Roots of polynomials 4A

1 a $3x^2+7x-4=0$
$a=3, b=7, c=-4$
$\alpha+\beta=-\frac{b}{a}=-\frac{7}{3}$

b $\alpha\beta=\frac{c}{a}=-\frac{4}{3}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
$=\left(\frac{-\frac{7}{3}}{-\frac{4}{3}}\right)$
$=\frac{7}{4}$

d $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=\left(-\frac{7}{3}\right)^2-2\left(-\frac{4}{3}\right)$
$=\frac{49}{9}+\frac{8}{3}$
$=\frac{73}{9}$

2 a $7x^2-3x+1=0$
$a=7, b=-3, c=1$
$\alpha+\beta=-\frac{b}{a}=\frac{3}{7}$

b $\alpha\beta=\frac{c}{a}=\frac{1}{7}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
$=\frac{\frac{3}{7}}{\frac{1}{7}}$
$=3$

d $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=\left(\frac{3}{7}\right)^2-2\left(\frac{1}{7}\right)$
$=\frac{9}{49}-\frac{2}{7}$
$=\frac{9}{49}-\frac{14}{49}$
$=-\frac{5}{49}$

3 a $6x^2-9x+2=0$
$a=6, b=-9, c=2$
$\alpha+\beta=-\frac{b}{a}=-\frac{-9}{6}=\frac{3}{2}$

b $\alpha^2\times\beta^2=(\alpha\beta)^2=\left(\frac{c}{a}\right)^2=\left(\frac{2}{6}\right)^2=\frac{1}{9}$

c $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{\frac{3}{2}}{\frac{1}{3}}=\frac{9}{2}$

d $(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$
So $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
$=\left(\frac{3}{2}\right)^3-3\left(\frac{1}{3}\right)\left(\frac{3}{2}\right)$
$=\frac{27}{8}-\frac{3}{2}$
$=\frac{27}{8}-\frac{12}{8}$
$=\frac{15}{8}$

4 $ax^2+bx+c=0$
$\alpha=2$ and $\beta=-3$
$\alpha+\beta=2+(-3)=-1=-\frac{b}{a}$
$\alpha\beta=(2)(-3)=-6=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-1\Rightarrow\frac{b}{a}=1$
$\frac{c}{a}=-6$
So equation is $x^2+x-6=0$,
with $a=1, b=1$ and $c=-6$

5 $ax^2+bx+c=0$
$\alpha=-\frac{1}{2}$ and $\beta=-\frac{1}{3}$
$\alpha+\beta=-\frac{1}{2}-\frac{1}{3}=-\frac{5}{6}=-\frac{b}{a}$
$\alpha\beta=\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)=\frac{1}{6}=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-\frac{5}{6}$, so $\frac{b}{a}=\frac{5}{6}$
$\frac{c}{a}=\frac{1}{6}$

5 So equation is $x^2+\frac{5}{6}x+\frac{1}{6}=0$
or $6x^2+5x+1=0$. So
$a=6, b=5$ and $c=1$.

6 $ax^2+bx+c=0$
$\alpha=\frac{-1+\mathrm{i}}{2}$ and $\beta=\frac{-1-\mathrm{i}}{2}$
$\alpha+\beta=-1=-\frac{b}{a}$
$\alpha\beta=\left(\frac{-1+\mathrm{i}}{2}\right)\left(\frac{-1-\mathrm{i}}{2}\right)=\frac{1-\mathrm{i}^2}{2}=\frac{2}{4}=\frac{1}{2}=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-1\Rightarrow\frac{b}{a}=1$
$\frac{c}{a}=\frac{1}{2}$
So equation is $x^2+x+\frac{1}{2}=0$
or $2x^2+2x+1=0$. So
$a=2, b=2$ and $c=1$.

