Roots of polynomials 4D

1 a $\alpha+\beta=4,\ \alpha\beta=3$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{4}{3}$$

b $\alpha^2\beta^2=(\alpha\beta)^2=3^2=9$

c $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=4^2-2(3)$$
$$=10$$

d $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=4^3-3(3)(4)$$
$$=64-36$$
$$=28$$

2 a $\alpha+\beta=-\frac{2}{3},\alpha\beta=\frac{3}{4}$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-\frac{2}{3}}{\frac{3}{4}}=-\frac{8}{9}$$

b $\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{3}{4}\right)^2=\frac{9}{16}$

c $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{2}{3}\right)^2-2\left(\frac{3}{4}\right)$$
$$=\frac{4}{9}-\frac{6}{4}=-\frac{38}{36}=-\frac{19}{18}$$

d $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$=\left(-\frac{2}{3}\right)^2-3\left(\frac{3}{4}\right)\left(-\frac{2}{3}\right)$$
$$=-\frac{8}{27}+\frac{3}{2}=\frac{65}{54}$$

3 a $\alpha+\beta=\frac{5}{4},\alpha\beta=-\frac{1}{3}$

$$(\alpha+2)(\beta+2)=\alpha\beta+2(\alpha+\beta)+4$$
$$=-\frac{1}{3}+2\left(\frac{5}{4}\right)+4$$
$$=\frac{37}{6}$$

3 b $(\alpha-4)(\beta-4)=\alpha\beta-4(\alpha+\beta)+16$

$$=-\frac{1}{3}-4\left(\frac{5}{4}\right)+16$$
$$=\frac{32}{3}$$

c $(\alpha^2+1)(\beta^2+1)=\alpha^2\beta^2+\alpha^2+\beta^2+1$

$$\alpha^2\beta^2=(\alpha\beta)^2=\left(-\frac{1}{3}\right)^2=\frac{1}{9}$$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=\left(\frac{5}{4}\right)^2-2\left(-\frac{1}{3}\right)$$
$$=\frac{25}{16}+\frac{2}{3}=\frac{107}{48}$$

So

$$\alpha^2\beta^2+(\alpha^2+\beta^2)+1=\frac{1}{9}+\frac{107}{48}+1=\frac{481}{144}$$

4 a $\alpha+\beta+\gamma=2,\alpha\beta+\beta\gamma+\gamma\alpha=-3$

and $\alpha\beta\gamma=4$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=-\frac{3}{4}$$

b $\alpha^2+\beta^2+\gamma^2$

$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma-\gamma\alpha)$$
$$=2^2-2(-3)$$
$$=10$$

c $\alpha^3+\beta^3+\gamma^3$

$$=(\alpha+\beta+\gamma)^3$$
$$-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$+3\alpha\beta\gamma$$
$$=2^3-3(2)(-3)+3(4)$$
$$=8+18+12=38$$

4 d $\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)^2$
$=\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$
$=\left(\alpha\beta\right)^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta^2\gamma+\left(\beta\gamma\right)^2$
$+\alpha\beta\gamma^2+\alpha^2\beta\gamma+\alpha\beta\gamma^2+\left(\gamma\alpha\right)^2$
$=\left(\alpha\beta\right)^2+\left(\beta\gamma\right)^2+\left(\gamma\alpha\right)^2$
$+2\left(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2\right)$
$=\left(\alpha\beta\right)^2+\left(\beta\gamma\right)^2+\left(\gamma\alpha\right)^2$
$+2\alpha\beta\gamma\left(\alpha+\beta+\gamma\right)$

Hence:
$\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)^2$
$=\left(\alpha\beta\right)^2+\left(\beta\gamma\right)^2+\left(\gamma\alpha\right)^2$
$+2\alpha\beta\gamma\left(\alpha+\beta+\gamma\right)$

So:
$\left(\alpha\beta\right)^2+\left(\beta\gamma\right)^2+\left(\gamma\alpha\right)^2$
$=\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)^2$
$-2\alpha\beta\gamma\left(\alpha+\beta+\gamma\right)$
$=\left(-3\right)^2-2\left(4\right)\left(2\right)$
$=9-16$
$=-7$

