Argand diagrams Mixed exercise 2

1 a $z^2 + 5z + 10 = 0$

Solve by completing the square.

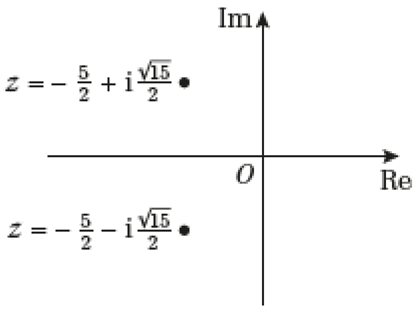
$$\left(z+\frac{5}{2}\right)^2 - \frac{25}{4} + 10 = 0$$

$$\left(z+\frac{5}{2}\right)^2 = -\frac{15}{4}$$

$$z+\frac{5}{2} = \pm\frac{\sqrt{15}}{2}\mathrm{i}$$

$$z = -\frac{5}{2} \pm \frac{\sqrt{15}}{2}\mathrm{i}$$

b



2 a If $f(-1+2\mathrm{i}) = 0$, then $(z-(-1+2\mathrm{i}))$ and $(z-(-1-2\mathrm{i}))$ are factors of $f(z)$.

Hence

$$(z-(-1+2\mathrm{i}))(z-(-1-2\mathrm{i}))$$
$$= z^2 - (-1-2\mathrm{i})z - (-1+2\mathrm{i})z + (-1+2\mathrm{i})(-1-2\mathrm{i})$$
$$= z^2 + z + 2\mathrm{i}z + z - 2\mathrm{i}z + 1 + 2\mathrm{i} - 2\mathrm{i} - 4\mathrm{i}^2$$
$$= z^2 + 2z + 5 \text{ is a quadratic factor of } f(z)$$

Use division to find the remaining factor.

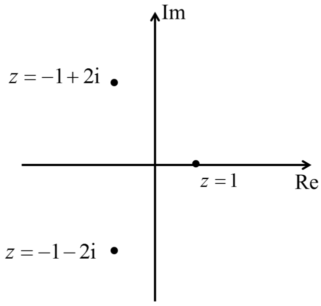
$$\begin{array}{r}
z-1 \\
z^2+2z+5\overline{\big)\, z^3 + z^2 + 3z - 5} \\
z^3 + 2z^2 + 5z \\
-z^2 - 2z - 5 \\
-z^2 - 2z - 5 \\
0
\end{array}$$

So, $z^3 + z^2 + 3z - 5 = (z-1)(z^2+2z+5)$

So the solutions of $f(z) = 0$ are

$-1+2\mathrm{i}$, $-1-2\mathrm{i}$ and 1.

b



c The vertices of the triangle are $A(-1,\ 2)$, $B(-1, -2)$ and $C(1,\ 0)$.

Use the distance formula to find the length of each side:

$$AB = \sqrt{(0)^2 + (4)^2} = 4$$

$$AC = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$BC = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

Then $AC^2 + BC^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2$

$$= 4^2$$
$$= AB^2$$

So by Pythagoras' theorem, ABC is a right-angled triangle which has a right-angle at C.

3 a $z = -1+4\text{i}$ is a solution implies that the complex conjugate $z = -1-4\text{i}$ is also a solution.

Hence

$$\left(z-(-1+4\text{i})\right)\left(z-(-1-4\text{i})\right)$$
$$= z^2 - (-1-4\text{i})z - (-1+4\text{i})z$$
$$+(-1+4\text{i})(-1-4\text{i})$$
$$= z^2 + z + 4\text{i}z + z - 4\text{i}z + 1 + 4\text{i} - 4\text{i} - 16\text{i}^2$$
$= z^2 + 2z + 17$ is a quadratic factor of $\text{f}(z)$

Use long division to find the remaining factors.

$$\begin{array}{r}
z^2 - 3z + 2 \\
z^2+2z+17\,\overline{\big)\, z^4 - z^3 + 13z^2 - 47z + 34} \\
z^4 + 2z^3 + 17z^2 \qquad\qquad \\
-3z^3 - 4z^2 - 47z \qquad \\
-3z^3 - 6z^2 - 51z \qquad \\
2z^2 + 4z + 34 \\
2z^2 + 4z + 34 \\
0
\end{array}$$

So $z^4 - z^3 + 13z^2 - 47z + 34$
$$= \left(z^2 + 2z + 17\right)\left(z^2 - 3z + 2\right)$$

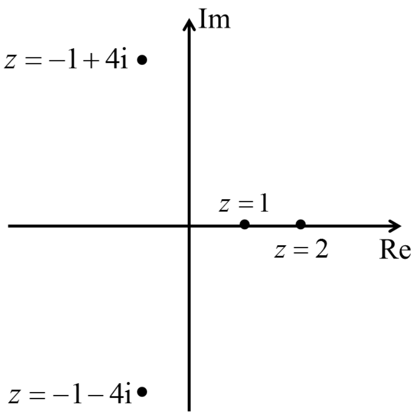
Either $z^2 + 2z + 17 = 0$
$\Rightarrow z = -1+4\text{i}$ or $z = -1-4\text{i}$

