Matrices Mixed exercise 6

1 $\mathbf{A}=\begin{pmatrix}1 & -3\\ 2 & 1\end{pmatrix} \Rightarrow \det \mathbf{A}=1+6=7$

$\therefore \quad \mathbf{A}^{-1}=\frac{1}{7}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}$

$\mathbf{A}^{-1}(\mathbf{AB})=\frac{1}{7}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & 1 & 9\\ 1 & 9 & 4\end{pmatrix}$

$\therefore \quad \mathbf{B}=\frac{1}{7}\begin{pmatrix}7 & 28 & 21\\ -7 & 7 & -14\end{pmatrix}$

$\therefore \quad \mathbf{B}=\begin{pmatrix}1 & 4 & 3\\ -1 & 1 & -2\end{pmatrix}$

2 a $\mathbf{A}=\begin{pmatrix}a & b\\ 2a & 3b\end{pmatrix}$

$\Rightarrow \det \mathbf{A}=3ab-2ab=ab$

$\therefore \quad \mathbf{A}^{-1}=\frac{1}{ab}\begin{pmatrix}3b & -b\\ -2a & a\end{pmatrix}$

$=\begin{pmatrix}\frac{3}{a} & -\frac{1}{a}\\ \frac{-2}{b} & \frac{1}{b}\end{pmatrix}$

b $\mathbf{XA}=\mathbf{Y} \Rightarrow \mathbf{X}=\mathbf{YA}^{-1}$

$\therefore \quad \mathbf{X}=\begin{pmatrix}a & 2b\\ 2a & b\end{pmatrix}\begin{pmatrix}\frac{3}{a} & -\frac{1}{a}\\ -\frac{2}{b} & \frac{1}{b}\end{pmatrix}$

$=\begin{pmatrix}-1 & 1\\ 4 & -1\end{pmatrix}$

3 a $\mathbf{XB}=\mathbf{BA}$

$\therefore \quad (\mathbf{XB})\mathbf{B}^{-1}=\mathbf{BAB}^{-1}$

i.e. $\mathbf{X}=\mathbf{BAB}^{-1} \qquad (\because \mathbf{BB}^{-1}=\mathbf{I})$

3 b $\mathbf{B}=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix} \Rightarrow \det \mathbf{B}=-2-(-1)=-1$

$\therefore \quad \mathbf{B}^{-1}=\frac{1}{-1}\begin{pmatrix}-1 & -1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$

$\therefore \quad \mathbf{X}=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}5 & 3\\ 0 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$

$=\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}2 & -1\\ 2 & 4\end{pmatrix}$

$\mathbf{X}=\begin{pmatrix}6 & 2\\ -4 & -3\end{pmatrix}$

4 $\mathbf{A}^2=\begin{pmatrix}a & 2 & -1\\ -1 & 1 & -1\\ b & 2 & 1\end{pmatrix}\times\begin{pmatrix}a & 2 & -1\\ -1 & 1 & -1\\ b & 2 & 1\end{pmatrix}$

$=\begin{pmatrix}a^2-2-b & 2a & -a-2-1\\ -a-1-b & -3 & -1\\ ab-2+b & 2b+4 & -b-1\end{pmatrix}$

Compare corresponding elements with given $\mathbf{A}^2$:

$\begin{pmatrix}a^2-2-b & 2a & -a-2-1\\ -a-1-b & -3 & -1\\ ab-2+b & 2b+4 & -b-1\end{pmatrix}=\begin{pmatrix}-4 & 2 & -4\\ -5 & -3 & -1\\ 4 & 10 & -4\end{pmatrix}$

$2a=2 \Rightarrow a=1$

$-b-1=-4 \Rightarrow b=3$

5 $\begin{vmatrix}1 & 0 & 2\\ t & 3 & 1\\ -2 & -1 & 1\end{vmatrix}=1\begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix}-0\begin{vmatrix}t & 1\\ -2 & 1\end{vmatrix}+2\begin{vmatrix}t & 3\\ -2 & -1\end{vmatrix}$