5 a $\sum\alpha=\frac{3}{2},\sum\alpha\beta=-\frac{4}{3}$ and $\alpha\beta\gamma=\frac{1}{2}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\sum\alpha\beta}{\alpha\beta\gamma}=\frac{\left(-\frac{4}{3}\right)}{\left(\frac{1}{2}\right)}=-\frac{8}{3}$

b

$\alpha^2+\beta^2+\gamma^2=\left(\alpha+\beta+\gamma\right)^2-$
$2\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$
$=\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$
$=\left(\frac{3}{2}\right)^2-2\left(-\frac{4}{3}\right)$
$=\frac{9}{4}+\frac{8}{3}=\frac{59}{12}$

5 c $\alpha^3+\beta^3+\gamma^3$
$=\left(\alpha+\beta+\gamma\right)^3$
$-3\left(\alpha+\beta+\gamma\right)\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$
$+3\alpha\beta\gamma$
$=\left(\sum\alpha\right)^3-3\left(\sum\alpha\right)\left(\sum\alpha\beta\right)+3\alpha\beta\gamma$
$=\left(\frac{3}{2}\right)^3-3\left(\frac{3}{2}\right)\left(-\frac{4}{3}\right)+3\left(\frac{1}{2}\right)$
$=\frac{27}{8}+6+\frac{3}{2}=\frac{87}{8}$

d $\alpha^3\beta^3\gamma^3=\left(\alpha\beta\gamma\right)^3=\left(\frac{1}{2}\right)^2=\frac{1}{8}$

6 a $\alpha+\beta+\gamma=-\frac{1}{2},\alpha\beta+\beta\gamma+\gamma\alpha=\frac{3}{4}$

and $\alpha\beta\gamma=-\frac{2}{5}$

$\left(\alpha+2\right)\left(\beta+2\right)\left(\gamma+2\right)$
$=\alpha\beta\gamma+2\left(\alpha\beta+\alpha\gamma+\beta\gamma\right)$
$+4\left(\alpha+\beta+\gamma\right)+8$
$=\left(-\frac{2}{5}\right)+2\left(\frac{3}{4}\right)+4\left(-\frac{1}{2}\right)+8$
$=-\frac{2}{5}+\frac{3}{2}-2+8=\frac{71}{10}.$

b $\left(\alpha-3\right)\left(\beta-3\right)\left(\gamma-3\right)$
$=\alpha\beta\gamma-3\left(\alpha\beta+\alpha\gamma+\beta\gamma\right)$
$+9\left(\alpha+\beta+\gamma\right)-27$
$=\left(-\frac{2}{5}\right)-3\left(\frac{3}{4}\right)+9\left(-\frac{1}{2}\right)-27$
$=-\frac{2}{5}-\frac{9}{4}-\frac{9}{2}-27=-\frac{683}{20}$

c $\left(1-\alpha\right)\left(1-\beta\right)\left(1-\gamma\right)$
$=1-\left(\alpha+\beta+\gamma\right)+\left(\alpha\beta+\alpha\gamma+\beta\gamma\right)-\alpha\beta\gamma$
$=1-\left(-\frac{1}{2}\right)+\left(\frac{3}{4}\right)-\left(-\frac{2}{5}\right)$
$=1+\frac{1}{2}+\frac{3}{4}+\frac{2}{5}=\frac{53}{20}$

6 d $(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(\alpha\beta)^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta^2\gamma+(\beta\gamma)^2$
$\quad+\alpha\beta\gamma^2+\alpha^2\beta\gamma+\alpha\beta\gamma^2+(\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$\quad+2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$\quad+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

Hence:

$(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$\quad+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

So:

$(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$=\left(\frac{3}{4}\right)^2-2\left(-\frac{2}{5}\right)\left(-\frac{1}{2}\right)$

$=\frac{9}{16}-\frac{2}{5}$

$=\frac{45}{80}-\frac{32}{80}$

$=\frac{13}{80}$

e $(\alpha\beta+\beta\gamma+\gamma\alpha)^3$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$$=(\alpha\beta+\beta\gamma+\gamma\alpha)\begin{pmatrix}(\alpha\beta)^2+(\beta\gamma)^2\\+(\gamma\alpha)^2\\+2\alpha\beta\gamma(\alpha+\beta+\gamma)\end{pmatrix}$$