Or $z^2 - 3z + 2 = 0$
$(z-2)(z-1) = 0$
$\Rightarrow z = 1$ or $z = 2$

So the roots of $\text{f}(z) = 0$ are

$1,\ 2,\ -1+4\text{i}$ and $-1-4\text{i}$.

b



4 a $(4-3\text{i})x - (1+6\text{i})y - 3 = 0$
$$4x - 3\text{i}x - y - 6\text{i}y - 3 = 0$$

Equate real parts:
$4x - y - 3 = 0$ **(1)**

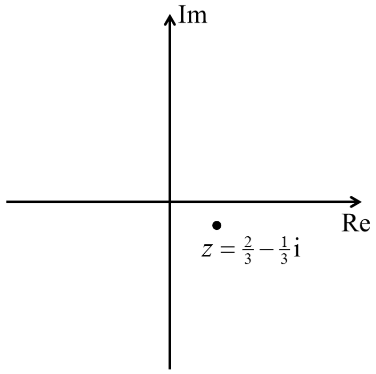
Equate imaginary parts:
$-3x - 6y = 0$ **(2)**

From **(2)**, $x = -2y$ **(3)**

Substitute $x = -2y$ into **(1)**:
$$4(-2y) - y - 3 = 0$$
$$-9y - 3 = 0$$
$$y = -\frac{1}{3}$$

Sub $y = -\frac{1}{3}$ into **(3)**
$$x = -2\left(-\frac{1}{3}\right) = \frac{2}{3}$$

b



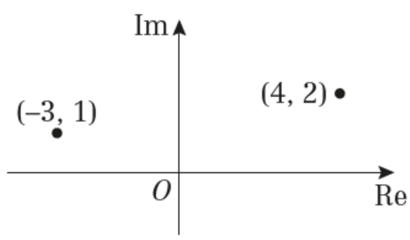
c $|z| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

d $\tan\alpha = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}$

$$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636...$$

As z is in the fourth quadrant, $\arg z = -\alpha = -0.46$ radians.

5 a



5 b $z_1 - z_2 = 4 + 2\mathrm{i} - (-3 + \mathrm{i})$

$= 4 + 2\mathrm{i} + 3 - \mathrm{i}$

$= 7 + \mathrm{i}$

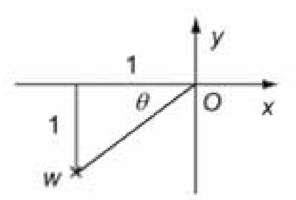
$|z_1 - z_2|^2 = 7^2 + 1^2 = 50$

$|z_1 - z_2| = \sqrt{50} = 5\sqrt{2}$

c $w = \dfrac{4+2\mathrm{i}}{-3+\mathrm{i}} \times \dfrac{-3-\mathrm{i}}{-3-\mathrm{i}} = \dfrac{-12-4\mathrm{i}-6\mathrm{i}+2}{(-3)^2+1^2}$

$= \dfrac{-10-10\mathrm{i}}{10} = -1 - \mathrm{i}$

d



$\tan\theta = \dfrac{1}{1} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

w is in the third quadrant.

$\arg w = -\left(\pi - \dfrac{\pi}{4}\right) = -\dfrac{3\pi}{4}$

6 a $z = a + 4\mathrm{i}$

$z^2 = (a + 4\mathrm{i})^2$

$= a^2 + 4a\mathrm{i} + 4a\mathrm{i} + 16\mathrm{i}^2$

$= (a^2 - 16) + 8a\mathrm{i}$

$2z = 2a + 8\mathrm{i}$

$z^2 + 2z = (a^2 - 16) + 8a\mathrm{i} + 2a + 8a\mathrm{i}$

$= (a^2 + 2a - 16) + (8a + 8)\mathrm{i}$

b If $z^2 + 2z$ is real, then $\mathrm{Im}(z^2 + 2z) = 0$

So, $8a + 8 = 0 \Rightarrow a = -1$

c $z = a + 4\mathrm{i}$, so $z = -1 + 4\mathrm{i}$

$|z| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17} \approx 4.12$

$\tan\alpha = \dfrac{4}{1}$

$\alpha = \tan^{-1} 4 = 1.3258...$

As z is in the second quadrant,

$\arg z = \pi - \alpha$.

$\arg z = \pi - 1.3258... \approx 1.82$

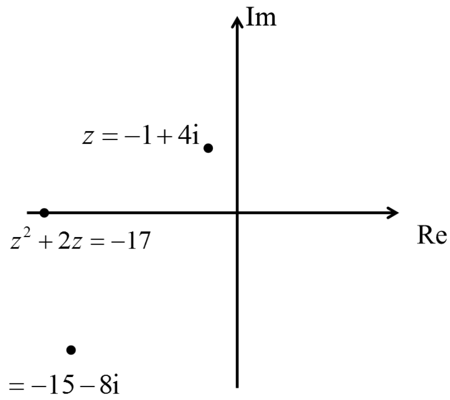
d Use $a = -1$ to find z, z^2 and $z^2 + 2z$

$z = a + 4\mathrm{i} = -1 + 4\mathrm{i}$

$z^2 = (a^2 - 16) + 8a\mathrm{i} = -15 - 8\mathrm{i}$

$z^2 + 2z = (a^2 + 2a - 16) + (8a + 8)\mathrm{i} = -17$

Show these points on an Argand diagram.



