(\mathbf{2})$

Substituting **(2)** and **(1)** gives $y=\sqrt{3}(4)=4\sqrt{3}$

The value of z satisfying both equations is $y=4+4\sqrt{3}\,\text{i}$



From part **b** $\arg(z-8)=\frac{2\pi}{3}$.

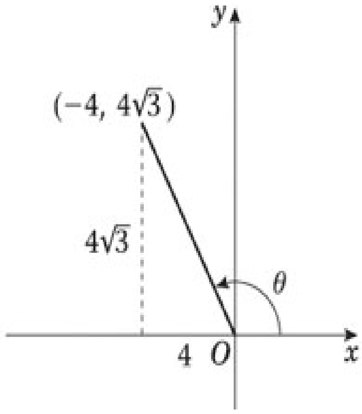
17 b $\arg(z-8\text{i}) = \arg(4+4\sqrt{3}\text{i}-8)$

$= \arg(-4+4\sqrt{3}\text{i}) = \theta$

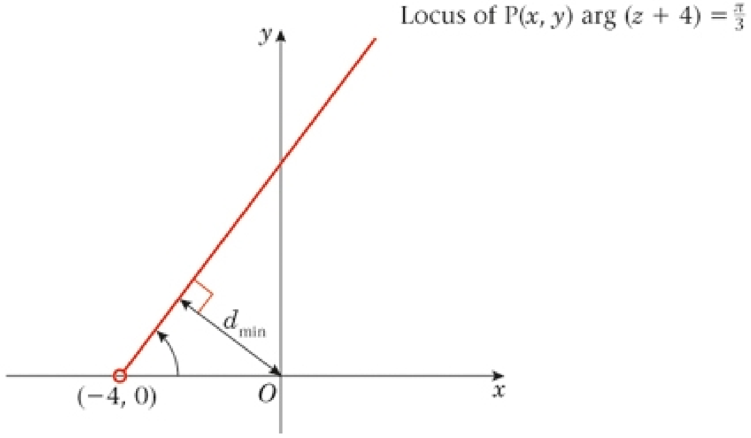
$\therefore \theta = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \pi - \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

Therefore, $\arg(z-8) = \frac{2\pi}{3}$.

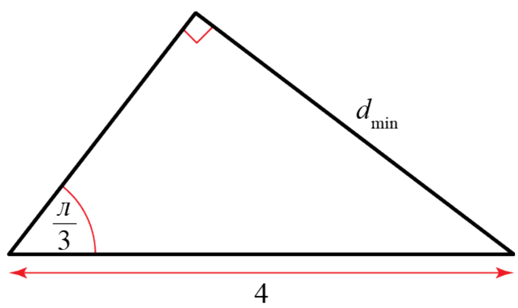


18 a



b $|z|_{\min}$ is the perpendicular distance from (0, 0) to the line $\arg(z+4) = \frac{\pi}{3}$

Consider the triangle shown.



$\frac{d_{\min}}{4} = \sin\left(\frac{\pi}{3}\right)$

$d_{\min} = 4\sin\left(\frac{\pi}{3}\right)$

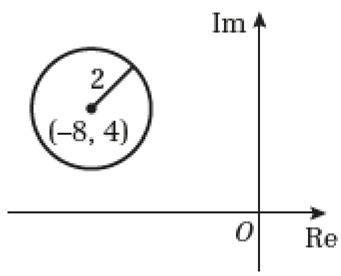
$d_{\min} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

Hence $|z|_{\min} = 2\sqrt{3}$.

19 a $|z+8-4\text{i}| = 2$

$|z-(-8+4\text{i})| = 2$

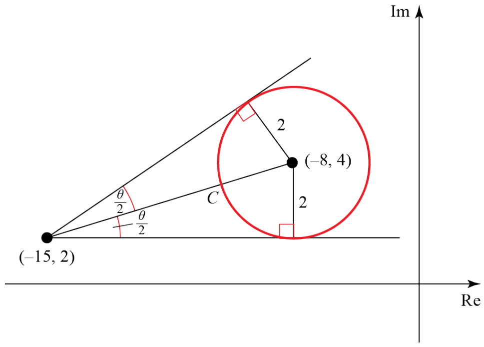
The locus of z is a circle with centre $(-8, 4)$ and radius 2.



19 b $\arg(z+15-2\mathrm{i})=\theta$

$\arg(z-(-15+2\mathrm{i}))=\theta$

The locus of z where $\arg(z-(-15+2\mathrm{i}))=\theta$ is a half-line from $(-15,\ 2)$ which makes an angle θ with the positive real axis.



