Roots of polynomials 4C

1 a $4x^4+3x^3+2x^2-5x+4=0$
$a=4, b=3, c=2, d=-5$ and $e=4$

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{3}{4}$$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{2}{4}=\frac{1}{2}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{5}{4}$

d
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$=\frac{\left(\frac{5}{4}\right)}{\left(\frac{e}{a}\right)}$$
$$=\frac{\left(\frac{5}{4}\right)}{\left(\frac{4}{4}\right)}$$
$$=\frac{5}{4}$$

2 a $2x^4+4x^3-3x^2-x+2=0$
Here $a=2, b=4, c=-3. d=-1$ and $e=2$

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{4}{2}=-2$$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=-\frac{3}{2}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{1}{2}$

d $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{2}{2}=1$

e
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$=\frac{\frac{1}{2}}{1}$$
$$=\frac{1}{2}$$

3 a $x^4+3x^3+2x^2-x+4=0$
Here $a=1, b=3, c=2, d=-1$ and $e=4$

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-3$$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{2}{1}=2$

3 c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=\frac{1}{1}=1$

d
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$=\frac{1}{\left(\frac{e}{1}\right)}$$
$$=\frac{1}{\left(\frac{4}{1}\right)}$$
$$=\frac{1}{4}$$

e $a^2\beta^2\gamma^2\delta^2=(\alpha\beta\gamma\delta)^2=\left(\frac{e}{a}\right)^2=4^2=16$

4 a $7x^4+6x^3-5x^2+4x+3=0$
Here $a=7, b=6, c=-5, d=4$ and $e=3$.

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-\frac{6}{7}$$

b $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=-\frac{5}{7}$

c $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=-\frac{4}{7}$

d
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$=\frac{\left(-\frac{4}{7}\right)}{\left(\frac{e}{a}\right)}$$
$$=\frac{\left(-\frac{4}{7}\right)}{\left(\frac{3}{7}\right)}$$
$$=-\frac{4}{3}$$

e $\alpha^3\beta^3\gamma^3\delta^3=(\alpha\beta\gamma\delta)^3=\left(\frac{3}{7}\right)^3=\frac{27}{343}$

5 $ax^4+bx^3+cx^2+dx+e=0$

The roots are $\alpha=-\frac{3}{2}, \beta=-\frac{1}{2}, \gamma=-2$ and $\delta=\frac{2}{3}$.

$\alpha+\beta+\gamma+\delta=-\frac{3}{2}-\frac{1}{2}-2+\frac{2}{3}=-\frac{10}{3}=-\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)+\left(-\frac{3}{2}\right)(-2)+\left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)$

$\quad+\left(-\frac{1}{2}\right)(-2)+\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)+(-2)\left(\frac{2}{3}\right)$

$=\frac{3}{4}+3-1+1-\frac{1}{3}-\frac{4}{3}$

$=\frac{25}{12}=\frac{c}{a}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)(-2)+\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)$

$\quad+\left(-\frac{3}{2}\right)(-2)\left(\frac{2}{3}\right)+\left(-\frac{1}{2}\right)(-2)\left(\frac{2}{3}\right)$

$=-\frac{3}{2}+\frac{1}{2}+2+\frac{2}{3}$

$=\frac{5}{3}=-\frac{d}{a}$

$\alpha\beta\gamma\delta=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)(-2)\left(\frac{2}{3}\right)=-1=\frac{e}{a}$

If $ax^4+bx^3+cx^2+dx+e=0$, then

$x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+e=0.$

$-\frac{b}{a}=-\frac{10}{3}\Rightarrow\frac{b}{a}=\frac{10}{3}$

$\frac{c}{a}=\frac{25}{12}$

$-\frac{d}{a}=\frac{5}{3}\Rightarrow\frac{d}{a}=-\frac{5}{3}$

$\frac{e}{a}=-1$

So $x^4+\frac{10}{3}x^3+\frac{25}{12}x^2-\frac{5}{3}x-1=0.$

Multiply by 12 for integer coefficients

$12x^4+40x^3+25x^2-20x-12=0$

So $a=12, b=40, c=25, d=-20$ and $e=-12$.