7 a $z = \dfrac{3+5\mathrm{i}}{2-\mathrm{i}}$

Multiply by the complex conjugate

$z = \dfrac{(3+5\mathrm{i})}{(2-\mathrm{i})} \times \dfrac{(2+\mathrm{i})}{(2+\mathrm{i})}$

$= \dfrac{6 + 3\mathrm{i} + 10\mathrm{i} + 5\mathrm{i}^2}{4 + 2\mathrm{i} - 2\mathrm{i} - \mathrm{i}^2}$

$= \dfrac{1 + 13\mathrm{i}}{5}$

$= \dfrac{1}{5} + \dfrac{13}{5}\mathrm{i}$

$|z| = \sqrt{\left(\dfrac{1}{5}\right)^2 + \left(\dfrac{13}{5}\right)^2} = \sqrt{\dfrac{170}{25}} = \dfrac{1}{5}\sqrt{170}$

b $\tan\alpha = \dfrac{\left(\frac{13}{5}\right)}{\left(\frac{1}{5}\right)} = 13$

$\alpha = \tan^{-1} 13 \approx 1.49$

As z is in the first quadrant,

$\arg z = \alpha \approx 1.49$

8 a $z = 1 + 2i$

$$z^2 = (1+2i)^2$$
$$= 1 + 2i + 2i + 4i^2$$
$$= -3 + 4i$$
$$z^2 - z = (-3+4i) - (1+2i) = -4 + 2i$$
$$|z^2 - z| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$$

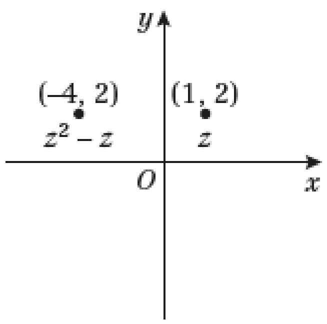
b $\tan\alpha = \frac{2}{4} = \frac{1}{2}$

$$\alpha = \tan^{-1}\frac{1}{2} = 0.46...$$

As $(z^2 - z)$ is in the second quadrant,

$$\arg(z^2 - z) = \pi - \alpha$$
$$\arg(z^2 - z) = \pi - 0.46... \approx 2.68$$

c



9 a i $z = \frac{1}{2+i} \times \frac{2-i}{2-i} = \frac{2-i}{5}$

$$= \frac{2}{5} - \frac{1}{5}i$$
$$z^2 = \left(\frac{2}{5} - \frac{1}{5}i\right)^2$$
$$= \frac{4}{25} - \frac{4}{25}i + \left(\frac{1}{5}i\right)^2$$
$$= \frac{4}{25} - \frac{4}{25}i - \frac{1}{25}$$
$$= \frac{3}{25} - \frac{4}{25}i$$

ii $z - \frac{1}{z} = -\frac{2}{5} - \frac{1}{5}i - (2+i)$

$$= \frac{2}{5} - \frac{1}{5}i - 2 - i$$
$$= -\frac{8}{5} - \frac{6}{5}i$$

b $|z^2|^2 = \left(\frac{3}{25}\right)^2 + \left(-\frac{4}{25}\right)^2$

$$= \frac{9}{625} + \frac{16}{625} = \frac{25}{625} = \frac{1}{25}$$
$$|z^2| = \frac{1}{5}$$

c $z - \frac{1}{z} = -\frac{8}{5} - \frac{6}{5}i$

$$\tan\alpha = \frac{\left(\frac{6}{5}\right)}{\left(\frac{8}{5}\right)} = \frac{6}{8} = \frac{3}{4}$$
$$\alpha = \tan^{-1}\frac{3}{4} = 0.6435...$$

As z is in the third quadrant,

$$\arg z = -(\pi - \alpha)$$
$$\arg z = -(\pi - 0.6435...) \approx -2.50$$

10 a $z = \frac{a+3i}{2+ai} = \frac{a+3i}{2+ai} \times \frac{2-ai}{2-ai}$

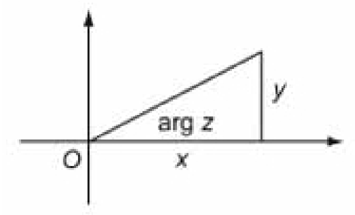
$$= \frac{2a - a^2 i + 6i + 3a}{4 + a^2}$$
$$= \frac{5a}{4+a^2} + \frac{6-a^2}{4+a^2}i \qquad (*)$$

Substitute $a = 4$ into (*)

$$z = \frac{20}{20} + \frac{-10}{20}i = 1 - \frac{1}{2}i$$
$$|z|^2 = 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$$
$$|z| = \frac{\sqrt{5}}{2}$$

10 b Now $\arg z = \frac{\pi}{4} \Rightarrow \tan(\arg z) = \tan\left(\frac{\pi}{4}\right) = 1$

If $z = x + \mathrm{i}\,y$, then $\tan(\arg z) = \frac{y}{x}$.