$=(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3$
$+\alpha\beta^3\gamma^2+\alpha^3\beta\gamma^2+2\alpha^2\beta^2\gamma(\alpha+\beta+\gamma)$
$+\alpha^2\beta^3\gamma+\alpha^2\beta\gamma^3+2\alpha\beta^2\gamma^2(\alpha+\beta+\gamma)$
$+\alpha^3\beta^2\gamma+\alpha\beta^2\gamma^3+2\alpha^2\beta\gamma^2(\alpha+\beta+\gamma)$

$=(\alpha\beta)^3+(\beta\gamma)^2+(\gamma\alpha)^3$
$+2\alpha^3\beta\gamma^2+2\alpha^2\beta^3\gamma+2\alpha^2\beta^2\gamma^2$
$+2\alpha^2\beta^2\gamma^2+2\alpha\beta^3\gamma^2+2\alpha\beta^2\gamma^3$
$+2\alpha^3\beta\gamma^2+2\alpha^2\beta^2\gamma^2+2\alpha^2\beta\gamma^3$

$=(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3+6(\alpha\beta\gamma)^2$
$+3\alpha^2\beta^3\gamma+3\alpha\beta^3\gamma^2+3\alpha\beta^2\gamma^3$
$+3\alpha^3\beta^2\gamma+3\alpha^3\beta\gamma^2+3\alpha^2\beta\gamma^3$

$$=(\alpha\beta)^3+(\beta\gamma)^3+(y\alpha)^3+6(\alpha\beta\gamma)^2$$
$$+3\alpha\beta\gamma\begin{pmatrix}\alpha\beta^2+\beta^2\gamma+\beta\gamma^2\\+\alpha^2\beta+\alpha^2\gamma+\alpha\gamma^2\end{pmatrix}$$

$$=(\alpha\beta)^3+(\beta\gamma)^3+(y\alpha)^3+6(\alpha\beta\gamma)^2$$
$$+3\alpha\beta\gamma\begin{pmatrix}\alpha(\beta^2+\gamma^2)\\+\beta(\alpha^2+\gamma^2)\\+\gamma(\alpha^2+\beta^2)\end{pmatrix}$$

However:

$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=\alpha^2\beta+\alpha^2\gamma+\alpha\beta^2$
$+\beta^2\gamma+\beta\gamma^2+\alpha\gamma^2+3\alpha\beta\gamma$

$=\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)$
$+\gamma(\alpha^2+\beta^2)+3\alpha\beta\gamma$

So

$$3\alpha\beta\gamma\begin{pmatrix}\alpha(\beta^2+\gamma^2)\\+\beta(\alpha^2+\gamma^2)\\+\gamma(\alpha^2+\beta^2)\end{pmatrix}$$

6 e $= 3\alpha\beta\gamma\begin{pmatrix}(\alpha+\beta+\gamma)\times \\ (\alpha\beta+\beta\gamma+\gamma\alpha) \\ -3\alpha\beta\gamma\end{pmatrix}$

Therefore

$(\alpha\beta)^3+(\beta\gamma)^3+(\gamma\alpha)^3$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^3-6(\alpha\beta\gamma)^2$

$-3\alpha\beta\gamma\begin{pmatrix}(\alpha+\beta+\gamma)\times \\ (\alpha\beta+\beta\gamma+\gamma\alpha) \\ -3\alpha\beta\gamma\end{pmatrix}$

$=\left(\frac{3}{4}\right)^3-6\left(-\frac{2}{5}\right)^2$

$-3\left(-\frac{2}{5}\right)\left(\left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)-3\left(-\frac{2}{5}\right)\right)$

$=\frac{27}{64}-\frac{24}{25}+\frac{6}{5}\left(\frac{33}{40}\right)$

$=\frac{27}{64}-\frac{24}{25}+\frac{99}{100}$

$=\frac{675}{1600}-\frac{1536}{1600}+\frac{1584}{1600}$

$=\frac{723}{1600}$

7 a $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$

$=\frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$

$=\frac{-4}{-2}=2$

b $\alpha^2+\beta^2+\gamma^2+\delta^2$

$=(\alpha+\beta+\gamma+\delta)^2$

$-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$=(3)^2-2(5)=9-10=-1.$

c $\alpha^4\beta^4\gamma^4\delta^4=(\alpha\beta\gamma\delta)^4=(-2)^4=16.$

8 a $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{\left(-\frac{1}{5}\right)}{\left(\frac{4}{3}\right)}=-\frac{3}{20}$

