Roots of polynomials 4B

1 a $2x^3+5x^2-2x+3=0$
$a=2, b=5, c=-2$ and $d=3$.
$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{5}{2}$

b $\alpha\beta\gamma=-\frac{d}{a}=-\frac{3}{2}$

c $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{2}{2}=-1$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
$=\frac{-1}{\left(-\frac{-3}{2}\right)}=\frac{2}{3}$

2 a $x^3+5x^2+17x+13=0$
$a=1, b=5, c=17$ and $d=13$
$\alpha+\beta+\gamma=-\frac{b}{a}=-5$

b $\alpha\beta\gamma=-\frac{d}{a}=-13$

c $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=17$

d $\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=(-13)^2=169$

3 a $7x^3-4x^2-x+6=0$
$a=7, b=-4, c=-1$ and $d=6$.
$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{-4}{7}=\frac{4}{7}$

b $\alpha\beta\gamma=-\frac{d}{9}=-\frac{6}{7}$

c $\alpha^3\beta^3\gamma^3=(\alpha\beta\gamma)^3=\left(-\frac{6}{7}\right)^3=-\frac{216}{343}$

d $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
$=\frac{\left(\frac{c}{9}\right)}{\left(-\frac{6}{7}\right)}$
$=\frac{\left(-\frac{1}{7}\right)}{\left(-\frac{6}{7}\right)}$
$=\frac{1}{6}$

4 $ax^3+bx^2+cx+d=0$
$\alpha=\frac{3}{2}, \beta=\frac{1}{2}$. and $\gamma=1$
$\alpha+\beta+\gamma=\frac{3}{2}+\frac{1}{2}+1=3=-\frac{b}{a}$
$\alpha\beta+\beta\gamma+\gamma\alpha=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)(1)+$
$(1)\left(\frac{3}{2}\right)=\frac{11}{4}=\frac{c}{a}$
$\alpha\beta\gamma=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(1)=\frac{3}{4}=-\frac{d}{a}$
$ax^3+bx^2+cx+d=0\Rightarrow$
$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$
$-\frac{b}{a}=3\Rightarrow\frac{b}{a}=-3$
$\frac{c}{a}=\frac{11}{4}$
$-\frac{d}{a}=\frac{3}{4}\Rightarrow\frac{d}{a}=-\frac{3}{4}$
So equation is $x^3-3x^2+\frac{11}{4}x-\frac{3}{4}=0$.
Multiply by 4 for integer coefficients gives
$4x^3-12x^2+11x-3=0$,
with $a=4, b=-12, c=11$ and $d=-3$.

5 $ax_3 + bx^2 + cx + d = 0$

$\alpha = 1+3\text{i}, \beta = 1-3\text{i}$ and $\gamma = \frac{1}{2}$

$\alpha+\beta+\gamma = (1+3\text{i}) + (1-3\text{i}) + \frac{1}{2} = \frac{5}{2} = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (1+3\text{i})(1-3\text{i}) + (1-3\text{i})\frac{1}{2} + \frac{1}{2}(1+3\text{i})$

$= 1-3i+3i-9i^2+\frac{1}{2}-\frac{3}{2}i+\frac{1}{2}+\frac{3}{2}i$

$= 11 = \frac{c}{a}$

$\alpha\beta\gamma = (1+3\text{i})(1-3\text{i})\left(\frac{1}{2}\right)$

$= \left(1-3\text{i}+3\text{i}+9\text{i}^2\right)\left(\frac{1}{2}\right)$

$= 5 = -\frac{d}{a}$

$ax^3 + bx^2 + cx + d = 0 \Rightarrow$

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

$-\frac{b}{a} = \frac{5}{2} \Rightarrow \frac{b}{a} = -\frac{5}{2}$

$\frac{c}{a} = 11$

$-\frac{d}{a} = 5 \Rightarrow \frac{d}{a} = -5$

So equation is $x^3 - \frac{5}{2}x^2 + 11x - 5 = 0.$

Multiplying by 2 for integer coefficients gives $2x^3 - 5x^2 + 22x - 10 = 0$, with $a = 2, b = -5, c = 22$ and $d = -10$.

6 $ax^3 + bx^2 + cx + d = 0$

$\alpha = \frac{5}{4}, \beta = -\frac{3}{2}$ and $\gamma = \frac{1}{2}$

$\alpha+\beta+\gamma = \frac{5}{4} + \left(-\frac{3}{2}\right) + \frac{1}{2} = \frac{1}{4} = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \left(\frac{5}{4}\right)\left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{4}\right)$

$= -\frac{15}{8} - \frac{3}{4} + \frac{5}{8}$

$= -\frac{15}{8} - \frac{6}{8} + \frac{5}{8}$

$= -\frac{16}{8} = -2 = \frac{c}{a}$

$\alpha\beta\gamma = \left(\frac{5}{4}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{15}{16} = -\frac{d}{a}$

$ax^3 + bx^2 + cx + d = 0 \Rightarrow$

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

$-\frac{b}{a} = \frac{1}{4} \Rightarrow \frac{b}{a} = -\frac{1}{4}$

$\frac{c}{a} = -2$

$-\frac{d}{a} = -\frac{15}{16} = \frac{d}{a} = \frac{15}{16}$

So equation is $x^3 - \frac{1}{4}x^2 - 2x + \frac{15}{16} = 0$

Multiply by 16 for integer coefficients gives $16x^2 - 4x^2 - 32x + 15 = 0$, with $a = 16, b = -4, c = -32$ and $d = 15$.

