Series ME 3

1 a $\sum_{r=1}^{10} r = \frac{1}{2}\times 10\times 11 = 55$

b $\sum_{r=10}^{50} r = \sum_{r=1}^{50} r - \sum_{r=1}^{9} r$

$= \frac{1}{2}\times 50\times 51 - \frac{1}{2}\times 9\times 10$

$= 1230$

c $\sum_{r=1}^{10} r^2 = \frac{1}{6}\times 10\times 11\times 21 = 385$

d $\sum_{r=1}^{10} r^3 = \frac{1}{4}\times 10^2\times 11^2 = 3025$

e $\sum_{r=26}^{50} r^2 = \sum_{r=1}^{50} r^2 - \sum_{r=1}^{25} r^2$

$= \frac{1}{6}\times 50\times 51\times 101 - \frac{1}{6}\times 25\times 26\times 51$

$= 37\,400$

f $\sum_{r=50}^{100} r^3 = \sum_{r=1}^{100} r^3 - \sum_{r=1}^{49} r^3$

$= \frac{1}{4}\times 100^2\times 101^2 - \frac{1}{4}\times 49^2\times 50^2$

$= 24\,001\,875$

g $\sum_{r=1}^{60} r + \sum_{r=1}^{60} r^2$

$= \frac{1}{2}\times 60\times 61 + \frac{1}{6}\times 60\times 61\times 121$

$= 75\,640$

2 a $\sum_{r=1}^{n} (3r-5) = 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5$

$= \frac{3}{2}n(n+1) - 5n$

$= \frac{3}{2}n^2 + \frac{3}{2}n - 5n$

$= \frac{3}{2}n^2 - \frac{7}{2}n$

2 b $\sum_{r=1}^{n} (r^2+r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1+3)$

$= \frac{1}{6}n(n+1)(2n+4)$

$= \frac{1}{3}n(n+1)(n+2)$

c $\sum_{r=1}^{n} (3r^2+7r) = 3\sum_{r=1}^{n} r^2 + 7\sum_{r=1}^{n} r$

$= \frac{3}{6}n(n+1)(2n+1) + \frac{7}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(6n+3+21)$

$= \frac{1}{6}n(n+1)(6n+24)$

$= n(n+1)(n+4)$

d $\sum_{r=1}^{n} (4r^3+6r^2) = 4\sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2$

$= n^2(n+1)^2 + n(n+1)(2n+1)$

$= n(n+1)(n^2+n+2n+1)$

$= n(n+1)(n^2+3n+1)$

e $\sum_{r=1}^{n} (r^2-2r) = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) - n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1-6)$

$= \frac{1}{6}n(n+1)(2n-5)$

f $\sum_{r=1}^{n} (r^2-3r) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1-9)$

$= \frac{1}{6}n(n+1)(2n-8)$

$= \frac{1}{3}n(n+1)(n-4)$

2 g $\sum_{r=1}^{n}(r^2-5)=\sum_{r=1}^{n}r^2-5\sum_{r=1}^{n}1$

$$=\frac{1}{6}n(n+1)(2n+1)-5n$$
$$=\frac{1}{6}n(2n^2+3n+1-30)$$
$$=\frac{1}{6}n(2n^2+3n-29)$$

h $\sum_{r=1}^{n}(2r^3+3r^2+r+4)$

$$=2\sum_{r=1}^{n}r^3+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r+4\sum_{r=1}^{n}1$$
$$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)$$
$$+\frac{1}{2}n(n+1)+4n$$
$$=\frac{1}{2}n(n^3+2n^2+n+n^2+3n+1+n+1+8)$$
$$=\frac{1}{2}n(n^3+3n^2+5n+10)$$

3 $\sum_{r=1}^{30}r(3r-1)=\sum_{r=1}^{30}3r^2-r$

$$=3\sum_{r=1}^{30}r^2-\sum_{r=1}^{30}r$$
$$=\frac{1}{2}\times30\times31\times61-\frac{1}{2}\times30\times31$$
$$=27\,900$$

4 a $\sum_{r=1}^{n}r^2(r-3)=\sum_{r=1}^{n}r^3-3r^2$

$$=\sum_{r=1}^{n}r^3-3\sum_{r=1}^{n}r^2$$
$$=\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)(2n+1)$$
$$=\frac{1}{4}n(n+1)(n(n+1)-2(2n+1))$$
$$=\frac{1}{4}n(n+1)(n^2-3n-2)$$

so $a=-3,\ b=-2.$

b $\sum_{r=1}^{20}r^2(r-3)=\frac{1}{4}\times20\times21\times(20^2-60-2)$

$$=35490$$

5 a $\sum_{r=1}^{n}(2r-1)^2=\sum_{r=1}^{n}4r^2-4r+1$

$$=4\sum_{r=1}^{n}r^2-4\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$
$$=\frac{2}{3}n(n+1)(2n+1)-2n(n+1)+n$$
$$=\frac{1}{3}n(n+1)(2(2n+1)-6)+n$$
$$=\frac{1}{3}n(n+1)(4n-4)+n$$
$$=\frac{1}{3}n(4n^2-4+3)$$
$$=\frac{1}{3}n(2n-1)(2n+1)$$

b $\sum_{r=1}^{2n}(2r-1)^2=\frac{1}{3}\times2n(4n-1)(4n+1)$

$$=\frac{2}{3}n(4n-1)(4n+1)$$

6 a $\sum_{r=1}^{n}r(r+2)=\sum_{r=1}^{n}r^2+2r$

$$=\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r$$
$$=\frac{1}{6}n(n+1)(2n+1)+n(n+1)$$
$$=\frac{1}{6}n(n+1)(2n+7)$$

b $\sum_{r=15}^{30}r(r+2)=\sum_{r=1}^{30}r(r+2)-\sum_{r=1}^{14}r(r+2)$

$$=\frac{1}{6}\times30\times31\times67-\frac{1}{6}\times14\times15\times35$$
$$=9160$$

