Matrices 6C

1 a $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3\times 2 - 4\times(-1) = 10$

b $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 4\times 2 - 2\times 1 = 6$

c $\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = (-2)\times 0 - 1\times 3 = -3$

d $\begin{vmatrix} -4 & -4 \\ 1 & 1 \end{vmatrix} = (-4)\times 1 - (-4)\times 1 = 0$

e $\begin{vmatrix} 7 & -4 \\ 0 & 3 \end{vmatrix} = 7\times 3 - (-4)\times 0 = 21$

f $\begin{vmatrix} -1 & -1 \\ -6 & -10 \end{vmatrix} = (-1)\times(-10) - (-1)\times(-6) = 4$

2 a $\det\begin{vmatrix} a & 1+a \\ 3 & 2 \end{vmatrix} = 2a - 3(1+a)$

$= 2a - 3 - 3a$

$= -3 - a$

Matrix is singular for $a = -3$

b Let $\mathbf{A} = \begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$

$\det \mathbf{A} = (1+a)(1-a) - (3-a)(a+2)$

$= 1 - a^2 - (-a^2 + a + 6)$

$= 1 - a^2 + a^2 - a - 6$

$= -a - 5$

$\det \mathbf{A} = 0 \Rightarrow a = -5$

c Let $\mathbf{B} = \begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

$\det \mathbf{B} = 2a + a^2 - (1-a)^2$

$= 2a + a^2 - 1 + 2a - a^2$

$= 4a - 1$

$\det \mathbf{B} = 0 \Rightarrow a = \frac{1}{4}$

3 $\det \mathbf{M} = \begin{vmatrix} -2 & 1-k \\ k-1 & k \end{vmatrix} = (-2)\times k - (1-k)\times(k-1)$

$= k^2 - 4k + 1$

For singular matrix $k^2 - 4k + 1 = 0$

$k = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2\times 1}$

$= \dfrac{4 \pm \sqrt{12}}{2}$

$k = 2 + \sqrt{3},\ k = 2 - \sqrt{3}$

4 $\det \mathbf{P} = \begin{vmatrix} 3k & 4-k \\ k-2 & -k \end{vmatrix}$

$= 3k\times(-k) - (4-k)\times(k-2)$

$= -2k^2 - 6k + 8$

For singular matrix $2k^2 + 6k - 8 = 0$

$(k+4)(k-1) = 0$

$k = -4$ and $k = 1$

5 a $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix} \Rightarrow \det \mathbf{A} = 2ab - 2ab = 0$

$\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \Rightarrow \det \mathbf{B} = 2ab - 2ab = 0$

b $\mathbf{AB} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$

$= \begin{pmatrix} 2ab - 2ab & -2a^2 + 2a^2 \\ 2b^2 - 2b^2 & -2ab + 2ab \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 a $\begin{vmatrix}1&0&0\\0&2&0\\0&0&3\end{vmatrix}=1\begin{vmatrix}2&0\\0&3\end{vmatrix}-0\begin{vmatrix}0&0\\0&3\end{vmatrix}+0\begin{vmatrix}0&2\\0&0\end{vmatrix}$

$=1(6-0)-0+0=6$

b $\begin{vmatrix}0&4&0\\5&-2&3\\2&1&4\end{vmatrix}=0\begin{vmatrix}-2&3\\1&4\end{vmatrix}-4\begin{vmatrix}5&3\\2&4\end{vmatrix}+0\begin{vmatrix}5-&2\\2&1\end{vmatrix}$

$=0-4(20-6)+0=-56$

c $\begin{vmatrix}1&0&1\\2&4&1\\3&5&2\end{vmatrix}=1\begin{vmatrix}4&1\\5&2\end{vmatrix}-0\begin{vmatrix}2&1\\3&2\end{vmatrix}+1\begin{vmatrix}2&4\\3&5\end{vmatrix}$

$=1(8-5)-0+1(10-12)$

$=3-2=1$

d $\begin{vmatrix}2&-3&4\\2&2&2\\5&5&5\end{vmatrix}=2\begin{vmatrix}2&2\\5&5\end{vmatrix}-(-3)\begin{vmatrix}2&2\\5&5\end{vmatrix}+4\begin{vmatrix}2&2\\5&5\end{vmatrix}$