7 a $16x^3 - kx^2 + 1 = 0$

Here $a = 16, b = -k, c = 0$ and $d = 1$.

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 0$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{1}{16}$

7 b i $\alpha = \beta \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \alpha^2 + 2\alpha\gamma$

$\alpha = \beta \Rightarrow a\beta\gamma = \alpha^2\gamma$

So $\alpha^2 + 2\alpha\gamma = 0$ and $a^2\gamma = -\frac{1}{16}$

So $\gamma = -\frac{1}{16\alpha^2}$

Substitute $\gamma = -\frac{1}{16\alpha^2}$ into

$\alpha^2 + 2\alpha\gamma = 0$ giving:

$\alpha^2 + 2\alpha\left(-\frac{1}{16\alpha^2}\right) = 0$

$\alpha^2 - \frac{1}{8\alpha} = 0$

$8\alpha^3 - 1 = 0$

$\alpha^3 = \frac{1}{8}$

$\alpha = \frac{1}{2}$

If $\alpha = \frac{1}{2}, \gamma = -\frac{1}{16\left(\frac{1}{2}\right)^2} = -\frac{1}{4}$

If $\alpha = \frac{1}{2}, \gamma = -\frac{1}{4}, \alpha\beta\gamma = -\frac{1}{16} \Rightarrow \beta = \frac{1}{2}.$

ii $\alpha + \beta + \gamma = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4} = -\frac{b}{a}$

$a = 16$ and $b = -k \Rightarrow \frac{k}{16} = \frac{3}{4}$, so

$k = 12.$

8 a $2x^3 - kx^2 + 30x - 13 = 0$

Here $a = 2, b = -k, c = 30$ and $d = -13.$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 15$

$\alpha\beta\gamma = -\frac{d}{a} = \frac{13}{2}$

$\alpha + \beta + \gamma = -\frac{b}{a} \Rightarrow \alpha + \beta + \gamma = \frac{k}{2},$

so $k = 2(\alpha + \beta + \gamma)$

b $\alpha = 2 - 3\text{i} \Rightarrow \beta = 2 + 3\text{i}$

$\alpha\beta\gamma = (2 - 3\text{i})(2 + 3\text{i})\gamma$

$= (4 + 6\text{i} - 6\text{i} - 9\text{i}^2)\gamma$

$= 13\gamma$

But $\alpha\beta\gamma = \frac{13}{2}$, so $\gamma = \frac{1}{2}.$

b We know $k = 2(\alpha + \beta + \gamma)$

$k = 2\left((2 - 3\text{i}) + (2 + 3\text{i}) + \frac{1}{2}\right)$

$k = 2\left(\frac{9}{2}\right) = 9.$

9 a $x^3 - mx + n = 0$

Here $a = 1, b = 0, c = -m$ and $d = n.$

Also $\alpha = \alpha, \beta = 1$ and $\gamma = -4.$

α must be real as if α was complex then α^* would also be a complex root and there are already 2 other roots. A cubic has 3 roots.

b $\alpha + \beta + \gamma = \alpha + 1 - 4 = \alpha - 3$

But, $\alpha - 3 = -\frac{b}{a} \Rightarrow \alpha - 3 = 0 \Rightarrow \alpha = 3$

$\alpha\beta + \beta\gamma + \gamma\alpha = (3)(1) + (1)(-4) + (-4)(3)$

$= -13$

But, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{m}{1} = -m$

So, $-m = -13 \Rightarrow m = 13.$

$\alpha\beta\gamma = (3)(1)(-4) = -12$

But $\alpha\beta\gamma = -\frac{a}{a} = -\frac{n}{1} = -n$

So, $-n = -12 \Rightarrow n = 12$

Therefore $m = 13, n = 12$ and $\alpha = 3.$

10 a $2x^3 - 10x^2 + 8x - k = 0$

Here $a = 2, b = -10, c = 8$ and $d = -k.$

One root is $x = 3 - \text{i},$ so another root is $3 + i.$

$\alpha + \beta + \gamma = (3 - \text{i}) + (3 + \text{i}) + \gamma = 6 + \gamma$

But $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{10}{2} = 5$

So $6 + \gamma = 5 \Rightarrow \gamma = -1$

Therefore the roots are $3 - \text{i}, 3 + \text{i}$ and -1.