8 b $\alpha^2+\beta^2+\gamma^2+\delta^2=\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$

$=\left(\frac{1}{2}\right)^2-2\left(-\frac{3}{4}\right)$

$=\frac{1}{4}+\frac{6}{4}=\frac{7}{4}$

c $\alpha^3\beta^3\gamma^3\delta^3=(\alpha\beta\gamma\delta)^3=\left(\frac{4}{3}\right)^3=\frac{64}{27}$

d $(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)\times$

$(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)$

$=\alpha^2\beta^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma$

$+\alpha\beta\gamma\delta+\alpha^2\beta\delta+\alpha\beta^2\delta$

$+\alpha\beta^2\gamma+\beta^2\gamma^2+\alpha\beta\gamma^2$

$+\beta\delta\gamma^2+\alpha\beta\gamma\delta+\beta^2\gamma\delta$

$+\alpha^2\beta\gamma+\alpha\beta\gamma^2+\alpha^2\gamma^2$

$+\alpha\gamma^2\delta+\alpha^2\gamma\delta+\alpha\beta\gamma\delta$

$+\alpha\beta\gamma\delta+\beta\gamma^2\delta+\alpha\gamma^2\delta$

$+\gamma^2\delta^2+\alpha\gamma\delta^2+\beta\gamma\delta^2$

$+\alpha^2\beta\delta+\alpha\beta\gamma\delta+\alpha^2\gamma\delta$

$+\alpha\gamma\delta^2+\alpha^2\delta^2+\alpha\beta\delta^2$

$+\alpha\beta^2\delta+\beta^2\gamma\delta+\alpha\beta\gamma\delta$

$+\beta\gamma\delta^2+\alpha\beta\delta^2+\beta^2\delta^2$

$=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$

$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$

$+\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\beta\gamma$

$+\alpha^2\gamma\delta+\alpha^2\beta\delta+\alpha^2\gamma\delta$

$+\alpha\beta^2\gamma+\alpha\beta^2\delta+\alpha\beta^2\gamma$

$+\beta^2\gamma\delta+\alpha\beta^2\delta+\beta^2\gamma\delta$

$+\alpha\beta\gamma^2+\alpha\beta\gamma^2+\alpha\gamma^2\delta$

$+\beta\gamma^2\delta+\alpha\gamma^2\delta+\beta\delta\gamma^2$

$+\alpha\beta\delta^2+\beta\gamma\delta^2+\alpha\gamma\delta^2$

$+\alpha\beta\delta^2+\beta\gamma\delta^2+\alpha\gamma\delta^2$

8 d $=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$
$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$
$+2\alpha^2\beta\gamma+2\alpha^2\beta\delta+2\alpha^2\gamma\delta$
$+2\alpha\beta^2\gamma+2\alpha\beta^2\delta+2\beta^2\gamma\delta$
$+2\alpha\beta\gamma^2+2\alpha\gamma^2\delta+2\beta\gamma^2\delta$
$+2\alpha\beta\delta^2+2\alpha\gamma\delta^2+2\beta\gamma\delta^2$

However:

$(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$=\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta$
$+\alpha\beta^2\gamma+\alpha\beta^2\delta+\beta^2\gamma\delta$
$+\alpha\beta\gamma^2+\alpha\gamma^2\delta+\beta\gamma^2\delta$
$+\alpha\beta\delta^2+\alpha\gamma\delta^2+\beta\gamma\delta^2+4\alpha\beta\gamma\delta$

So

$(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$
$=(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2+(\gamma\delta)^2$
$+(\alpha\delta)^2+(\beta\delta)^2+6\alpha\beta\gamma\delta$
$+2\begin{pmatrix}\alpha+\beta\\+\gamma+\delta\end{pmatrix}\begin{pmatrix}\alpha\beta\gamma+\alpha\beta\delta\\+\alpha\gamma\delta+\beta\gamma\delta\end{pmatrix}$
$-8\alpha\beta\delta\delta$

So:

$(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2$
$+(\gamma\delta)^2+(\alpha\delta)^2+(\beta\delta)^2$
$=(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)^2$
$+2\alpha\beta\gamma\delta$
$-2\begin{pmatrix}\alpha+\beta\\+\gamma+\delta\end{pmatrix}\begin{pmatrix}\alpha\beta\gamma+\alpha\beta\delta\\+\alpha\gamma\delta+\beta\gamma\delta\end{pmatrix}$
$=\left(-\frac{3}{4}\right)^2+2\left(\frac{4}{3}\right)-2\left(\frac{1}{2}\right)\left(-\frac{1}{5}\right)$
$=\frac{9}{16}+\frac{8}{3}+\frac{1}{5}$
$=\frac{135}{240}+\frac{640}{240}+\frac{48}{240}$
$=\frac{823}{240}$

e $(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$=(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$\times(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$
$=(\alpha\beta\gamma)^2+\alpha^2\beta^2\gamma\delta+\alpha^2\beta\gamma^2\delta+\alpha\beta^2\gamma^2\delta$
$+\alpha^2\beta^2\gamma\delta+(\alpha\beta\delta)^2+\alpha^2\beta\gamma\delta^2+\alpha\beta^2\gamma\delta^2$
$+\alpha^2\beta\gamma^2\delta+\alpha^2\beta\gamma\delta^2+(\alpha\gamma\delta)^2+\alpha\beta\gamma^2\delta^2$
$+\alpha\beta^2\gamma^2\delta+\alpha\beta^2\gamma\delta^2+\alpha\beta\gamma^2\delta^2+(\beta\gamma\delta)^2$
$=(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$+\alpha\beta\gamma\delta\begin{pmatrix}\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\beta+\alpha\delta+\beta\delta\\+\alpha\gamma+\alpha\delta+\gamma\delta+\beta\gamma+\beta\delta+\gamma\delta\end{pmatrix}$

So $(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$=(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$+2\alpha\beta\gamma\delta(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

Hence

$(\alpha\beta\gamma)^2+(\alpha\beta\delta)^2+(\alpha\gamma\delta)^2+(\beta\gamma\delta)^2$
$=(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2$
$-2\alpha\beta\gamma\delta(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$=\left(-\frac{1}{5}\right)^2-2\left(\frac{4}{3}\right)\left(-\frac{3}{4}\right)$
$=\frac{1}{25}+2=\frac{51}{25}$

9 a $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$
$=\alpha\beta\gamma\delta+(\alpha+\beta+\gamma+\delta)$
$+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$+(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+1$
$=\frac{3}{2}+\left(-\frac{1}{2}\right)+\left(-\frac{1}{3}\right)+\left(\frac{1}{4}\right)+1$
$=\frac{23}{12}$

b $(2-\alpha)(2-\beta)(2-\gamma)(2-\delta)$
$=16-8(\alpha+\beta+\gamma+\delta)$
$+4(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$
$-2(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+\alpha\beta\gamma\delta$
$=16-8\left(-\frac{1}{2}\right)+4\left(-\frac{1}{3}\right)-2\left(\frac{1}{4}\right)+\frac{3}{2}$
$=16+4-\frac{4}{3}-\frac{1}{2}+\frac{3}{2}=\frac{59}{3}$

10 a $x^3 - 6x^2 + 9x - 15 = 0$
So $a = 1, b = -6, c = 9$ and $d = -15$.
$\alpha + \beta + \gamma = -\frac{b}{a} = 6$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 9$
$\alpha\beta\gamma = -\frac{d}{a} = 15$

b i $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{9}{15} = \frac{3}{5}$

ii
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 -$
$2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (6)^2 - 2(9) = 18$

iii $(\alpha - 1)(\beta - 1)(\gamma - 1)$
$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha)$
$+ (\alpha + \beta + \gamma) - 1$
$= 15 - 9 + 6 - 1 = 11.$