Hence

$$1 = \tan(\arg z) = \frac{\left(\frac{6-a^2}{4+a^2}\right)}{\left(\frac{5a}{4+a^2}\right)} = \frac{6-a^2}{5a}$$

$$5a = 6 - a^2$$

$$a^2 + 5a - 6 = 0$$

$$(a+6)(a-1) = 0$$

If $a = -6$, substituting into the result (*) in part **a** gives

$$z = \frac{30}{40} - \frac{30}{40}\mathrm{i} = \frac{3}{4} - \frac{3}{4}\mathrm{i}$$

This is in the third quadrant and has a negative argument $\left(-\frac{3\pi}{4}\right)$, so $a = -6$ does not give $\arg z = \frac{\pi}{4}$

If $a = 1$, substituting into the result (*) in part **a** gives

$$z = \frac{5}{5} + \frac{5}{5}\mathrm{i} = 1 + \mathrm{i}$$

This is in the first quadrant and does have an argument $\frac{\pi}{4}$.

Therefore $a = 1$ is the only possible value of a.

11 a $|z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\tan\alpha = \frac{1}{1} \Rightarrow \alpha = \frac{\pi}{4}$

As z_1 is in the third quadrant,

$$\arg z_1 = -\left(\pi - \frac{\pi}{4}\right) - \frac{3\pi}{4}$$

$$z_1 = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

$|z_2| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3}$

As z_2 is in the first quadrant

$$\arg z_2 = \frac{\pi}{3}$$

$$z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

b i $|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$

ii $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{2}$

c i
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = -\frac{3\pi}{4} + \frac{\pi}{3} = -\frac{9\pi}{12} + \frac{4\pi}{12} = -\frac{5\pi}{12}$$

ii
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = -\frac{3\pi}{4} - \frac{\pi}{3} = -\frac{9\pi}{12} - \frac{4\pi}{12} = -\frac{13\pi}{12}$$

11 c ii $-\dfrac{13\pi}{12}$ is not in the range $-\pi < \theta < \pi$

$$-\frac{13\pi}{12} + 2\pi = -\frac{13\pi}{12} + \frac{24\pi}{12} = \frac{11\pi}{12}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{11\pi}{12}$$

12 a $z = 2 - 2i\sqrt{3}$

$$|z| = \sqrt{(2)^2 + \left(-2\sqrt{3}\right)^2}$$
$$= \sqrt{4+12}$$
$$= \sqrt{16}$$
$$= 4$$

b $\tan\alpha = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

As z is in the fourth quadrant, $\arg z = -\alpha$

$$\arg z = -\frac{\pi}{3}$$

c $z = 4\left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$

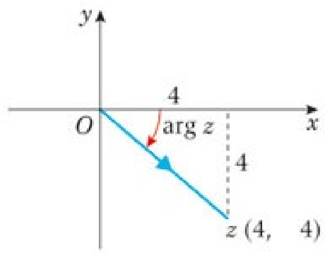
$$w = 4\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{4}{4} = 1$$

d $\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z$

$$= -\frac{\pi}{4} - \left(-\frac{\pi}{3}\right)$$
$$= -\frac{3\pi}{12} + \frac{4\pi}{12}$$
$$= \frac{\pi}{12}$$

13 $4 - 4i$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16+16}$

$$= \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

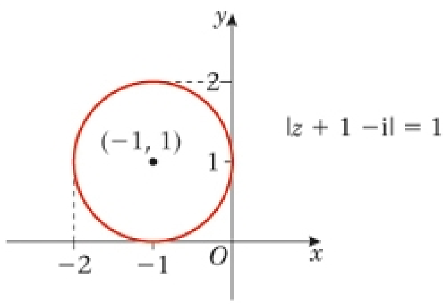
argument $= \theta = -\tan^{-1}\left(\dfrac{4}{4}\right) = -\dfrac{\pi}{4}$

$$4 - 4i = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 4\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(-\frac{\pi}{4}\right)\right)$$

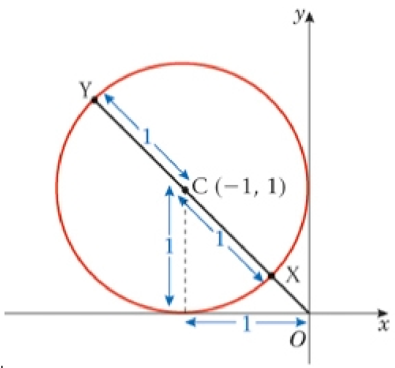
14 a

$$|z + 1 - i| = 1$$
$$|x + iy + 1 - i| = 1$$
$$(x+1)^2 + (y-1)^2 = 1$$

b This is a circle, centre $(-1, 1)$, radius 1.



14 c



$|z|$ is the distance from (0, 0) to the locus of P.

From the Argand diagram,

$|z|_{\max}$ is the distance OY

$|z|_{\min}$ is the distance OX

Note that radius $= CX = CY = 1$

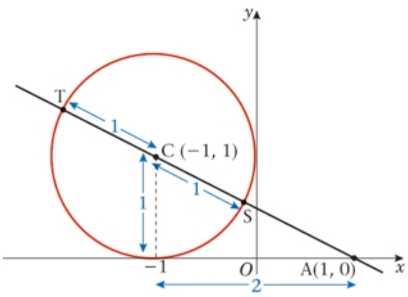
and $OC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z|_{\max} = OC + CY = \sqrt{2} + 1$

$|z|_{\min} = OC - CX = \sqrt{2} - 1$

The greatest value of $|z|$ is $\sqrt{2}+1$ and the least value of $|z|$ is $\sqrt{2}-1$.

d



$|z-1|$ is the distance from $A(1,0)$ to the locus of points.