6 $ax^4+bx^3+cx^2+dx+e=0$

The roots are $\alpha=-\frac{1}{2}, \beta=\frac{1}{3}, \gamma=1+\mathrm{i}$ and $\delta=1-\mathrm{i}$.

$\alpha+\beta+\gamma+\delta$

$=-\frac{1}{2}+\frac{1}{3}+(1+\mathrm{i})+(1-\mathrm{i})$

$=\frac{11}{6}=-\frac{b}{a}$

$\alpha\beta+\alpha1\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{2}\right)(1+\mathrm{i})+\left(-\frac{1}{2}\right)(1-\mathrm{i})$

$\quad+\left(\frac{1}{3}\right)(1+\mathrm{i})+\frac{1}{3}(1-\mathrm{i})+(1+\mathrm{i})(1-\mathrm{i})$

$=-\frac{1}{6}-\frac{1}{2}-\frac{1}{2}\mathrm{i}-\frac{1}{2}+\frac{1}{2}\mathrm{i}+\frac{1}{3}$

$\quad+\frac{1}{3}\mathrm{i}+\frac{1}{3}-\frac{1}{3}\mathrm{i}+1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2$

$=-\frac{1}{6}-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+1+1=\frac{3}{2}=\frac{c}{a}$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1+\mathrm{i})+\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1-\mathrm{i})$

$\quad+\left(-\frac{1}{2}\right)(1+\mathrm{i})(1-\mathrm{i})+\left(\frac{1}{3}\right)(1+\mathrm{i})(1-\mathrm{i})$

$=-\frac{1}{6}-\frac{1}{6}\mathrm{i}-\frac{1}{6}+\frac{1}{6}\mathrm{i}-\frac{1}{2}(2)+\frac{1}{3}(2)$

$=-\frac{1}{3}-1+\frac{2}{3}=-\frac{2}{3}=-\frac{d}{a}$

$\alpha\beta\gamma\delta=\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)(1+\mathrm{i})(1-\mathrm{i})$

$\quad=\left(-\frac{1}{6}\right)(2)$

$\quad=-\frac{1}{3}=\frac{e}{a}$

If $ax^4+bx^3+cx^2+dx+e=0$,

the $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$

$-\frac{b}{a}=\frac{11}{6}\Rightarrow\frac{b}{a}=-\frac{11}{6}$

$\frac{c}{a}=\frac{3}{2}$

$-\frac{d}{a}=-\frac{2}{3}\Rightarrow\frac{d}{a}=\frac{2}{3}$

$\frac{e}{a}=-\frac{1}{3}$

So $x^4-\frac{11}{6}x^3+\frac{3}{2}x^2+\frac{2}{3}x-\frac{1}{3}=0$

Multiply by 6 for integer coefficients

$6x^4-11x^3+9x^2+4x-2=0$

So $a=6, b=-11, c=9, d=4$ and $e=-2$.

7 $ax^4+bx^3+cx^2+dx+e=0$

$\sum\alpha=\frac{17}{12}\Rightarrow-\frac{b}{a}=\frac{17}{12}\Rightarrow\frac{b}{a}=-\frac{17}{12}$

$\sum\alpha\beta=-\frac{25}{72}\Rightarrow\frac{c}{a}=-\frac{25}{72}$

$\sum\alpha\beta\gamma=-\frac{53}{72}\Rightarrow-\frac{d}{a}=-\frac{53}{72}\Rightarrow\frac{d}{a}=\frac{53}{72}$

$\alpha\beta\gamma\delta=-\frac{1}{6}\Rightarrow\frac{e}{a}=-\frac{1}{6}.$

If $ax^4+bx^3+cx^2+dx+e=0,$

then $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+e=0$

So $x^4-\frac{17}{12}x^3-\frac{25}{72}x^2+\frac{53}{72}x-\frac{1}{6}=0$

Multiply by 72 for integer coefficients:

$72x^4-102x^3-25x^2+53x-12=0.$

So $a=72, b=-102, c=-25, d=53$ and $e=-12.$

8 $x^4-16x^3+86x^2-176x+105=0$

$a=1, b=-16, c=86, d=-176$ and $e=105.$

The roots are $\alpha=\alpha, \beta=\alpha+k, \gamma=\alpha+2k$ and $\delta=\alpha+3k.$

$\alpha+\beta+\gamma+\delta$

$=\alpha+(\alpha+k)+(\alpha+2k)+(\alpha+3k)$

$=4\alpha+6k$

But $\alpha+\beta+\gamma+\delta=-\frac{b}{a}=16$

So $4\alpha+6k=16\Rightarrow 2\alpha+3k=8.$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\alpha(\alpha+k)+\alpha(\alpha+2k)+\alpha(\alpha+3k)$

$+(\alpha+k)(\alpha+2k)+(\alpha+k)(\alpha+3k)$

$+(\alpha+2k)(\alpha+3k)$

$=\alpha^2+k\alpha+\alpha^2+2k\alpha+\alpha^2+3k\alpha+\alpha^2$

$+3k\alpha+2k^2+\alpha^2+4k\alpha$

$+3k^2+\alpha^2+5k\alpha+6k^2$

$=6\alpha^2+18k\alpha+11k^2$

But $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=86$

So $6\alpha^2+18k\alpha+11k^2=86$

If $2\alpha+3k=8\Rightarrow\alpha=\frac{8}{2}-\frac{3}{2}k=4-\frac{3}{2}k.$

Substitute $\alpha=4-\frac{3}{2}k$ into

$6\alpha^2+18k\alpha+11k^2=86.$

$6\left(4-\frac{3}{2}k\right)^2+18k\left(4-\frac{3}{2}k\right)+11k^2=86$

$6\left(16-12k+\frac{9}{4}k^2\right)+72k-27k^2+11k^2=86$

$96-72k+\frac{27}{2}k^2+72k-27k^2+11k^2=86$

$-\frac{5}{2}k^2=-10$

$k^2=4$

$k=\pm 2$

If $k=2,$ then $\alpha=4-\frac{3}{2}k\Rightarrow\alpha=1.$

So the roots are $1, 1+2, 1+2(2)$ and $1+3(2)$

Hence the roots are $1, 3, 5$ and $7.$

Note if $k=-2,$ then $\alpha=7,$ which would yield the same roots.

9 $3072x^4 - 2880x^3 + 840x^2 - 90x + 3 = 0$

Here $a = 3072, b = -2880, c = 840, d = -90$ and $e = 3.$

The roots are $\alpha = \alpha, \beta = r\alpha, \gamma = r^2\alpha$ and $\delta = r^3\alpha$

$\alpha + \beta + \gamma + \delta = \alpha + r\alpha + r^2\alpha + r^3\alpha$

$= \alpha(1 + r + r^2 + r^3)$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{2880}{3072} = \frac{15}{16}$

So $\alpha(1 + r + r^2 + r^3) = \frac{15}{16}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= \alpha(r\alpha) + \alpha(r^2\alpha) + \alpha(r^3\alpha) +$

$(r\alpha)(r^2\alpha) + (r\alpha)(r^3\alpha) + (r^2\alpha)(r^3\alpha)$

$= \alpha^2(r + r^2 + r^3 + r^3 + r^4 + r^5)$

$= \alpha^2 r(1 + r + 2r^2 + r^3 + r^4)$

But $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta =$

$\frac{c}{a} = \frac{840}{3072} = \frac{35}{128}$

So $\alpha^2 r(1 + r + 2r^2 + r^3 + r^4) = \frac{35}{128}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= (r\alpha)(r^2\alpha) + \alpha(r\alpha)(r^3\alpha) +$

$\alpha(r^2\alpha)(r^3\alpha) + (r\alpha)(r^2\alpha)(r^3\alpha)$

$= r^3\alpha^3 + r^4\alpha^3 + r^5\alpha^3 + r^6\alpha^3$

$\alpha^3 r^3(1 + r + r^2 + r^3)$

But

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = \frac{90}{3072} = \frac{15}{512}$

So $\alpha^3 r^3(1 + r + r^2 + r^3) = \frac{15}{512}$

$\alpha\beta\gamma\delta = \alpha(r\alpha)(r^2\alpha)(r^3\alpha) = \alpha^4 r^6$

But $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{3}{3072} = \frac{1}{1024}$

So $\alpha^4 r^6 = \frac{1}{1024}$

Taking square root of both sides

$\alpha^2 r^3 = \frac{1}{32} = \frac{1}{2^5}$

So $\alpha = \frac{1}{2}$ and $r = \frac{1}{2}$ are possible roots.

