## Matrices 6F

**1 a** Write the system of equations using matrices:

$$\begin{pmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 2 & -6 & 4 \\ 3 & 2 & -9 \\ -2 & 4 & 1 \end{vmatrix} = 50$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & -9 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -9 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 2 & -9 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & -9 \end{vmatrix} & \begin{vmatrix} 2 & -6 \\ 3 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 38 & -15 & 16 \\ -22 & 10 & -4 \\ 46 & -30 & 22 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 38 & 15 & 16 \\ 22 & 10 & 4 \\ 46 & 30 & 22 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{50} \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 38 & 22 & 46 \\ 15 & 10 & 30 \\ 16 & 4 & 22 \end{pmatrix} \begin{pmatrix} 32 \\ -49 \\ -3 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 0 \\ -100 \\ 250 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$$

$x = 0, y = -2, z = 5$

**b** Write the system of equations using matrices:

$$\begin{pmatrix} -4 & 6 & -2 \\ 3 & 3 & -2 \\ -6 & -7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -22 \\ 1 \\ 3 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} -4 & 6 & -2 \\ 3 & 3 & -2 \\ -6 & -7 & 3 \end{vmatrix} = 44$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 3 & -2 \\ -7 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -6 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ -6 & -7 \end{vmatrix} \\ \begin{vmatrix} 6 & -2 \\ -7 & 3 \end{vmatrix} & \begin{vmatrix} -4 & -2 \\ -6 & 3 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ -6 & -7 \end{vmatrix} \\ \begin{vmatrix} 6 & -2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -4 & -2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ 3 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -3 & -3 \\ 4 & -24 & 64 \\ -6 & 14 & -30 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -5 & 3 & -3 \\ -4 & -24 & -64 \\ -6 & -14 & -30 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{44} \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{44} \begin{pmatrix} -5 & -4 & -6 \\ 3 & -24 & -14 \\ -3 & -64 & -30 \end{pmatrix} \begin{pmatrix} -22 \\ 1 \\ 3 \end{pmatrix}$$

$$= \frac{1}{44} \begin{pmatrix} 88 \\ -132 \\ -88 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$$

$x = 2, y = -3, z = -2$

**1 c** Write the system of equations using matrices:

$$\begin{pmatrix} 4 & 7 & -2 \\ -10 & -1 & -7 \\ -2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 0 \\ 9 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 4 & 7 & -2 \\ -10 & -1 & -7 \\ -2 & 1 & -4 \end{vmatrix} = -114$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & -7 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} -10 & -7 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} -10 & -1 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 7 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} 4 & 7 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 7 & -2 \\ -1 & -7 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -10 & -7 \end{vmatrix} & \begin{vmatrix} 4 & 7 \\ -10 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 26 & -12 \\ -26 & -20 & 18 \\ -51 & -48 & 66 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 11 & -26 & -12 \\ 26 & -20 & -18 \\ -51 & 48 & 66 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{-114} \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{-114} \begin{pmatrix} 11 & 26 & -51 \\ -26 & -20 & 48 \\ -12 & -18 & 66 \end{pmatrix} \begin{pmatrix} 21 \\ 0 \\ 9 \end{pmatrix}$$

$$= \frac{1}{-114} \begin{pmatrix} -228 \\ -114 \\ 342 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$x = 2, y = 1, z = -3$

**d** Write the system of equations using matrices:

$$\begin{pmatrix} -3 & -6 & 4 \\ -3 & 6 & -10 \\ 3 & 7 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -23 \\ -1 \\ 27 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} -3 & -6 & 4 \\ -3 & 6 & -10 \\ 3 & 7 & -3 \end{vmatrix} = -78$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -6 & -10 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -10 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -3 & 4 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} -6 & 4 \\ -6 & -10 \end{vmatrix} & \begin{vmatrix} -3 & 4 \\ -3 & -10 \end{vmatrix} & \begin{vmatrix} -3 & -6 \\ -3 & -6 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 88 & 39 & -3 \\ -10 & -3 & -3 \\ 84 & 42 & 0 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 88 & -39 & -3 \\ 10 & -3 & 3 \\ 84 & -42 & 0 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{-42} \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{-42} \begin{pmatrix} 88 & 10 & 84 \\ -39 & -3 & -42 \\ -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} -23 \\ -1 \\ 27 \end{pmatrix}$$

$$= \frac{1}{-42} \begin{pmatrix} 234 \\ -234 \\ 66 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{234}{42} \\ \frac{234}{42} \\ -\frac{66}{42} \end{pmatrix} = \begin{pmatrix} -\frac{39}{7} \\ \frac{39}{7} \\ -\frac{11}{7} \end{pmatrix}$$

$x = -\frac{39}{7}, y = \frac{39}{7}, z = -\frac{11}{7}$

**2** Write the system of equations using matrices:

$$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\30\\-3\end{pmatrix},\ \text{where } \mathbf{A}=\begin{pmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{pmatrix}$$

$$\text{so } \begin{pmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\30\\-3\end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det\begin{vmatrix}1&-3&-4\\6&5&-7\\1&4&6\end{vmatrix}=111$$