From the Argand diagram,

$|z-1|_{\max}$ is the distance AT

$|z-1|_{\min}$ is the distance AS

d Note that radius $= CS = CT = 1$

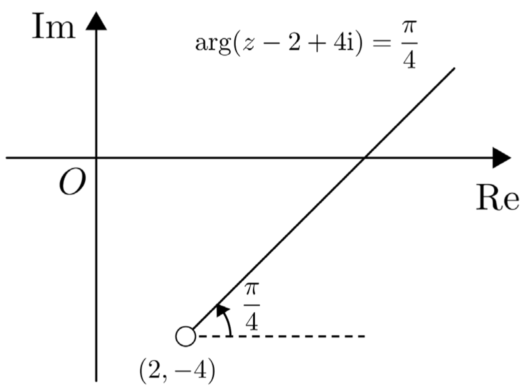
and $AC = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|z-1|_{\max} = AC + CT = \sqrt{5} + 1$

$|z-1|_{\min} = AC - CS = \sqrt{5} - 1$

The greatest value of $|z-1|$ is $\sqrt{5}+1$ and the least value of $|z-1|$ is $\sqrt{5}-1$.

15 a $\arg(z-2+4\text{i}) = \frac{\pi}{4}$ is a half-line from $(2,-4)$ which makes an angle $\frac{\pi}{4}$ with the positive real axis, as shown



b First find a Cartesian form for the locus:

$\arg(z-2+4\text{i}) = \frac{\pi}{4}$

$\Rightarrow \arg(x+\text{i}y-2+4\text{i}) = \frac{\pi}{4}$

$\Rightarrow \arg((x-2)+\text{i}(y+4)) = \frac{\pi}{4}$

$\Rightarrow \frac{y+4}{x-2} = \tan\frac{\pi}{4} = 1$

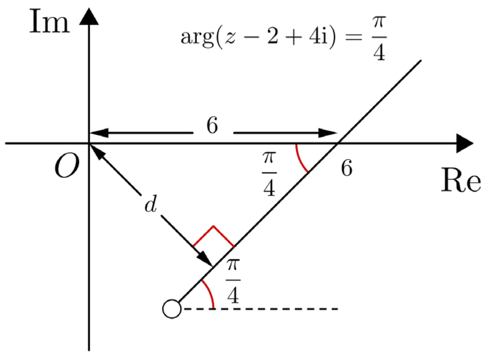
$\Rightarrow y+4 = x-2$

$\Rightarrow y = x-6, x>2, y>-4$

$|z|_{\min}$ is a line from the origin which meets the line line $y = x-6$ at right-angles, labelled d in the diagram.

$y = x-6$ cuts x-axis at $0 = x-6 \Rightarrow x = 6$.

15 b



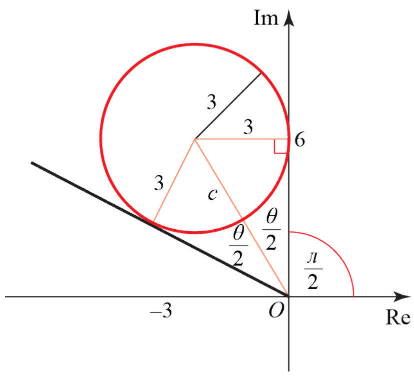
$$|z|_{\min}=d \Rightarrow \frac{d}{6}=\sin\left(\frac{\pi}{4}\right) \Rightarrow d=6\sin\left(\frac{\pi}{4}\right)$$
$$=6\left(\frac{1}{\sqrt{2}}\right)=\frac{6\sqrt{2}}{2}=3\sqrt{2}.$$

Therefore $|z|_{\min}=3\sqrt{2}$

16 $|z+3-6\mathrm{i}|=3$

$|z-(-3+6\mathrm{i})|=3$

The locus of z is a circle with centre $(-3,6)$ and radius 3.



The maximum value of $\arg z$ occurs at the point A.

16 Find the length of c using Pythagoras' Theorem.

$c^2=3^2+6^2$

$c^2=45$

$c=3\sqrt{5}$

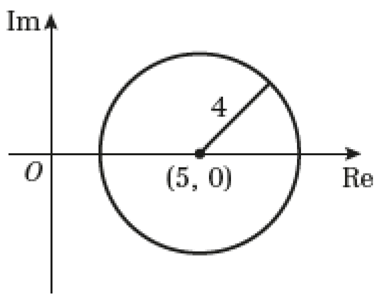
$$\sin\left(\frac{\theta}{2}\right)=\frac{3}{3\sqrt{5}}=\frac{1}{\sqrt{5}}$$
$$\frac{\theta}{2}=\arcsin\left(\frac{1}{\sqrt{5}}\right)$$
$$\theta=2\arcsin\left(\frac{1}{\sqrt{5}}\right)$$