9 Check by substituting into

$\alpha(1 + r + r^2 + r^3) = \frac{15}{16}$

$\frac{1}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$

$= \frac{1}{2}\left(1 + \frac{4}{8} + \frac{2}{8} + \frac{1}{8}\right)$

$= \frac{1}{2}\left(\frac{15}{8}\right) = \frac{15}{16}$

So $\alpha = \frac{1}{2}$ and $r = \frac{1}{2}$

Substituting these values in the root expressions then gives $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$.

10 a $40x^3 + 90x^3 - 115x^2 + mx + n = 0$

So $a = 40, b = 90, c = -115, d = m$ and $e = n.$

The roots are $\alpha = 1, \beta = -3, \gamma = \frac{1}{2}$ and $\delta = \delta$

$\alpha + \beta + \gamma + \delta = 1 - 3 + \frac{1}{2} + \delta = -\frac{3}{2} + \delta$

But $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{90}{40} = -\frac{9}{4}$

So $-\frac{3}{2} + \delta = -\frac{9}{4}$

$\delta = -\frac{9}{4} + \frac{6}{4}$

$\delta = -\frac{3}{4}$

b $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= (1)(-3)\left(\frac{1}{2}\right) + (1)(-3)\left(-\frac{3}{4}\right)$

$+ (1)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right) + (-3)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right)$

$= -\frac{3}{2} + \frac{9}{4} - \frac{2}{8} + \frac{9}{8}$

$= -\frac{12}{8} + \frac{18}{8} - \frac{3}{8} + \frac{9}{8}$

$= \frac{12}{8} = \frac{3}{2}$

But $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{m}{40}$

So $-\frac{m}{40} = \frac{3}{2} \Rightarrow m = -60.$

$\alpha\beta\gamma\delta = (1)(-3)\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right) = \frac{9}{8}$

10 b But $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{n}{40}$

So $\frac{n}{40}=\frac{9}{8}\Rightarrow n=45$

11 a $2x^4-34x^3+202x^2+dx+e=0$

So $a=2, b=-34, c=202, d=d$ and $e=e$.

The roots are $\alpha=\alpha, \beta=\alpha+1, \gamma=2\alpha+1$ and $\delta=3\alpha+1$.

$\alpha+\beta+\gamma+\delta$

$=\alpha+(\alpha+1)+(2\alpha+1)+(3\alpha+1)$

$=7\alpha+3$

But $\alpha+\beta+\gamma+\delta=-\frac{b}{a}=\frac{34}{2}=17$

So $7\alpha+3=17$

$7\alpha=14$

$\alpha=2$

b So the roots are $\alpha=2, \beta=3, \gamma=5$ and $\delta=7$.

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=(2)(3)(5)+(2)(3)(7)$

$\quad+(2)(5)(7)+(3)(5)(7)$

$=30+42+70+105=247.$

But $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=-\frac{d}{2}$

So $-\frac{d}{2}=247\Rightarrow d=-494$

$\alpha\beta\gamma\delta=(2)(3)(5)(7)=210$

But $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{e}{2}$.

So $\frac{e}{2}=210\Rightarrow e=420.$

12 a $4x^4-19x^3+px^2+qx+10=0$

So $a=4, b=-19, c=p, d=q$ and $e=10$.

