Roots of polynomials 4E

1 a $x^3-7x^2+6x+5=0$

$a=1, b=-7, c=6$ and $d=5$

Method 1:

$\alpha+\beta+\gamma=-\frac{b}{a}=7$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=6$

$\alpha\beta\gamma=-\frac{d}{a}=-5$

Sum $=(\alpha+1)+(\beta+1)+(\gamma+1)$
$=\alpha+\beta+\gamma+3$
$=10$

Pair sum $=(\alpha+1)(\beta+1)+(\beta+1)(\gamma+1)$
$+(\gamma+1)(\alpha+1)$
$=\alpha\beta+\beta\gamma+\gamma\alpha+2(\alpha+\beta+\gamma)+3$
$=6+2(7)+3$
$=23$

Product $=(\alpha+1)(\beta+1)(\gamma+1)$
$=\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)$
$+(\alpha+\beta+\gamma)+1$
$=-5+6+7+1$
$=9$

Hence the new equation is:

$w^3-10w^2+23w-9=0$

Method 2:

Let $w=x+1$, so $x=w-1$

Substitute $x=w-1$ into

$x^3-7x^2+6x+5=0$

$(w-1)^3-7(w-1)^2+6(w-1)+5=0$

$w^3-3w^2+3w-1-7(w^2-2w+1)$
$+6w-6+5=0$

$w^3-3w^2+3w-1-7w^2+14w-7$
$+6w-6+5=0$

Simplify to obtain:

$w^3-10w^2+23w-9=0$

b Remember:

$\alpha+\beta+\gamma=7$

$\alpha\beta+\beta\gamma+\gamma\alpha=6$

$\alpha\beta\gamma=-5$

Method 1:

Sum $=2\alpha+2\beta+2\gamma$
$=2(\alpha+\beta+\gamma)$
$=14$

Pair sum $=(2\alpha)(2\beta)+(2\beta)(2\gamma)$
$+(2\gamma)(2\alpha)$
$=4(\alpha\beta+\beta\gamma+\gamma\alpha)$
$=24$

Product $=(2\alpha)(2\beta)(2\gamma)=8\alpha\beta\gamma=-40$

Hence the new equation is:

$w^3-14w^2+24w+40=0$

Method 2:

$w=2x$, so $x=\frac{w}{2}$

Substitute $x=\frac{w}{2}$ into

$x^3-7x^2+6x+5=0$

$\left(\frac{w}{2}\right)^3-7\left(\frac{w}{2}\right)^2+6\left(\frac{w}{2}\right)+5=0$

$\frac{w^3}{8}-\frac{7}{4}w^2+3w+5=0$

Multiply by 8:

$w^3-14w^2+24w+40=0$

2 a $3x^3-4x^2-5x+1=0$

$a=3, b=-4, c=-5$ and $d=1$

$\alpha+\beta+\gamma=-\frac{b}{a}=\frac{4}{3}$

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{5}{3}$

$\alpha\beta\gamma=-\frac{d}{a}=-\frac{1}{3}$

Sum $=(\alpha-3)+(\beta-3)+(\gamma-3)$

$=\alpha+\beta+\gamma-9$

$=-\frac{23}{3}$

Pair sum $=(\alpha-3)(\beta-3)+(\beta-3)(\gamma-3)+(\gamma-3)(\alpha-3)$

$=\alpha\beta+\beta\gamma+\gamma\alpha-6(\alpha+\beta+\gamma)+27$

$=-\frac{5}{3}-6\left(\frac{4}{3}\right)+27$

$=-\frac{5}{3}-\frac{24}{3}+\frac{81}{3}$

$=\frac{52}{3}$

Product $=(\alpha-3)(\beta-3)(\gamma-3)$

$=\alpha\beta\gamma-3(\alpha\beta+\beta\gamma+\gamma\alpha)+9(\alpha+\beta+\gamma)-27$

$=-\frac{1}{3}-3\left(-\frac{5}{3}\right)+9\left(\frac{4}{3}\right)-27$

$=-\frac{1}{3}+5+12-27$

$=-\frac{31}{3}$

Hence the new equation is:

$w^3+\frac{23}{3}w^2+\frac{52}{3}w+\frac{31}{3}=0$

Multiply by 3:

$3w^3+23w^2+52w+31=0$

b $w=\frac{x}{2}$, so $x=2w$

Substitute $x=2w$ into

$3x^3-4x^2-5x+1=0$

$3(2w)^3-4(2w)^2-5(2w)+1=0$

$3(8w^3)-4(4w^2)-10w+1=0$

$24w^3-16w^2-10w+1=0$

3 $x^3-3x^2+4x-7=0$

Let $w=2x+1$, so $x=\frac{w-1}{2}$

Substitute $x=\frac{w-1}{2}$ into $x^3-3x^2+4x-7=0$

$\left(\frac{w-1}{2}\right)^3-3\left(\frac{w-1}{2}\right)^2+4\left(\frac{w-1}{2}\right)-7=0$

Multiply through by $2^3=8$ to remove fractions:

$(w-1)^3-6(w-1)^2+16(w-1)-56=0$

$w^3-3w^2+3w-1-6(w^2-2w+1)+16w-16-56=0$

$w^3-3w^2+3w-1-6w^2+12w-6+16w-16-56=0$

$w^3-9w^2+31w-79=0$

4 $x^3+4x^2-4x+2=0$

Let $w=2x-1$, so $x=\frac{w+1}{2}$

Substitute $x=\frac{w+1}{2}$ into $x^3+4x^2-4x+2=0$

$\left(\frac{w+1}{2}\right)^3+4\left(\frac{w+1}{2}\right)^2-4\left(\frac{w+1}{2}\right)+2=0$

Multiply through by $2^3=8$ to remove fractions:

$(w+1)^3+8(w+1)^2-16(w+1)+16=0$

$w^3+3w^2+3w+1+8(w^2+2w+1)-16w-16+16=0$

$w^3+3w^2+3w+1+8w^2+16w+8-16w=0$

$w^3+11w^2+3w+9=0$

5 $3x^3 - x^2 + 2x - 5 = 0$

Let $w = 3x + 1$, so $x = \frac{w-1}{3}$

Substitute $x = \frac{w-1}{3}$ into $3x^3 - x^2 + 2x - 5 = 0$

$3\left(\frac{w-1}{3}\right)^3 - \left(\frac{w-1}{3}\right)^2 + 2\left(\frac{w-1}{3}\right) - 5 = 0$

Multiply through by $3^3 = 27$ to remove fractions:

$3(w-1)^3 - 3(w-1)^2 + 18(w-1) - 135 = 0$

$3(w^3 - 3w^2 + 3w - 1) - 3(w^2 - 2w + 1)$
$+ 18w - 18 - 135 = 0$

$3w^3 - 9w^2 + 9w - 3 - 3w^2 + 6w - 3$
$+ 18w - 18 - 135 = 0$

$3w^3 - 12w^2 + 33w - 159 = 0$

So $w^3 - 4w^2 + 11w - 53 = 0$ as all terms are divisible by 3.

6 a $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$

$w = 3x$, so $x = \frac{w}{3}$

Substitute $x = \frac{w}{3}$ into

$2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$

$2\left(\frac{w}{3}\right)^4 + 4\left(\frac{w}{3}\right)^3 - 5\left(\frac{w}{3}\right)^2 + 2\left(\frac{w}{3}\right) - 1 = 0$

$\frac{2w^4}{81} + \frac{4w^3}{27} - \frac{5w^2}{9} + \frac{2w}{3} - 1 = 0$

Multiply through by 81 to remove fractions:

$2w^4 + 12w^3 - 45w^2 + 54w - 81 = 0$

b $w = x - 1$, so $x = w + 1$

Substitute $x = w + 1$ into

$2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$

$2(w+1)^4 + 4(w+1)^3 - 5(w+1)^2$
$+ 2(w+1) - 1 = 0$

$2(w^4 + 4w^3 + 6w^2 + 4w + 1)$
$+ 4(w^3 + 3w^2 + 3w + 1) - 5(w^2 + 2w + 1)$
$+ 2w + 2 - 1 = 0$

$2w^4 + 8w^3 + 12w^2 + 8w + 2$
$+ 4w^3 + 12w^2 + 12w + 4 - 5w^2 - 10w - 5$
$+ 2w + 2 - 1 = 0$

$2w^4 + 12w^3 + 19w^2 + 12w + 2 = 0$

7 a $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$

$w = 2x$, so $x = \frac{w}{2}$

Substitute $x = \frac{w}{2}$ into

$x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$

$\left(\frac{w}{2}\right)^4 + 2\left(\frac{w}{2}\right)^3 - 3\left(\frac{w}{2}\right)^2$
$+ 4\left(\frac{w}{2}\right) + 5 = 0$