7 **a** The other root, β, is the complex conjugate. So $\beta=-1+4\mathrm{i}$

b $\alpha=-1-4\mathrm{i}$ and $\beta=-1+4\mathrm{i}$
$\alpha+\beta=-2=-\frac{b}{a}$
$\alpha\beta=(-1-4\mathrm{i})(-1+4\mathrm{i})=1-16\mathrm{i}^2=17=\frac{c}{a}$
$ax^2+bx+c=0\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$
$-\frac{b}{a}=-2\Rightarrow\frac{b}{a}=2$
$\frac{c}{a}=17$
So equation is $x^2+2x+17=0$,
with $a=1, b=2$ and $c=17$.

8 The sum of the roots is 4,
so $\alpha+\beta=4$
Also $-\frac{b}{a}=4$, but $-\frac{b}{a}=\frac{-(k-3)}{k}=\frac{3-k}{k}$
So $\frac{3-k}{k}=4$
$3-k=4k$
$3=5k$
$k=\frac{3}{5}$

9 The roots are α and $-\alpha$.
$\alpha+-\alpha=0=-\frac{b}{a}$
If $-\frac{b}{a}=0$, then $b=0$
$b=0\Rightarrow 16+n=0\Rightarrow n=-16$

10 The roots are reciprocals, so the roots are α and $\frac{1}{\alpha}$.
$\alpha\left(\frac{1}{\alpha}\right)=1=\frac{c}{a}$
But $\frac{c}{a}=\frac{k}{6}$
So $\frac{k}{6}=1\Rightarrow k=6$

11 The roots are k and $2k$.
$k+2k=3k=-\frac{b}{a}$
But $-\frac{b}{a}=-\frac{4}{m}$, so $3k=-\frac{4}{m}$
$(k)(2k)=2k^2=\frac{c}{a}$
But $\frac{c}{a}=\frac{4m}{m}=4$, so $2k^2=4$
If $2k^2=4, k^2=2$ and $k=\pm\sqrt{2}$.
$3k=-\frac{4}{m}$, so $m=-\frac{4}{3k}$
If $k=\sqrt{2}, m=-\frac{4}{3\sqrt{2}}=-\frac{2\sqrt{2}}{3}$
If $k=-\sqrt{2}, m=\frac{4}{3\sqrt{2}}=\frac{2\sqrt{2}}{3}$

12 **a** Let $\alpha=p+q\mathrm{i}$ and $\alpha^*=p-q\mathrm{i}$
$\alpha+\alpha^*=(p+q\mathrm{i})+(p-q\mathrm{i})=2p$
$\alpha\alpha^*=(p+q\mathrm{i})(p-q\mathrm{i})=p^2+q^2$
Also $2p=-\frac{b}{a}=-\frac{8}{a}$
and $p^2+q^2=\frac{c}{a}$
If $\mathrm{Re}(\alpha)=2$ then $p=2$, so $2p=4$.
$2p=-\frac{8}{a}\Rightarrow 4=-\frac{8}{a}\Rightarrow a=-2$

12 b If $\text{Im}(\alpha) = 3i$, then $q = 3$

$p^2 + q^2 = \frac{c}{a}$,

So $(2)^2 + (3)^2 = \frac{c}{-2}$

$13 = \frac{c}{-2}$

$c = -26$

13 a Let $\alpha = m + n\text{i}$ and $\alpha^* = m - n\text{i}$

$\alpha + \alpha^* = (m + n\text{i}) + (m - n\text{i}) = 2m$

$\alpha\alpha^* = (m + n\text{i})(m - n\text{i}) = m^2 + n^2$

Also $2m = -\frac{b}{a} = -\frac{p}{4}$

and $m^2 + n^2 = \frac{c}{a} = \frac{q}{4}$

If $\text{Re}(\alpha) = -3$, then $m = -3$, so $2m = -6$

$2m = -\frac{p}{4} \Rightarrow -6 = -\frac{p}{4} \Rightarrow p = 24$

b If $\text{Im}(\alpha) \neq 0$, then $n \neq 0$

$m^2 + n^2 = \frac{c}{a} \Rightarrow (-3)^2 + n^2 = \frac{q}{4}$

So $q = 36 + 4n^2$

As $n \neq 0$, $4n^2 > 0$, so $q > 36$