**A** is invertible, there is a unique solution to the set of equations

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}5&-7\\4&6\end{vmatrix}&\begin{vmatrix}6&-7\\1&6\end{vmatrix}&\begin{vmatrix}6&5\\1&4\end{vmatrix}\\\begin{vmatrix}-3&-4\\4&6\end{vmatrix}&\begin{vmatrix}1&-4\\1&6\end{vmatrix}&\begin{vmatrix}1&-3\\1&4\end{vmatrix}\\\begin{vmatrix}-3&-4\\5&-7\end{vmatrix}&\begin{vmatrix}1&-4\\6&-7\end{vmatrix}&\begin{vmatrix}1&-3\\6&5\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}58&43&19\\-2&10&7\\41&17&23\end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix}58&-43&19\\2&10&-7\\41&-17&23\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}$$

$$\text{Inverse}=\frac{1}{111}\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}$$

$\mathbf{A}^{-1}\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ is the coordinates of point of intersection

$$=\frac{1}{111}\begin{pmatrix}58&2&41\\-43&10&-17\\19&-7&23\end{pmatrix}\begin{pmatrix}3\\30\\-3\end{pmatrix}$$

$$=\frac{1}{111}\begin{pmatrix}111\\222\\-222\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

**3** Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.01 & 1.025 & 0.985 \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = \begin{pmatrix} 3000 \\ 190 \\ 3041 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.01 & 1.025 & 0.985 \end{vmatrix} = 0.065$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 1.025 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1.01 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1.01 & 1.025 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1.025 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.01 & 0.985 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.01 & 1.025 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -0.985 & 0.985 & 2.035 \\ -0.04 & -0.025 & 0.015 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -0.985 & -0.985 & 2.035 \\ 0.04 & -0.025 & -0.015 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{0.065} \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix}$$

$$= \frac{1}{0.065} \begin{pmatrix} -0.985 & 0.04 & 1 \\ -0.985 & -0.025 & 1 \\ 2.035 & -0.015 & -2 \end{pmatrix} \begin{pmatrix} 3000 \\ 190 \\ 3041 \end{pmatrix}$$

$$= \frac{1}{0.065} \begin{pmatrix} 93.6 \\ 81.25 \\ 20.15 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1440 \\ 1250 \\ 310 \end{pmatrix}$$

$\mathrm{A} = £1440,\ \mathrm{B} = £1250,\ \mathrm{C} = £310$

**4** Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.99 & 0.98 & 1.04 \end{pmatrix} \begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix} = \begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.99 & 0.98 & 1.04 \end{vmatrix} = -0.11$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0.98 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0.99 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0.99 & 0.98 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0.98 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0.99 & 1.04 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0.99 & 0.98 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1.04 & 1.04 & 1.97 \\ 0.06 & 0.05 & -0.01 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1.04 & -1.04 & 1.97 \\ -0.06 & 0.05 & 0.01 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}$$

$$\text{Inverse} = -\frac{1}{0.11}\begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix}$$

$$= -\frac{1}{0.11}\begin{pmatrix} -1.04 & -0.06 & 1 \\ -1.04 & 0.05 & 1 \\ 1.97 & 0.01 & -2 \end{pmatrix}\begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix}$$

$$= -\frac{1}{0.11}\begin{pmatrix} -55 \\ -27.5 \\ -137.5 \end{pmatrix}$$

$$\begin{pmatrix} \text{Br} \\ \text{G} \\ \text{Bl} \end{pmatrix} = \begin{pmatrix} 500 \\ 250 \\ 1250 \end{pmatrix}$$

Brown = 500, Grey = 250, Black = 1250

**5 a** Write the system of equations using matrices :

$$\begin{pmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix}$$

$$\det \begin{vmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{vmatrix} = -5(a-2)$$

$-5(a-2) = -5a - 10$

$5a = 10 \Rightarrow a = 2$

**b** The system of equations is inconsistent and has no solution. The three planes meet in three distinct parallel lines to form a prism.

**6 a**

$$\begin{vmatrix} 1 & 4 & q \\ 2 & 3 & -3 \\ q & q & -2 \end{vmatrix} = 1\begin{vmatrix} 3 & -3 \\ q & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ q & -2 \end{vmatrix} + q\begin{vmatrix} 2 & 3 \\ q & q \end{vmatrix}$$

$= (-6 + 3q) - 4(-4 + 3q) + q(2q - 3q)$

$= -6 + 3q + 16 - 12q - q^2$

$= -q^2 - 9q + 10$

$\det \mathbf{M} = 0 \Rightarrow -q^2 - 9q + 10$

$= 0 \Rightarrow q^2 + 9q - 10 = 0$

**b i** Consistent, infinity of solutions, planes meet in a sheaf

**ii** Consistent, unique solution, planes meet in a point

**iii** Inconsistent, no solutions, planes meet in a prism