Matrices 6D

1 a det $\begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} = 6-(-4)\times(-1)$

$= 6-4$

$= 2 \quad \neq 0$

$\therefore$ the matrix is non-singular

So inverse is $\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

or $\begin{pmatrix} 1 & 0.5 \\ 2 & 1.5 \end{pmatrix}$

b det $\begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} = -3-(-1)\times 3$

$= -3+3$

$= 0$

$\therefore$ matrix is singular.

c det $\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} = 0-0$

$= 0$

$\therefore$ matrix is singular

d det $\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5-6$

$= -1 \quad \neq 0$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

e det $\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12-12$

$= 0$

$\therefore$ matrix is singular

f det $\begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = 8-18$

$= -10 \quad \neq 0$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-10}\begin{pmatrix} 2 & -3 \\ -6 & 4 \end{pmatrix}$

$= \begin{pmatrix} -0.2 & 0.3 \\ 0.6 & -0.4 \end{pmatrix}$

2 a Let $\mathbf{A} = \begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$

det $\mathbf{A} = 2a+a^2-(1+a)^2$

$= 2a+a^2-1-2a-a^2$

$= -1$

$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 2+a & -(1+a) \\ -(1+a) & a \end{pmatrix}$

$= \begin{pmatrix} -(2+a) & (1+a) \\ (1+a) & -a \end{pmatrix}$

b Let $\mathbf{B} = \begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

det $\mathbf{B} = -2ab-(-a)\times 3b$

$= -2ab+3ab$

$= ab$

$\mathbf{B}^{-1} = \frac{1}{ab}\begin{pmatrix} -b & -3b \\ a & 2a \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{a} & -\frac{3}{a} \\ \frac{1}{b} & \frac{2}{b} \end{pmatrix}$ **provided that** $ab \neq 0$

3 a
$$\mathbf{ABC} = 1$$
$$\Rightarrow \mathbf{A}^{-1}\mathbf{ABC} = \mathbf{A}^{-1}\mathbf{I}$$
$$\Rightarrow \mathbf{BC} = \mathbf{A}^{-1}$$
$$\Rightarrow \mathbf{BCC}^{-1} = \mathbf{A}^{-1}\mathbf{C}^{-1}$$
$$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}^{-1} = (\mathbf{CA})^{-1}$$
$$\therefore \mathbf{B}^{-1} = \mathbf{CA}$$

b
$$\mathbf{CA} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$
$$\therefore (\mathbf{CA})^{-1} = \frac{1}{-3+4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

4 a
$$\mathbf{AB} = \mathbf{C}$$
$$\Rightarrow \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{C}$$
$$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

b
$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \Rightarrow \det \mathbf{A} = 6 - -4 = 10$$
$$\therefore \mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$
$$\therefore \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$
$$= \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$$
$$= \frac{1}{10}\begin{pmatrix} 10 & 40 \\ -10 & 20 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$$

5 a
$$\mathbf{BAC} = \mathbf{B}$$
$$\Rightarrow \mathbf{B}^{-1}\mathbf{BAC} = \mathbf{B}^{-1}\mathbf{B}$$
$$\Rightarrow \mathbf{AC} = \mathbf{I}$$
$$\Rightarrow \mathbf{A} = \mathbf{C}^{-1}.$$

b
$$\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$
$$\det \mathbf{C} = 10 - 9 = 1$$
$$\therefore \mathbf{C}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$
$$\therefore \mathbf{A} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

6
$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \Rightarrow$$
$$\det \mathbf{A} = 6 - (-4)\times(-1) = 2$$
$$\therefore \mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$
$$\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix} \quad (\times \text{ on left by } \mathbf{A}^{-1})$$
$$\Rightarrow \mathbf{B} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$$
$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} 4 & 8 & -6 \\ 0 & 2 & 4 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 1 & 2 \end{pmatrix}$$

7
$$\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{B} = 5 + 8 = 13$$
$$\mathbf{B}^{-1} = \frac{1}{13}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$$
$$\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix} \quad (\times \text{ on right by } \mathbf{B}^{-1})$$
$$\Rightarrow \mathbf{ABB}^{-1} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\mathbf{B}^{-1}$$
$$\therefore \mathbf{A} = \frac{1}{13}\begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$$
$$= \frac{1}{13}\begin{pmatrix} 13 & 39 \\ -26 & 13 \\ 0 & -13 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & -1 \end{pmatrix}$$

8 a $\mathbf{A}=\begin{pmatrix}3a & b\\ 4a & 2b\end{pmatrix}\Rightarrow$

det $\mathbf{A}=6ab-4ab=2ab$

$\therefore\ \mathbf{A}^{-1}=\dfrac{1}{2ab}\begin{pmatrix}2b & -b\\ -4a & 3a\end{pmatrix}.$

b $\mathbf{B}=\mathbf{XA}$

$\Rightarrow \mathbf{BA}^{-1}=\mathbf{XAA}^{-1}$

$\therefore\ \mathbf{X}=\mathbf{BA}^{-1}$

So $\mathbf{X}=\begin{pmatrix}-a & b\\ 3a & 2b\end{pmatrix}\begin{pmatrix}2b & -b\\ -4a & 3a\end{pmatrix}\times\dfrac{1}{2ab}$