If $\gamma=3+\mathrm{i}$ is a root then the complex conjugate is also a root. So $\delta=3-\mathrm{i}$

The roots are $\alpha, \beta, \gamma=3+\mathrm{i}$ and $\delta=3-\mathrm{i}$

$\alpha+\beta+\gamma+\delta$

$=\alpha+\beta+(3+\mathrm{i})+(3-\mathrm{i})$

$=\alpha+\beta+6.$

But $\alpha+\beta+\gamma+\delta=-\frac{b}{a}=\frac{19}{4}$

So, $\alpha+\beta+6=\frac{19}{4}$

$4\alpha+4\beta+24=19$

$4\alpha+4\beta+5=0$

12 a $\alpha\beta\gamma\delta=\alpha\beta(3+\mathrm{i})(3-\mathrm{i})=10\alpha\beta$

But $\alpha\beta\gamma\delta=\frac{e}{a}=\frac{10}{4}=\frac{5}{2}$

So $10\alpha\beta=\frac{5}{2}$

$20\alpha\beta-5=0$

$4\alpha\beta-1=0$

b If $4\alpha\beta-1=0$, then $4\alpha\beta=1$

and $\alpha=\frac{1}{4\beta}$

Substitute $\alpha=\frac{1}{4\beta}$ into $4\alpha+4\beta+5=0$

$4\left(\frac{1}{4\beta}\right)+4\beta+5=0$

$\frac{1}{\beta}+4\beta+5=0$

Multiply by β

$1+4\beta^2+5\beta=0$

So $4\beta^2+5\beta+1=0$

$(4\beta+1)(\beta+1)=0$

$\beta=-\frac{1}{4}$ or $\beta=-1$

If $\beta=-\frac{1}{4}, \alpha=\frac{1}{4(-\frac{1}{4})}=-1$

If $\beta=-1, \alpha=\frac{1}{4(-1)}=-\frac{1}{4}$.

So the roots are $-\frac{1}{4}, -1, 3+\mathrm{i}$ and $3-\mathrm{i}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

$=\left(-\frac{1}{4}\right)(-1)+\left(-\frac{1}{4}\right)(3+\mathrm{i})+\left(-\frac{1}{4}\right)(3-\mathrm{i})$

$\quad+(-1)(3+\mathrm{i})+(-1)(3-\mathrm{i})+(3+\mathrm{i})(3-\mathrm{i})$

$=\frac{1}{4}-\frac{3}{4}-\frac{3}{4}-3-3+10$

$=\frac{11}{4}$

But $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{p}{4}$

So, $\frac{p}{4}=\frac{11}{4}\Rightarrow p=11$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

$=\left(-\frac{1}{4}\right)(-1)(3+\mathrm{i})+\left(-\frac{1}{4}\right)(-1)(3-\mathrm{i})$

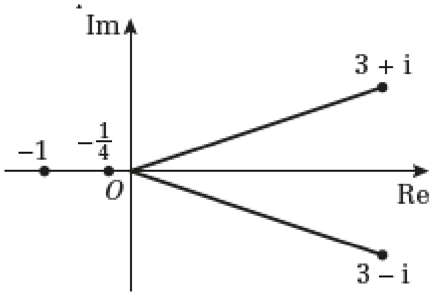
$\quad+\left(-\frac{1}{4}\right)(3+\mathrm{i})(3-\mathrm{i})+(-1)(3+\mathrm{i})(3-\mathrm{i})$

12 b $=\frac{3}{4}+\frac{3}{4}+\left(-\frac{1}{4}\right)(10)+(-1)(10)=-11$

But $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}=-\frac{q}{4}$

$-\frac{q}{4}=-11 \Rightarrow q=44$

c



13 a $6x^4-10x^3+3x^2+6x--40=0$

So, $a=6, b=-10, c=3, d=6$ and $e=-40.$

Substitute $x=\frac{1-3\text{i}}{2}$ into the equation

$$6\left(\frac{1-3\text{i}}{2}\right)^4-10\left(\frac{1-3\text{i}}{2}\right)^3+3\left(\frac{1-3\text{i}}{2}\right)^2+6\left(\frac{1-3\text{i}}{2}\right)-40=0$$

$$\frac{6}{2^4}(1-3\text{i})^4-\frac{10}{2^3}(1-3\text{i})^3+\frac{3}{2^2}(1-3\text{i})^2+3(1-3\text{i})-40=0$$

$$\frac{3}{8}(1-3\text{i})^4-\frac{5}{4}(1-3\text{i})^3+\frac{3}{4}(1-3\text{i})^2+3-9\text{i}-40=0$$

