Volumes of revolution ME 5

1 $V = \pi\int_0^3 y^2 \, dx$

$= \pi\int_0^3 x^4\left(9 - x^2\right) dx$

$= \pi\int_0^3 \left(9x^4 - x^6\right) dx$

$= \pi\left[\frac{9x^5}{5} - \frac{x^7}{7}\right]_0^3$

$= \pi\left(\frac{2187}{5} - \frac{2187}{7}\right)$

$= \frac{4374}{35}\pi$

2 a $2y^2 - 6\sqrt{x} + 3 = 0$

$y = 0 \Rightarrow 3 = 6\sqrt{x} \Rightarrow x = \frac{1}{4}$

Curve cuts the x-axis when $x = \frac{1}{4}$.

b $V = \pi\int_{0.25}^4 y^2 \, dx$

$= \pi\int_{0.25}^4 \left(3\sqrt{x} - \frac{3}{2}\right) dx$

$= 3\pi\int_{0.25}^4 \left(\sqrt{x} - \frac{1}{2}\right) dx$

$= 3\pi\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x\right]_{0.25}^4$

$= 3\pi\left(\left(\frac{16}{3} - 2\right) - \left(\frac{1}{12} - \frac{1}{8}\right)\right)$

$= 3\pi\left(\frac{10}{3} + \frac{1}{24}\right)$

$= 3\pi \times \frac{81}{24}$

$= \frac{81}{8}\pi$

3 a $f(x) = x^2 + 4x + 4$

$y = f(x) \Rightarrow y = x^2 + 4x + 4$

$y = (x+2)^2$

$\sqrt{y} = x + 2$

$x = \sqrt{y} - 2$

$x^2 = \left(\sqrt{y} - 2\right)^2$

$x^2 = 4 - 4\sqrt{y} + y$

b $V = \pi\int_4^9 x^2 \, dy$

$= \pi\int_4^9 \left(4 - 4\sqrt{y} + y\right) dy$

$= \pi\left[4y - \frac{8}{3}y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_4^9$

$= \pi\left(\left(36 - 72 + \frac{81}{2}\right) - \left(16 - \frac{64}{3} + 8\right)\right)$

$= \pi\left(\frac{9}{2} - \frac{8}{3}\right)$

$= \frac{11}{6}\pi$

4 a $V = \pi\int_0^1 y^2 \, dx$

$= \pi\int_0^1 \left(x^2 + 3\right)^2 dx$

$= \pi\int_0^1 \left(x^4 + 6x^2 + 9\right) dx$

$= \pi\left[\frac{1}{5}x^5 + 2x^3 + 9x\right]_0^1$

$= \frac{56}{5}\pi$

4 b $x=1 \Rightarrow y=4$ and $x=0 \Rightarrow y=3$

Volume generated by curve when rotated about the y-axis

$$= \pi\int_3^4 x^2 \ \mathrm{d}y$$

$$= \pi\int_3^4 (y-3) \ \mathrm{d}y$$

$$= \pi\left[\frac{1}{2}y^2 - 3y\right]_3^4$$

$$= \pi\left((8-12)-\left(\frac{9}{2}-9\right)\right)$$

$$= \frac{1}{2}\pi$$

Volume of cylinder generated when the line $x=1$ is rotated about the y-axis
$= \pi\times1\times4 = 4\pi$

Volume obtained when region is rotated about the y-axis = $4\pi - \frac{1}{2}\pi = \frac{7}{2}\pi$

5 a Substituting $x=2$ and $y=4.5$ into the equation of the line gives
$3\times2+4\times4.5=24$
Substituting $x=2$ into the equation of the gives $y=\frac{1}{4}\times2\times(2+1)^2=4.5$
So (2, 4.5) are the coordinates of the point of intersection A.

5 b Volume V_1 is generated by the curve between O and A.

$$V_1 = \pi\int_0^2 y^2 \ \mathrm{d}x$$

$$= \frac{\pi}{16}\int_0^2 x^2(x+1)^4 \ \mathrm{d}x$$

$$= \frac{\pi}{16}\int_0^2 x^2\left(x^4+4x^3+6x^2+4x+1\right) \mathrm{d}x$$

$$= \frac{\pi}{16}\int_0^2 \left(x^6+4x^5+6x^4+4x^3+x^2\right) \mathrm{d}x$$

$$= \frac{\pi}{16}\left[\frac{1}{7}x^7+\frac{2}{3}x^6+\frac{6}{5}x^5+x^4+\frac{1}{3}x^3\right]_0^2$$