$=2(10-10)+3(10-10)$

$+4(10-10)=0$

7 a $\begin{vmatrix}4&3&-1\\2&-2&0\\0&4&-2\end{vmatrix}$

$=4\begin{vmatrix}-2&0\\4&-2\end{vmatrix}-3\begin{vmatrix}2&0\\0&-2\end{vmatrix}+(-1)\begin{vmatrix}2&-2\\0&4\end{vmatrix}$

$=4(4-0)-3(-4-0)-1(8-0)$

$=16+12-8=20$

b $\begin{vmatrix}3&-2&1\\4&1&-3\\7&2&-4\end{vmatrix}$

$=3\begin{vmatrix}1&-3\\2&-4\end{vmatrix}-(-2)\begin{vmatrix}4&-3\\7&-4\end{vmatrix}+1\begin{vmatrix}4&1\\7&2\end{vmatrix}$

$=3(-4+6)+2(-16+21)+1(8-7)$

$=6+10+1=17$

7 c $\begin{vmatrix}5&-2&-3\\6&4&2\\-2&-4&-3\end{vmatrix}$

$=5\begin{vmatrix}4&2\\-4&-3\end{vmatrix}-(-2)\begin{vmatrix}6&2\\-2&-3\end{vmatrix}+(-3)\begin{vmatrix}6&4\\-2&-4\end{vmatrix}$

$=5(-12+8)+2(-18+4)-3(-24+8)$

$=5\times(-4)+2\times(-14)-3\times(-16)$

$=-20-28+48=0$

8 $\det(\mathbf{A})=\begin{vmatrix}2&1&-4\\2k+1&3&k\\1&0&1\end{vmatrix}$

$=2\begin{vmatrix}3&k\\0&1\end{vmatrix}-1\begin{vmatrix}2k+1&k\\1&1\end{vmatrix}+(-4)\begin{vmatrix}2k+1&3\\1&0\end{vmatrix}$

$=2(3-0)-1(2k+1-k)-4(0-3)$

$=6-k-1+12=17-k$

As **A** is singular,

$\det(\mathbf{A})=0$

$17-k=0$

$k=17$

9 $\det(\mathbf{A})=\begin{vmatrix}2&-1&3\\k&2&4\\-2&1&k+3\end{vmatrix}$

$=2\begin{vmatrix}2&4\\1&k+3\end{vmatrix}-(-1)\begin{vmatrix}k&4\\-2&k+3\end{vmatrix}+3\begin{vmatrix}k&2\\-2&1\end{vmatrix}$

$=2(2k+6-4)+1(k^2+3k+8)+3(k+4)$

$=4k+4+k^2+3k+8+3k+12$

$=k^2+10k+24$

As $\det(\mathbf{A})=8$

$k^2+10k+24=8$

$k^2+10k+16=0$

$(k+8)(k+2)=0$

$k=-8,-2$

10 a $\det(\mathbf{A}) = \begin{vmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{vmatrix}$

$= 2\begin{vmatrix} 0 & 4 \\ 10 & 8 \end{vmatrix} - 5\begin{vmatrix} -2 & 4 \\ 3 & 8 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ 3 & 10 \end{vmatrix}$

$= 2(0-40) - 5(-16-12) + 3(-20-0)$

$= -80 + 140 - 60 = 0$

Hence **A** is singular.

b $\mathbf{AB} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2+5+0 & 2+10-6 & 0+10-3 \\ -2+0+0 & -2+0-8 & 0+0-4 \\ 3+10+0 & 3+20-16 & 0+20-8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{pmatrix}$

c $\det(\mathbf{AB}) = \begin{vmatrix} 7 & 6 & 7 \\ -2 & -10 & -4 \\ 13 & 7 & 12 \end{vmatrix}$

$= 7\begin{vmatrix} -10 & -4 \\ 7 & 12 \end{vmatrix} - 6\begin{vmatrix} -2 & -4 \\ 13 & 12 \end{vmatrix} + 7\begin{vmatrix} -2 & -10 \\ 13 & 7 \end{vmatrix}$

$= 7(-120+28) - 6(-24+52) + 7(-14+130)$

$= 7\times(-92) - 6\times 28 + 7\times 116$

$= -644 - 168 + 812 = 0$

Hence **AB** is also singular.