11 a $2x^3 + 4x^2 + 7 = 0$
So $a = 2, b = 4, c = 0$ and $d = 7$.
$\alpha + \beta + \gamma = -\frac{b}{a} = -2$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 0$
$\alpha\beta\gamma = -\frac{d}{a} = -\frac{7}{2}$

b i $\alpha^2 + \beta^2 + \gamma^2$
$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (-2)^2 - 2(0)$
$= 4$

ii $\alpha^3\beta^3\gamma^3 = (\alpha\beta\gamma)^3 = \left(-\frac{7}{2}\right)^3 = -\frac{343}{8}$

iii $(\alpha + 2)(\beta + 2)(\gamma + 2)$
$= \alpha\beta\gamma + 4(\alpha + \beta + \gamma)$
$+ 2(\alpha\beta + \beta\gamma + \gamma\alpha) + 8$
$= -\frac{7}{2} + 4(-2) + 2(0) + 8 = -\frac{7}{2}$

12 $(\alpha + \beta + \gamma)^3$
$\equiv (\alpha + \beta + \gamma)\begin{pmatrix} \alpha^2 + \beta^2 + \gamma^2 \\ +2(\alpha\beta + \beta\gamma + \gamma\alpha) \end{pmatrix}$
$\equiv \alpha^3 + \beta^3 + \gamma^3$
$+\alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2) + \gamma(\alpha^2 + \beta^2)$
$+2(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$
Calculate $(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$:
$(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$
$\equiv \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma$
$+\gamma^2\alpha + \beta^2\gamma + \gamma^2\beta + 3\alpha\beta\gamma$
$\equiv \alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2)$
$+\gamma(\alpha^2 + \beta^2) + 3\alpha\beta\gamma$
So:
$(\alpha + \beta + \gamma)^3$
$\equiv \alpha^3 + \beta^3 + \gamma^3$
$+3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma$
Therefore
$\alpha^3 + \beta^3 + \gamma^3$
$\equiv (\alpha + \beta + \gamma)^3$
$-3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

13 a $3x^3 - px + 11 = 0$
So $a = 3, b = 0, c = -p$ and $d = 11$.
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{p}{3}$
But $\alpha\beta + \beta\gamma + \gamma\alpha = 4$
So $-\frac{p}{3} = 4 \Rightarrow p = -12$

b $\alpha + \beta + \gamma = -\frac{b}{a} = 0$
$\alpha\beta\gamma = -\frac{d}{a} = -\frac{11}{3}$

c $(3 - a)(3 - \beta)(3 - \gamma)$
$= 27 - 9(\alpha + \beta + \gamma) + 3(\alpha\beta + \beta\gamma + \gamma\alpha)$
$-\alpha\beta\gamma$
$= 27 - 9(0) + 3(4) - \left(-\frac{11}{3}\right)$
$= 27 + 12 + \frac{11}{3} = \frac{128}{3}$

14 a $x^4+2x^2-x+3=0$

So $a=1, b=0, c=2, d=-1$ and $e=3$.

$\sum\alpha=-\frac{b}{a}=0$

$\sum\alpha\beta=\frac{c}{a}=2$

$\sum\alpha\beta\gamma=-\frac{d}{a}=1$

$\alpha\beta\gamma\delta=\frac{e}{a}=3$

b i $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{1}{3}$

ii

$\alpha^2+\beta^2+\gamma^2+\delta^2=$

$\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$

$=0^2-2(2)=-4$

iii $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$

$=\alpha\beta\gamma\delta+(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$

$+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$+(\alpha+\beta+\gamma+\delta)+1$

$=3+1+2+0+1=7$

15 a $ax^4+3x^3+2x^2+x-6=0$

So $a=a, b=3, c=2, d=1$ and $e=-6$.

$\alpha\beta\gamma\delta=-3$

But $\alpha\beta\gamma\delta=\frac{e}{a}=-\frac{6}{a}$

So $-\frac{6}{a}=-3\Rightarrow a=2$

b $\sum\alpha=-\frac{b}{a}=-\frac{3}{2}$

$\sum\alpha\beta=\frac{c}{a}=\frac{3}{2}=1$

$\sum\alpha\beta\gamma=-\frac{d}{a}=-\frac{1}{2}$

c $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{\left(-\frac{1}{2}\right)}{-3}=\frac{1}{6}$

16 $\left(\sum\alpha\right)^2\equiv(\alpha+\beta+\gamma+\delta)^2$

$\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$

$+2(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta)$

Therefore:

$\alpha^2+\beta^2+\gamma^2+\delta^2$

$\equiv\left(\sum\alpha\right)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta)$