$$= \frac{\pi}{16}\left(\frac{128}{7}+\frac{128}{3}+\frac{192}{5}+16+\frac{8}{3}\right)$$

$$= \pi\left(\frac{8}{7}+\frac{8}{3}+\frac{12}{5}+1+\frac{1}{6}\right)$$

$$= \pi\left(\frac{240+560+504+210+35}{210}\right)$$

$$= \frac{1549}{210}\pi$$

Volume V_2 generated by the line $3x+4y=24$ is a cone with $r=4.5$ and $h=6$

$$V_2 = \frac{1}{3}\pi\times\left(\frac{9}{2}\right)^2\times6$$

$$= \frac{81}{2}\pi$$

Total volume $= \frac{1549}{210}\pi + \frac{81}{2}\pi = \frac{5027}{105}\pi$

6 a y coordinate of points A and B
$= 0.1 \times 2^2 + 4 = 4.4$

Volume of cylinder with $r = 2$ and $h = 4.4$
$= \pi \times 2^2 \times 4.4 = 17.6\pi \text{ cm}^3$

Volume generated by curve
$$= \pi \int_4^{4.4} x^2 \, dy$$
$$= 10\pi \int_4^{4.4} (y - 4) \, dy$$
$$= 10\pi \left[\frac{1}{2} y^2 - 4y \right]_4^{4.4}$$
$$= 10\pi \left((9.68 - 17.6) - (8 - 16) \right)$$
$$= 10\pi \left(-7.92 - (-8) \right)$$
$$= 0.8\pi$$

Volume of bronze required
$= 17.6\pi - 0.8\pi$
$= 16.8\pi \text{ cm}^3$

b For example, the shape of the holder is unlikely to exactly follow the curve, or it doesn't allow for any wastage.

7 Volume of the cap of the mushroom
$$= \pi \int_0^4 x^2 \, dy$$
$$= 4\pi \int_0^4 \left(8\sqrt{y} - 4y \right) \, dy$$
$$= 4\pi \left[\frac{16}{3} y^{\frac{3}{2}} - 2y^2 \right]_0^4$$
$$= 4\pi \left(\frac{128}{3} - 32 \right)$$
$$= \frac{128}{3}\pi$$

The stem of mushroom is a cylinder with $r = 1$ and $h = 4$

Volume of stem $= \pi \times 1 \times 4 = 4\pi$

Total volume of mushroom
$$= \frac{128}{3}\pi + 4\pi = \frac{140}{3}\pi \text{ cm}^3$$

8 The y coordinate of the points of intersection of the two curves will be given by solving simultaneously

$3y^2 + x^2 - 11y = 0$ (1)

$y = 2x^2$ (2)

$$3y^2 + \frac{y}{2} - 11y = 0$$
$$6y^2 - 21y = 0$$
$$2y^2 - 7y = 0$$
$$y(2y - 7) = 0$$
$$y = 0 \text{ or } y = \frac{7}{2}$$

Volume V_1 is generated by the curve $3y^2 + x^2 - 11y = 0$ between $y = 0$ and $y = \frac{7}{2}$

$$V_1 = \pi \int_0^{\frac{7}{2}} \left(11y - 3y^2 \right) \, dy$$
$$= \pi \left[\frac{11}{2} y^2 - y^3 \right]_0^{\frac{7}{2}}$$
$$= \pi \left(\frac{539}{8} - \frac{343}{8} \right)$$
$$= \frac{49}{2}\pi$$

Volume V_2 is generated by the curve $y = 2x^2$ between $y = 0$ and $y = \frac{7}{2}$

$$V_2 = \pi \int_0^{\frac{7}{2}} \frac{y}{2} \, dy$$
$$= \pi \left[\frac{y^2}{4} \right]_0^{\frac{7}{2}}$$
$$= \frac{49}{16}\pi$$

Volume generated by shaded region
$$= V_1 - V_2$$
$$= \frac{49}{2}\pi - \frac{49}{16}\pi$$
$$= \frac{343}{16}\pi$$

Challenge

a Using Pythagoras' theorem:

$r^2 = x^2 + y^2$

So $y^2 = r^2 - x^2$

Area of disc $= \pi y^2 = \pi\left(r^2 - x^2\right)$

b Volume of sphere will be obtained by integrating this disc area between the limits $x = -r$ to $x = r$

$$= \pi\int_{-r}^{r}\left(r^2 - x^2\right)\,\mathrm{d}x$$

$$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{-r}^{r}$$

$$= \pi\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right)$$

$$= \frac{4}{3}\pi r^3$$