7 a $\sum_{r=n+1}^{2n}r^2=\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2$

$$=\frac{1}{6}(2n)(2n+1)(2(2n)+1)-\frac{1}{6}n(n+1)(2n+1)$$
$$=\frac{1}{6}n(2n+1)(2(4n+1)-n-1)$$
$$=\frac{1}{6}n(2n+1)(7n+1)$$

b $\sum_{r=16}^{30}r^2=\frac{1}{6}\times15\times31\times106=8215$

8 a $\sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)-n$

$=\frac{1}{3}n(n^2-4)$

b $\sum_{r=10}^{40}(r^2-r-1)$

$=\sum_{r=1}^{40}(r^2-r-1)-\sum_{r=1}^{9}(r^2-r-1)$

$=\frac{1}{3}\times 40\times(40^2-4)-\frac{1}{3}\times 9\times(9^2-4)$

$=21\,049$

c $\sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{2n}r$

$\Rightarrow \frac{1}{3}n(n^2-4)=\frac{1}{2}\times 2n\times(2n+1)$

$\Rightarrow \frac{1}{3}n(n^2-4)=n(2n+1)$

$\Rightarrow \frac{1}{3}(n^2-4)=2n+1$

$\Rightarrow n^2-4=6n+3$

$\Rightarrow n^2-6n-7=0$

$\Rightarrow (n-7)(n+1)=0$

$\Rightarrow n=7$

9 a $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}2r^3+r$

$=2\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$

$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)$

$=\frac{1}{2}n(n+1)(n(n+1)+1)$

$=\frac{1}{2}n(n+1)(n^2+n+1)$

9 b $\sum_{r=1}^{n}(100r^2-r)=100\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$

$=\frac{1}{6}n(n+1)(200n+97)$

Now if $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}(100r^2-r)$, then

$\frac{1}{2}n(n+1)(n^2+n+1)=\frac{1}{6}n(n+1)(200n+97)$

$\frac{1}{6}n(n+1)(3(n^2+n+1)-(200n+97))=0$

$\frac{1}{6}n(n+1)(3n^2-197n-94)=0$

But $n\neq 0$, $n\neq -1$ and $3n^2-197n-94$ has discriminant 39 937 which is not square.

10 a $\sum_{r=1}^{n}r(r+1)^2=\sum_{r=1}^{n}r^3+2r^2+r$

$=\sum_{r=1}^{n}r^3+2\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r$

$=\frac{1}{4}n^2(n+1)^2+\frac{1}{3}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$

$=\frac{1}{12}n(n+1)(n+2)(3n+5)$

b $\sum_{r=1}^{n}70r=35n(n+1)$

$\Rightarrow \frac{1}{12}n(n+1)(n+2)(3n+5)=35n(n+1)$

$\Rightarrow \frac{1}{12}(n+2)(3n+5)=35$

$\Rightarrow 3n^2+11n+10=420$

$\Rightarrow 3n^2+11n-410=0$

$\Rightarrow (3n+41)(n-10)=0$

$\Rightarrow n=10$

11 $\frac{1}{6}n(n+1)(2n+1)=\frac{9}{2}(n+1)(n+2)+n+1$

$\Rightarrow n(n+1)(2n+1)=27(n+1)(n+2)+6(n+1)$

$\Rightarrow (n+1)(2n^2+n)=(n+1)(27n+54+6)$

$\Rightarrow 2n^2+n=27n+60$

$\Rightarrow 2n^2-26n-60=0$

$\Rightarrow n^2-13n-30=0$

$\Rightarrow (n-15)(n+2)=0$

$\Rightarrow n=15$

Challenge

a $$\sum_{i=1}^{n}\left(\sum_{r=1}^{i} r^2\right) = \sum_{i=1}^{n}\frac{1}{6}i(i+1)(2i+1)$$
$$= \frac{1}{6}\sum_{i=1}^{n} 2i^3 + 3i^2 + i$$
$$= \frac{1}{6}\left(\frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right)$$
$$= \frac{1}{12}\left(n(n+1)(n^2 + n + 2n + 1 + 1)\right)$$
$$= \frac{1}{12}n(n+1)(n^2 + 3n + 2)$$
$$= \frac{1}{12}n(n+1)(n+2)(n+1)$$
$$= \frac{1}{12}n(n+1)^2(n+2)$$

b $$\sum_{j=1}^{n}\left(\sum_{i=1}^{j}\left(\sum_{r=1}^{i} r\right)\right) = \sum_{j=1}^{n}\left(\frac{1}{2}\sum_{i=1}^{j} i(i+1)\right)$$
$$= \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i^2\right) + \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i\right)$$
$$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{4}\sum_{j=1}^{n} j^2 + \frac{1}{4}\sum_{j=1}^{n} j$$
$$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{24}n(n+1)(2n+1)$$
$$+ \frac{1}{8}n(n+1)$$
$$= \frac{1}{24}n(n+1)(n^2 + 5n + 6)$$
$$= \frac{1}{24}n(n+1)(n+2)(n+3)$$