The max value of $\arg(z-(-15+2\mathrm{i}))=\theta$ is found when the half-line is tangent to the top of the circle $|z-(-8+4\mathrm{i})|=2$

Find the length of c shown in the diagram:

$c^2=2^2+7^2$

$c^2=53$

$c=\sqrt{53}$

Then

$\sin\left(\frac{\theta}{2}\right)=\frac{2}{\sqrt{53}}$

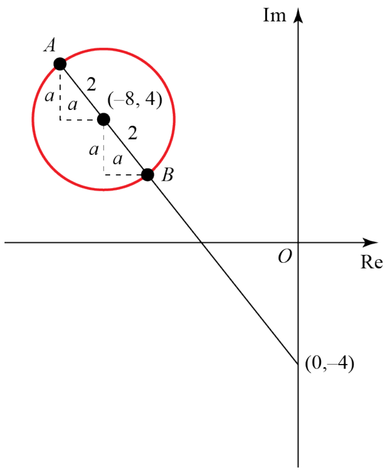
$\frac{\theta}{2}=\arcsin\frac{2}{\sqrt{53}}$

$\theta=2\arcsin\frac{2}{\sqrt{53}}$

c $\arg(z+4\mathrm{i})=\frac{3\pi}{4}$

$\arg\left(z-(-4\mathrm{i})=\frac{3\pi}{4}\right)$

The locus of z is a half-line from $(0,-4)$ with angle $\frac{3\pi}{4}$.



19 c Find a:

$$a^2 + a^2 = 2^2$$
$$2a^2 = 4$$
$$a = \pm\sqrt{2}$$

a is a distance, so $a > 0 \Rightarrow a = \sqrt{2}$.

Coordinates of A: $\left(-8-\sqrt{2}, 4+\sqrt{2}\right)$

Coordinates of B: $\left(-8+\sqrt{2}, 4-\sqrt{2}\right)$

So the complex numbers satisfying both $|z+8-4\text{i}| = 2$ and $\arg(z+4\text{i}) = \frac{3\pi}{4}$ are

$\left(-8-\sqrt{2}\right)+\left(4+\sqrt{2}\right)\text{i}$ and $\left(-8+\sqrt{2}\right)+\left(4-\sqrt{2}\right)\text{i}$

Challenge

$|z+\text{i}| = 5$

The locus of z is a circle with centre $(0,-1)$ and radius 5.

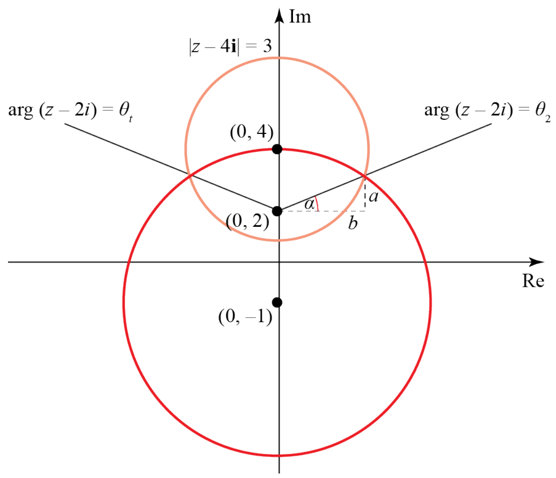
The Cartesian equation of the locus is $x^2 + (y+1)^2 = 5^2$.

$\arg(z-2\text{i}) = \theta$

The locus of z is a half-line from point $(0,2)$ which makes an angle θ with the positive real axis

$|z-4\text{i}| = 3$ is the circle $x^2 + (y-4)^2 = 3^2$

$|z-4\text{i}| < 3$ is the set of points inside the circle $x^2 + (y-4)^2 = 3^2$



Let the two limiting values of θ be θ_1 and θ_2

These values are found when $\arg(z-2\text{i}) = \theta$ meets the points where the circles $x^2 + (y+1)^2 = 5^2$ and $x^2 + (y-4)^2 = 3^2$ intersect.

Challenge (cont.)

First, find the points where $x^2+(y+1)^2=5^2$ and $x^2+(y-4)^2=3^2$ intersect:

$$25-(y+1)^2=9-(y-4)^2$$
$$25-(y^2+2y+1)=9-(y^2-8y+16)$$
$$25-y^2-2y-1=9-y^2+8y-16$$
$$10y=31$$
$$y=\frac{31}{10}$$

Substitute $y=\frac{31}{10}$ into $x^2+(y+1)^2=25$:

$$x^2+\left(\frac{31}{10}+1\right)^2=25$$
$$x^2+\left(\frac{41}{10}\right)^2=25$$
$$x^2=\frac{819}{100}$$
$$x=\pm\sqrt{\frac{819}{100}}=\pm\frac{3\sqrt{91}}{10}$$

Hence $x^2+(y+1)^2=5^2$ and $x^2+(y-4)^2=3^2$ intersect at $\left(-\frac{3\sqrt{91}}{10},\frac{31}{10}\right)$ and $\left(\frac{3\sqrt{91}}{10},\frac{31}{10}\right)$

Calculate the length a on the diagram:

$$a=\frac{31}{10}-2=\frac{11}{10}$$

Calculate the length b on the diagram:

$$b=\frac{3\sqrt{91}}{10}$$

Then $\tan\alpha=\frac{\frac{11}{10}}{\frac{3\sqrt{91}}{10}}\frac{11}{3\sqrt{91}}$

$$\alpha=\tan^{-1}\frac{11}{3\sqrt{91}}=0.3669...$$

So $\theta_1=\pi-0.3669=2.77$

and $\theta_2=0.3669=0.37$

The range of values for θ are $0.37<\theta<2.77$