Expand $(1-3i)^4$ using $(a+b)^4$ with $a=1$ and $b=-3\text{i}$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(1-3\text{i})^4=1+4(-3\text{i})+6(-3\text{i})^2+4(-3\text{i})^3+(-3\text{i})^4$$

$$(1-3\text{i})^4=1-12\text{i}+54\text{i}^2-108\text{i}^3+81\text{i}^4$$

Use $\text{i}=\text{i}, \text{i}^2=-1, \text{i}^3=-\text{i}$ and $\text{i}^4=1$ to simplify

$$(1-3\text{i})^4=1-12\text{i}-54+108\text{i}+81$$
$$=28+96\text{i}$$

Expand $(1-3\text{i})^3$ using $(a+b)^3$ with $a=1$ and $b=3\text{i}$

13 a $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$(1-3\text{i})^3=1+3(-3\text{i})+3(-3\text{i})^2+(-3\text{i})^3$$
$$=1-9\text{i}+27\text{i}^2-27\text{i}^3$$
$$=1-9\text{i}-27+27\text{i}$$
$$=-26+18\text{i}$$

$$(1-3\text{i})^2=1-3\text{i}-3\text{i}+9\text{i}^2$$
$$=-8-6\text{i}$$

Substituting

$$\frac{3}{8}(28+96\text{i})-\frac{5}{4}(-26+18\text{i})+\frac{3}{4}(-8-6\text{i})+3-9\text{i}-40$$

$$=\frac{21}{2}+36\text{i}+\frac{65}{2}-\frac{45}{2}\text{i}-6-\frac{9}{2}\text{i}+3-9\text{i}-40$$

$$=\left(\frac{21}{2}+\frac{65}{2}-6+3-40\right)+\left(36-\frac{45}{2}-\frac{9}{2}-9\right)\text{i}$$

$$=0$$

b If $x=\frac{1-3\text{i}}{2}$ is a root, then $x=\frac{1+3\text{i}}{2}$ is also a root.

$$\left(x-\left(\frac{1+3\text{i}}{2}\right)\right)\left(x-\left(\frac{1-3\text{i}}{2}\right)\right)$$

$$=x^2-x\left(\frac{1-3\text{i}}{2}\right)-x\left(\frac{1+3\text{i}}{2}\right)+\left(\frac{1+3\text{i}}{2}\right)\left(\frac{1-3\text{i}}{2}\right)$$

$$=x^2-\frac{1}{2}x-\frac{3}{2}x\text{i}-\frac{1}{2}x+\frac{3}{2}x\text{i}+\frac{1}{4}-\frac{3}{4}\text{i}+\frac{3}{4}\text{i}-\frac{9}{4}\text{i}^2$$

$$=x^2-x+\frac{5}{2}$$

Multiply by 4:

$$=2x^2-2x+5$$

Find the other factors using division:

$$\begin{array}{r} 3x^2-2x-8 \\ 2x^2-2x+5\overline{\big)\,6x^4-10x^3+3x^2+6x-40} \\ 6x^4-6x^3+15x^2 \qquad\qquad \\ -4x^3-12x^2+6x \qquad \\ -4x^3+\ 4x^2-10x \qquad \\ -16x^2+16x-40 \\ -16x^2+16x-40 \\ 0 \end{array}$$

13 b continued

So

$6x^4 - 10x^3 + 3x^2 + 6x - 40$

$= (2x^2 - 2x + 5)(3x^2 - 2x - 8) = 0$

Let $(3x^2 - 2x - 8) = 0$

$(3x + 4)(x - 2) = 0$

$x = -\frac{4}{3}$ or $x = 2$

So the other roots are $x = -\frac{4}{3}$, $x = 2$ and $x = \frac{1+3i}{2}$.

c