The maximum value of $\arg z$ is

$$\theta=\frac{\pi}{2}+2\arcsin\left(\frac{1}{\sqrt{5}}\right)$$

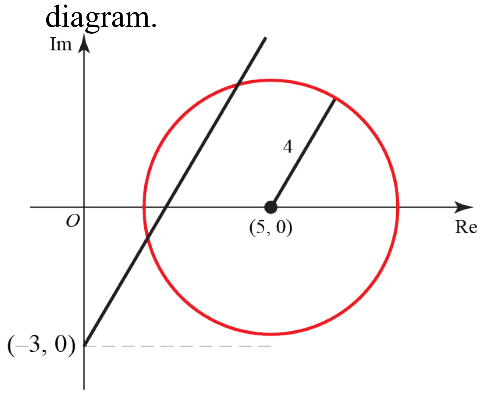
17 a $|z-5|=4$

The locus of z is a circle with centre $(5,0)$ and radius 4. The Cartesian equation of the circle is $(x-5)^2+y^2=4^2$



b $\arg(z+3\mathrm{i})=\frac{\pi}{3}$

The locus of z is the half-line beginning at $(0,-3)$ which forms an angle $\theta=\frac{\pi}{3}$ with the positive real axis. The half-line will intersect the circle as shown in the diagram.



17 b As $\theta = \frac{\pi}{3}$, the gradient of the half-line is

$\tan\frac{\pi}{3} = \sqrt{3}$.

The equation of the half-line is

$y = (\sqrt{3})x - 3$

Substitute $y = (\sqrt{3})x - 3$ into

$(x-5)^2 + y^2 = 4^2$:

$$(x-5)^2 + \left((\sqrt{3}x) - 3\right)^2 = 4^2$$

$$x^2 - 10x + 25 + 3x^2 - 6\sqrt{3}x + 9 = 16$$

$$4x^2 + \left(-10 - 6\sqrt{3}\right)x + 18 = 0$$

$$2x^2 + \left(-5 - 3\sqrt{3}\right)x + 9 = 0$$

Use the quadratic formula to solve:

$$x = \frac{5 + 3\sqrt{3} \pm \sqrt{\left(5 + 3\sqrt{3}\right)^2 - (4)(2)(9)}}{2(2)}$$

$$x = \frac{5 + 3\sqrt{3} \pm 5.653...}{4}$$

$x \approx 3.96$ or $x \approx 1.14$

If $x \approx 3.96$, then

$y = (\sqrt{3})(3.96) - 3 \approx 3.86$

If $x \approx 1.14$, then

$y = (\sqrt{3})(1.14) - 3 \approx -1.03$

So the coordinates are $(1.14, -1.03)$ and $(3.96, 3.86)$.

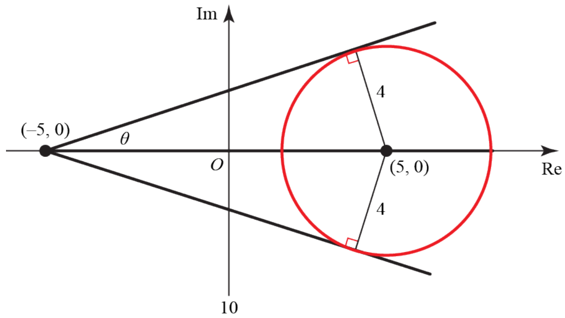
c $|z - 5| = 4$

The locus of z is a circle with centre $(5, 0)$ and radius 4.

$\arg(z + 5) = \theta$

The locus of z is the half-line starting at $(-5, 0)$ which forms an angle θ with the positive real axis.

c



$\sin\theta = \frac{4}{10} = \frac{2}{5}$

$\theta = \arcsin\frac{2}{5} = 0.4115...$

So the half-line and the circle will have common solutions in the interval $(-0.41, 0.41)$.

They will have no common solutions in the interval $-\pi < \theta < -0.41$ and $0.41 < \theta < \pi$.

18 a $|z + 5 - 5\text{i}| = |z - 6 - 3\text{i}|$

$|z - (-5 + 5\text{i})| = |z - (6 + 3\text{i})|$

The locus of z is the perpendicular bisector of $(-5, 5)$ and $(6, 3)$.

The gradient of the line joining $(-5, 5)$ and $(6, 3)$ is $-\frac{2}{11}$.

The gradient of the perpendicular bisector is $\frac{11}{2}$.

The midpoint of $(-5, 5)$ and $(6, 3)$ is $\left(\frac{1}{2}, 4\right)$.

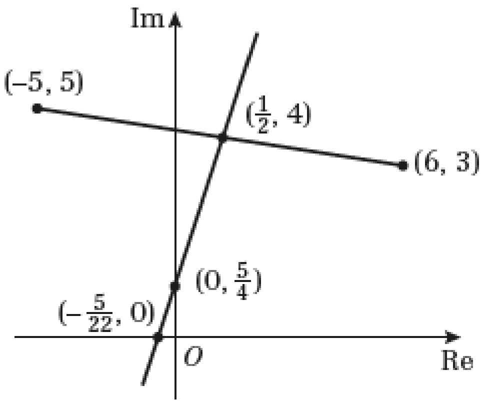
Use $y - y_1 = m(x - x_1)$ with $m = \frac{11}{2}$ and $(x_1, y_1) = \left(\frac{1}{2}, 4\right)$:

$$y - 4 = \frac{11}{2}\left(x - \frac{1}{2}\right)$$

$$y - 4 = \frac{11}{2}x - \frac{11}{4}$$

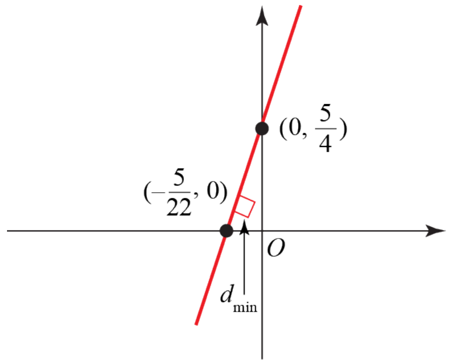
$$y = \frac{11}{2}x + \frac{5}{4}$$

18 a



b From part **a**, the Cartesian equation of the locus is $y=\frac{11}{2}x+\frac{5}{4}$

c



The equation of the line labelled $d_{\min}$ is $y=-\frac{2}{11}x$ as it is parallel to the line joining $(-5,5)$ and $(6,3)$ and passes through the origin.

The lines $y=-\frac{2}{11}x$ and $y=\frac{11}{2}x+\frac{5}{4}$ intersect where:

$$-\frac{2}{11}x=\frac{11}{2}x+\frac{5}{4}$$
$$\left(-\frac{2}{11}-\frac{11}{2}\right)x=\frac{5}{4}$$
$$\left(-\frac{4}{22}-\frac{121}{22}\right)x=\frac{5}{4}$$
$$-\frac{125}{22}x=\frac{5}{4}$$
$$x=\frac{5}{4}\times-\frac{22}{125}$$
$$x=-\frac{11}{50}$$

c If $x=-\frac{11}{50}$, $y=-\frac{2}{11}\left(-\frac{11}{50}\right)=\frac{1}{25}$

$$d_{\min}=\sqrt{\left(-\frac{11}{50}\right)^2+\left(\frac{1}{25}\right)^2}$$
$$=\sqrt{\frac{121}{2500}+\frac{1}{625}}$$
$$=\sqrt{\frac{125}{2500}}$$
$$=\sqrt{\frac{1}{20}}$$
$$=\frac{1}{2\sqrt{5}}$$
$$=\frac{\sqrt{5}}{10}$$

So the least possible value of $|z|$ is $\frac{\sqrt{5}}{10}$

19 a $|z-4|=|z-8\text{i}|$

The locus of z is the perpendicular bisector of $(4,0)$ and $(0,8)$.

The gradient of the line joining $(4,0)$ and $(0,8)$ is $-\frac{8}{4}=-2$.

The gradient of the perpendicular bisector is $\frac{1}{2}$.

The midpoint of $(4,0)$ and $(0,8)$ is $(2,4)$

Use $y-y_1=m(x-x_1)$ with $m=\frac{1}{2}$ and $(x_1,y_1)=(2,4)$:

$$y-4=\frac{1}{2}(x-2)$$
$$y-4=\frac{1}{2}x-1$$
$$y=\frac{1}{2}x+3$$

b $\arg z=\frac{\pi}{4}$

The locus of z is the half-line from $(0,0)$ which forms an angle of $\frac{\pi}{4}$ with the positive real axis.

For a half-line to have angle $\frac{\pi}{4}$, its gradient must be 1. As the half-line begins at the origin, the Cartesian equation of the locus is $y=x$.

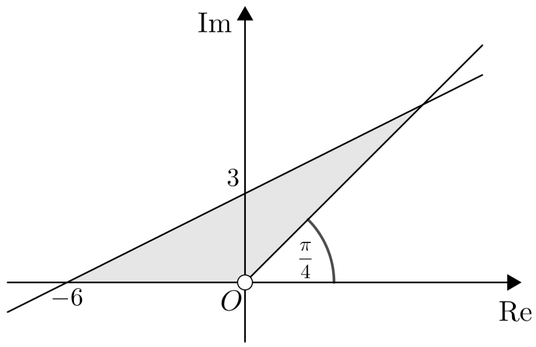
19 b Equate $y = x$ and $y = \frac{1}{2}x + 3$:

$$x = \frac{1}{2}x + 3$$
$$\frac{1}{2}x = 3$$
$$x = 6$$

If $x = 6$, $y = 6$
So $z = 6 + 6\text{i}$

c $|z - 4| \leqslant |z - 8\text{i}|$
Shade the region that is below the line found in part **a** as $(4, 0)$ lies below the line.

$\frac{\pi}{4} \leqslant \arg z \leqslant \pi$
Shade the region between the half-lines with starting point $(0, 0)$ and angles $\frac{\pi}{4}$ and π.



20 a i Method 1: Substitution
Let $|z - 3 + \text{i}| = |z - 1 - \text{i}|$

$$\Rightarrow |x + \text{i}y - 3 + \text{i}| = |x + \text{i}y - 1 - \text{i}|$$
$$\Rightarrow |(x-3) + \text{i}(y+1)| = |(x-1) + \text{i}(y-1)|$$
$$\Rightarrow |(x-3) + \text{i}(y+1)|^2 = |(x-1) + \text{i}(y-1)|^2$$
$$\Rightarrow (x-3)^2 + (y+1)^2 = (x-1)^2 + (y-1)^2$$
$$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1$$
$$= x^2 - 2x + 1 + y^2 - 2y + 1$$
$$\Rightarrow -6x + 2y + 10 = -2x - 2y + 2$$
$$\Rightarrow -4x + 4y + 8 = 0$$
$$\Rightarrow 4y = 4x - 8$$
$$\Rightarrow y = x - 2$$

a i The Cartesian equation of the locus of points representing
$|z - 3 + \text{i}| = |z - 1 - \text{i}|$ is $y = x - 2$.