b $\alpha\beta\gamma = (3 - \text{i})(3 + \text{i})(-1)$

$= (9 + 3\text{i} - 3\text{i} - \text{i}^2)(-1)$

$= (10)(-1)$

$= -10$

But $\alpha\beta\gamma = -\frac{d}{a} = \frac{k}{2}$

So $\frac{k}{2} = -10 \Rightarrow k = -20.$

11 $x^3 - 14x^2 + 56x - 64 = 0$

Here $a = 1, b = -14, c = 56$ and $d = -64$

The roots are $\alpha = \alpha, \beta = k\alpha$ and $\gamma = k^2\alpha$

$\alpha + \beta + \gamma = \alpha + k\alpha + k^2\alpha$

$= \alpha\left(k^2 + k + 1\right)$

But $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{14}{1} = 14$

So $\alpha\left(k^2 + k + 1\right) = 14$

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (\alpha)(k\alpha) + (k\alpha)\left(k^2\alpha\right) + \left(k^2\alpha\right)(\alpha)$

$= k\alpha^2 + k^3\alpha^2 + k^2\alpha^2$

$= \alpha^2 k\left(1 + k + k^2\right)$

But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 56$

So $\alpha^2 k\left(1 + k + k^2\right) = 56$

$\alpha\beta\gamma = (\alpha)(k\alpha)\left(k^2\alpha\right) = k^3\alpha^3$

But $\alpha\beta\gamma = -\frac{d}{a} = 64$

So $k^3\alpha^3 = 64 = 4^3$

To satisfy both equations $\alpha k = 4$

The possible solutions are therefore:

$\alpha = 2, k = 2$

or

$\alpha = 8, k = \frac{1}{2}$

12 $8x^3 + 12x^2 - cx + d = 0$

Here $a = 8, b = 12, c = -c$ and $d = d$.

The roots are $\alpha = \alpha, \beta = \frac{\alpha}{2}$ and $\gamma = \alpha - 4$.

$\alpha + \beta + \gamma = \alpha + \frac{\alpha}{2} + \alpha - 4 = \frac{5\alpha}{2} - 4$

But $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{12}{8} = -\frac{3}{2}$

So $\frac{5\alpha}{2} - 4 = -\frac{3}{2}$

$\frac{5\alpha}{2} = \frac{5}{2}$

$\alpha = 1$

Therefore the roots are $\alpha = 1, \beta = \frac{1}{2}$

and $\gamma = -3$.

$\alpha\beta + \beta\gamma + \gamma\alpha = (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(-3) + (-3)(1)$

$= \frac{1}{2} - \frac{3}{2} - 3 = -4$

12 But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{c}{8}$

So $-\frac{c}{8} = -4 \Rightarrow c = 32.$

$\alpha\beta\gamma = (1)\left(\frac{1}{2}\right)(-3) = -\frac{3}{2}$

But $\alpha\beta\gamma = -\frac{d}{a} = -\frac{d}{8}$

So $-\frac{d}{8} = -\frac{3}{2} \Rightarrow d = 12.$

13 $2x^3 + 48x^2 + cx + d = 0$

Here $a = 2, b = 48, c = c$ and $d = d$.

The roots are $\alpha = \alpha, \beta = 2\alpha$ and $\gamma = 3\alpha$.

$\alpha + \beta + \gamma = \alpha + 2\alpha + 3\alpha = 6\alpha$

But $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{48}{2} = -24$

So $6\alpha = -24 \Rightarrow \alpha = -4.$

Therefore the roots are $\alpha = -4, \beta = -8$

and $\gamma = -12.$

$\alpha\beta + \beta\gamma + \gamma\alpha$

$= (-4)(-8) + (-8)(-12) + (-12)(-4)$

$= 32 + 96 + 48$

$= 176$

But $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{c}{2}$

So $\frac{c}{2} = 176 \Rightarrow c = 352.$

$\alpha\beta\gamma = (-4)(-8)(-12) = -384$

But $\alpha\beta\gamma = -\frac{d}{a} = -\frac{d}{2}$

So $-\frac{d}{2} = -384 \Rightarrow d = 768.$

Challenge

$ax^2 + bx^2 + cx + d = 0$

Roots are α, β and γ.

If α, β and γ are all real, then

$a\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$

must clearly all be real.

So assume $\alpha = a + b\mathrm{i}$

Let $\beta = a - b\mathrm{i},$ the complex conjugate

γ must be real as there can be a maximum of 3 roots.

Let $\alpha = a + b\mathrm{i}, \beta = a - b\mathrm{i}$ and $\gamma = r,$ where $r \in R$

Challenge

a $\alpha+\beta+\gamma=(a+b\text{i})+(a-b\text{i})+r$
$=2a+r\in IR$

b $\alpha\beta+\beta\gamma+\gamma\alpha$
$=(a+b\text{i})(a-b\text{i})+(a-b\text{i})r+r(a+b\text{i})$
$=a^2-ab\text{i}+ab\text{i}-b^2\text{i}^2+ra-rb\text{i}+ra+rb\text{i}$
$=a^2+b^2+2ar\in R$

c $\alpha\beta\gamma=(a+b\text{i})(a-b\text{i})r$
$=(a^2+b^2)r\in R.$