$=\dfrac{1}{2ab}\begin{pmatrix}-6ab & 4ab\\ -2ab & 3ab\end{pmatrix}$

$\therefore\ \mathbf{X}=\begin{pmatrix}-3 & 2\\ -1 & \frac{3}{2}\end{pmatrix}$

9 a Given $\mathbf{AB}=\mathbf{BA}$
and $\mathbf{ABA}=\mathbf{B}$

$\Rightarrow\ \mathbf{A}(\mathbf{AB})=\mathbf{B}$

$\Rightarrow\ \mathbf{A}^2\mathbf{B}=\mathbf{B}$

$\Rightarrow\ \mathbf{A}^2\mathbf{BB}^{-1}=\mathbf{BB}^{-1}$

$\Rightarrow\ \mathbf{A}^2=\mathbf{I}$

b $\mathbf{AB}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}c & d\\ a & b\end{pmatrix}$

$\mathbf{BA}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}b & a\\ d & c\end{pmatrix}$

$\mathbf{AB}=\mathbf{BA}\Rightarrow b=c$

$d=a$

i.e. $a=d$ and $b=c$

10 a det $\mathbf{M}=\begin{vmatrix}2 & 3\\ k & -1\end{vmatrix}=2\times(-1)-3\times k$

$=-2-3k\neq 0$ for inverse to exist.

Hence $-2-3k\Rightarrow k\neq-\dfrac{2}{3}$

b $\mathbf{M}^{-1}=\dfrac{1}{-(2+3k)}\begin{pmatrix}-1 & -3\\ -k & 2\end{pmatrix}$

$=\dfrac{1}{3k+2}\begin{pmatrix}1 & 3\\ k & -2\end{pmatrix}$

11 a det $\mathbf{A}=\begin{vmatrix}4 & p\\ -2 & -2\end{vmatrix}=4\times(-2)-p\times(-2)$

$=-8+2p\neq 0$ for inverse to exist.

Hence $-8+2p\Rightarrow p\neq 4$

$\mathbf{A}^{-1}=\dfrac{1}{2p-8}\begin{pmatrix}-2 & -p\\ 2 & 4\end{pmatrix}$

$=\dfrac{1}{p-4}\begin{pmatrix}-1 & \frac{-p}{2}\\ 1 & 2\end{pmatrix}$

b $\mathbf{A}+\mathbf{A}^{-1}=\begin{pmatrix}4 & p\\ -2 & -2\end{pmatrix}+\begin{pmatrix}-\frac{1}{p-4} & \frac{p}{2p-8}\\ \frac{1}{p-4} & \frac{2}{p-4}\end{pmatrix}$

$=\begin{pmatrix}\frac{4p-17}{p-4} & \frac{2p^2-9p}{2p-8}\\ \frac{9-2p}{p-4} & \frac{10-2p}{p-4}\end{pmatrix}=\begin{pmatrix}5 & \frac{9}{2}\\ -3 & -4\end{pmatrix}$

Compare a corresponding element to find p

$\dfrac{4p-17}{p-4}=5$

$4p-17=5(p-4)$

$4p-17=5p-20$

$5p-4p=-17+20$

$p=3$

12 a det $\mathbf{M}=\begin{vmatrix}k & -3\\ 4 & k+3\end{vmatrix}=k\times(k+3)-(-3)\times 4$

$=k^2+3k+12$

12 b Complete the square

$$k^2+3k+12=\left(k^2+3k+\frac{9}{4}\right)+12-\frac{9}{4}$$

$$=\left(k+\frac{3}{2}\right)^2+9.75\geq 9.75 \text{ for all } k$$

Hence, $\mathbf{M}$ is non-singular

c $10\mathbf{M}^{-1}+\mathbf{M}=\mathbf{I}$

$$\mathbf{M}^{-1}=\frac{1}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}$$

$$10\mathbf{M}^{-1}=\frac{10}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}$$

$$10\mathbf{M}^{-1}+\mathbf{M}=\frac{10}{k^2+3k+12}\begin{pmatrix}k+3 & 3\\ -4 & k\end{pmatrix}+\begin{pmatrix}k & -3\\ 4 & k+3\end{pmatrix}$$

Top right element is $\dfrac{30}{k^2+3k+12}-3$

Compare with top right element of $\mathbf{I}$

$$\frac{30}{k^2+3k+12}-3=0$$

$$\frac{30}{k^2+3k+12}=3$$

$$3\left(k^2+3k+12\right)=30$$

$$3k^2+9k+6=0$$

$$(3k+6)(k+1)=0$$

$k=-2$ inadmissible, does not give $\mathbf{I}$

$k=-1$

13 a $\det \mathbf{A}=\begin{vmatrix}a & 2\\ 3 & 2a\end{vmatrix}=a\times 2a-2\times 3$

$$=2a^2-6$$

$$\mathbf{A}^{-1}=\frac{1}{2a^2-6}\begin{pmatrix}2a & -2\\ -3 & a\end{pmatrix}$$

b $2a^2-6$

$2a^2=6$

$a^2=3$

$a=\sqrt{3},\ a=-\sqrt{3}$