11 $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$

$- a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} + (-b)\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$

$= 0 - a(0 - cb) - b(ac - 0)$

$= abc - abc = 0$

Hence the matrix is singular for all a, b and c.

12 $\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix}$

$- (-2)\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$

$= 2(x^2+6) + 2(3x-2) + 4(9+x)$

$= 2x^2 + 12 + 6x - 4 + 36 + 4x$

$= 2x^2 + 10x + 44$

$= 2(x^2+5)x + 44$

$= 2\left(x^2 + 5x + \left(\frac{5}{2}\right)^2\right) + 44 - 2\times\left(\frac{5}{2}\right)^2$

$= 2\left(x + \frac{5}{2}\right)^2 + 31\frac{1}{2} \geq 31\frac{1}{2}$, for all real x

Hence the determinant cannot be zero and the matrix is non-singular for all real x.

13 $\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix}$

$= (x-3)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$

$= (x-3)(x^2+x-2) + 2(x+1-4) + 0$

$= x^3 + x^2 - 2x - 3x^2 - 3x + 6 + 2x - 6$

$= x^3 - 2x^2 - 3x$

From the matrix to be singular, the determinant must be zero.

$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$

$= x(x-3)(x+1)$

$= 0$

$x = -1, 0, 3$

14 a $\det \mathbf{M} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 1\times 1 - (-3)\times 2 = 7$

14 b $\det \mathbf{N} = \begin{vmatrix} -1 & k \\ 4 & 3 \end{vmatrix} = (-1)\times 1 - k\times 4 = -3 - 4k$

$\det \mathbf{N} = 7$

$-3 - 4k = 7$

$4k = -10 \Rightarrow k = -2.5$

c $\mathbf{MN}=\begin{pmatrix}1 & -3\\ 2 & 1\end{pmatrix}\times\begin{pmatrix}-1 & -2.5\\ 4 & 3\end{pmatrix}$

$=\begin{pmatrix}-13 & -11.5\\ 2 & -2\end{pmatrix}$

d $\det \mathbf{MN}=\begin{vmatrix}-13 & -11.5\\ 2 & -2\end{vmatrix}$

$=(-13)\times(-2)-(-11.5)\times 2=49$

$\det \mathbf{M}=7$

$\det \mathbf{N}=-3+(-4)\times(-2.5)=7$

$\det \mathbf{MN}=7\times 7=49$

15 a $\det \mathbf{A}=\begin{vmatrix}2 & 1 & -1\\ 1 & 0 & 4\\ -4 & 2 & 1\end{vmatrix}$

$=2(0-8)-1(1+16)-1(2-0)$

$=-16-17-2$

$=-35$

b $\det \mathbf{B}=\begin{vmatrix}3 & 1 & 2\\ k & 4 & 5\\ 0 & 2 & 3\end{vmatrix}$

$=3(12-10)-1(3k-0)+2(2k-0)$

$=6-3k+4k$

$=6+k$

$\det \mathbf{B}=2$

$6+k=2\Rightarrow k=-4$

c $\mathbf{AB}=\begin{pmatrix}2 & 1 & -1\\ 1 & 0 & 4\\ -4 & 2 & 1\end{pmatrix}\times\begin{pmatrix}3 & 1 & 2\\ -4 & 4 & 5\\ 0 & 2 & 3\end{pmatrix}$

$=\begin{pmatrix}2 & 4 & 6\\ 3 & 9 & 14\\ -20 & 6 & 5\end{pmatrix}$

15 d $\det \mathbf{AB}=\begin{vmatrix}2 & 4 & 6\\ 3 & 9 & 14\\ -20 & 6 & 5\end{vmatrix}$

$=2(45-84)-4(15+280)+6(18+180)$

$=-78-1180+1188$

$=-70$

$=(-35)\times 2=\det \mathbf{A}\times\det \mathbf{B}$

Challenge

a $\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc=0$ for singular matrix

$ad-bc=0\Rightarrow ad=bc$

The possibilities are:

$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix},\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix},$

$\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix},\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix},\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}$

b $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$

$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$