$\frac{w^4}{16} + \frac{2w^3}{8} - \frac{3w^2}{4} + 2w + 5 = 0$

Multiply through by 16 to remove fractions:

$w^4 + 4w^3 - 12w^2 + 32w + 80 = 0$

b $w = x - 2$, so $x = w + 2$

Substitute $x = w + 2$ into

$x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$

$(w+2)^4 + 2(w+2)^3 - 3(w+2)^2$
$+ 4(w+2) + 5 = 0$

$w^4 + 8w^3 + 24w^2 + 32w + 16$
$+ 2(w^3 + 6w^2 + 12w + 8)$
$- 3(w^2 + 4w + 4) + 4w + 8 + 5 = 0$

$w^4 + 8w^3 + 24w^2 + 32w + 16$
$+ 2w^3 + 12w^2 + 24w + 16$
$- 3w^2 - 12w - 12 + 4w + 8 + 5 = 0$

$w^4 + 10w^3 + 33w^2 + 48w + 33$

8 a $3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$

$w = 3x$, so $x = \frac{w}{3}$

Substitute $x = \frac{w}{3}$ into

$3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$

$3\left(\frac{w}{3}\right)^4 + 5\left(\frac{w}{3}\right)^3 - 4\left(\frac{w}{3}\right)^2 - 3\left(\frac{w}{3}\right) + 1 = 0$

$\frac{3w^4}{81} + \frac{5w^3}{27} - \frac{4w^2}{9} - w + 1 = 0$

Multiply through by 81 to remove fractions:

$3w^4 + 15w^3 - 36w^2 - 81w + 81 = 0$

All terms are divisible by 3:

$w^4 + 5w^3 - 12w^2 - 27w + 27 = 0$

8 b $w = x + 1$, so $x = w - 1$

Substitute $x = w - 1$ into

$3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$

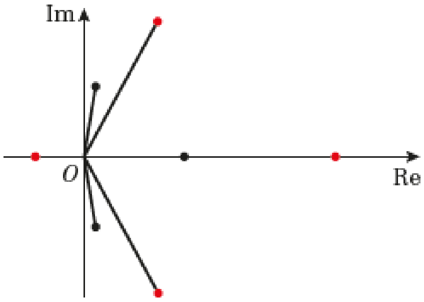
$3(w-1)^4 + 5(w-1)^3 - 4(w-1)^2$
$\quad -3(w-1) + 1 = 0$

$3(w^4 - 4w^3 + 6w^2 - 4w + 1)$
$\quad +5(w^3 - 3w^2 + 3w - 1)$
$\quad -4(w^2 - 2w + 1) - 3w + 3 + 1 = 0$

$3w^4 - 12w^3 + 18w^2 - 12w + 3$
$\quad +5w^3 - 15w^2 + 15w - 5$
$\quad -4w^2 + 8w - 4 - 3w + 3 + 1 = 0$

$3w^4 - 7w^3 - w^2 + 8w - 2 = 0$

b



Challenge

a $2x^4 - 3x^3 + x^2 - 2x - 6 = 0$

$w = 2x + 1$, so $x = \frac{w-1}{2}$

Substitute $x = \frac{w-1}{2}$ into

$2\left(\frac{w-1}{2}\right)^4 - 3\left(\frac{w-1}{2}\right)^3 + \left(\frac{w-1}{2}\right)^2$
$\quad -2\left(\frac{w-1}{2}\right) - 6 = 0$

Multiply through by 8 to remove fractions:

$(w-1)^4 - 3(w-1)^3 + 2(w-1)^2$
$\quad -8(w-1) - 48 = 0$

$w^4 - 4w^3 + 6w^2 - 4w + 1$
$\quad -3(w^3 - 3w^2 + 3w - 1)$
$\quad +2(w^2 - 2w + 1) - 8w + 8 - 48 = 0$

$w^4 - 4w^3 + 6w^2 - 4w + 1$
$\quad -3w^3 + 9w^2 - 9w + 3$
$\quad +2w^2 - 4w + 2 - 8w + 8 - 48 = 0$

$w^4 - 7w^3 + 17w^2 - 25w - 34 = 0$