Method 2: Geometrical approach
$|z - 3 + \text{i}| = |z - 1 - \text{i}|$
As $|z - 3 + \text{i}| = |z - 1 - \text{i}|$ is a perpendicular bisector of the line joining
$A(3, -1)$ to $B(1, 1)$,
then $m_{AB} = \frac{1 - -1}{1 - 3} = \frac{2}{-2} = -1$
and perpendicular gradient $= \frac{-1}{-1} = 1$
mid-point of AB is
$\left(\frac{3+1}{2}, \frac{-1+1}{2}\right) = (2, 0)$
$\Rightarrow y - 0 = 1(x - 2)$
$y = x - 2$

a ii $|z - 2| = 2\sqrt{2}$
$\Rightarrow$ circle centre (2, 0), radius $2\sqrt{2}$.
$\Rightarrow$ equation of circle is
$(x-2)^2 + y^2 = (2\sqrt{2})^2$
$\Rightarrow (x-2)^2 + y^2 = 8$

b $|z - 3 + \text{i}| = |z - 1 + \text{i}| \Rightarrow y = x - 2$ **(1)**
$|z - 2| = 2\sqrt{2} \Rightarrow (x-2)^2 + y^2 = 8$ **(2)**
Sub **(1)** in **(2)**:

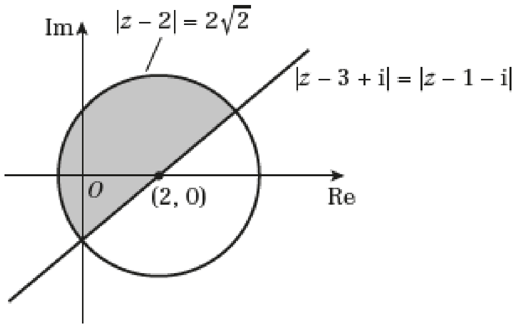
$$\Rightarrow (x-2)^2 + (x-2)^2 = 8$$
$$\Rightarrow 2(x-2)^2 = 8$$
$$\Rightarrow (x-2)^2 = 4$$
$$\Rightarrow x - 2 = \pm\sqrt{4}$$
$$\Rightarrow x - 2 = \pm 2$$
$$\Rightarrow x = 2 \pm 2$$
$$\Rightarrow x = 0, 4$$

when $x = 0, y = 0 - 2 = -2 \Rightarrow z = 0 - 2\text{i}$
when $x = 4, y = 4 - 2 = 2 \Rightarrow z = 4 + 2\text{i}$
The values of z are -2i and $4 + 2\text{i}$

c $|z-2| \leqslant 2\sqrt{2}$ is the region inside and on the circumference of the circle $|z-2| = 2\sqrt{2}$

$|z-3+\mathrm{i}| \geqslant |z-1-\mathrm{i}|$ is the set of points which are closer to $(1,\ 1)$ than to $(3,\ -1)$.
The boundary is the line $y = x-2$.

The region which satisfies both inequalities is shaded below



Challenge

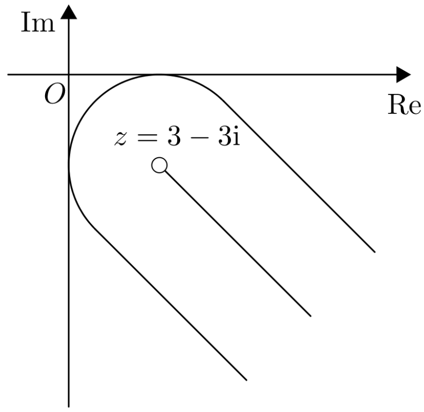
a $\arg(z-3+3\mathrm{i}) = -\dfrac{\pi}{4}$

$\arg(z-(3-3\mathrm{i})) = -\dfrac{\pi}{4}$

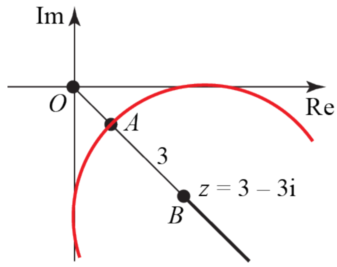
The locus of *z* is the half-line starting at $(3,-3)$ which makes an angle $-\dfrac{\pi}{4}$ with the positive real axis.

$|w-z| = 3$

The distance from w to z is always 3.
This will be formed by two lines parallel to $\arg(z-(3-3\mathrm{i})) = -\dfrac{\pi}{4}$ at a perpendicular distance of 3, and a semi-circular curve of radius 3 around the point 3 – 3i.



b



The distance from $(0,0)$ to $(3,-3)$ is

$OB = \sqrt{(3)^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$

The distance $AB = 3$, since it is a radius of the circle $(x-3)^2+(y+3)^2 = 3^2$

$OA = OB - AB = 3\sqrt{2}-3$

Therefore, the minimum distance of $|w|$ is $3\sqrt{2}-3$