$=1(3+1)+2(-t+6)=16-2t$

As $\mathbf{A}$ is singular

$\det(\mathbf{A})=16-2t=0 \Rightarrow t=8$

6 $\det(\mathbf{M}) = \begin{vmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} x & 0 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix}$

$= 2 - 0 + 0 = 2$

The matrix of minors is

$$\begin{pmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} x & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ x & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ x & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & x & x-6 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 2 & -x & x-6 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ -x & 1 & 0 \\ x-6 & -1 & 2 \end{pmatrix}$$

7 a $\det \mathbf{A} = \begin{vmatrix} k & -2 \\ -4 & k \end{vmatrix} = k \times (k) - (-2) \times (-4)$

$= k^2 - 8 \neq 0$ for inverse to exist.

Hence $k^2 - 8 \Rightarrow k \neq \pm 2\sqrt{2}$

b $\mathbf{A}^{-1} = \dfrac{1}{k^2 - 8}\begin{pmatrix} k & 2 \\ 4 & k \end{pmatrix}$

8 a $\det \mathbf{B} = \begin{vmatrix} k & 6 \\ -1 & k-2 \end{vmatrix} = k \times (k-2) - 6 \times (-1)$

$= k^2 - 2k + 6$

b Complete the square on the determinant

$\det \mathbf{B} = k^2 - 2k + 6 = \left(k^2 - 2k + 1\right) + 6 - 1$

$= (k-1)^2 + 5 \geq 5$ for all k,

so $\det \mathbf{B} \neq 0$ and $\mathbf{B}$ is non-singular.

8 c $21\mathbf{B}^{-1} + \mathbf{B} = -8\mathbf{I}$

$\mathbf{B}^{-1} = \dfrac{1}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21\mathbf{B}^{-1} = \dfrac{21}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix}$

$21\mathbf{B}^{-1} + \mathbf{B} = \dfrac{21}{k^2 - 2k + 6}\begin{pmatrix} k-2 & -6 \\ 1 & k \end{pmatrix} + \begin{pmatrix} k & 6 \\ -1 & k-2 \end{pmatrix}$

Top right element is $\dfrac{21}{k^2 - 2k + 6} \times (-6) + 6$

$= 6 - \dfrac{126}{k^2 - 2k + 6}$

Compare with top right element of $\mathbf{I} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

$6 - \dfrac{126}{k^2 - 2k + 6} = 0$

$6\left(k^2 - 2k + 6\right) - 126 = 0$

$6k^2 - 12k + 36 - 126 = 0$

$(6k - 30)(k + 3) = 0$

$k = 5$ inadmissible, does not give $\mathbf{I}$

$k = -3$

9 a $\mathbf{M}$ is singular when $\det \mathbf{M} = 0$

$\det \mathbf{M} \begin{vmatrix} 2 & -m \\ m & -1 \end{vmatrix} = 2 \times (-1) - (-m) \times m$

$= m^2 - 2$

$m^2 - 2 = 0 \Rightarrow m^2 = 2$

$m = \sqrt{2}, m = -\sqrt{2}$

b $\mathbf{M}^{-1} = \dfrac{1}{m^2 - 2}\begin{pmatrix} -1 & m \\ -m & 2 \end{pmatrix}$

10 a $\det \mathbf{A} = \begin{vmatrix} 3 & 4 & 5 \\ 1 & a & -1 \\ -2 & 1 & 1 \end{vmatrix}$

$$= 3\begin{vmatrix} a & -1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & a \\ -2 & 1 \end{vmatrix}$$

$$= 3(a+1) - 4(1-2) + 5(1+2a)$$

$= 13a + 12 \neq 0$ for $\mathbf{A}$ to have an inverse.

Hence, $a \neq -\frac{12}{13}$

b $\mathbf{M} = \begin{pmatrix} \begin{vmatrix} a & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & a \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ a & -1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & a \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} a+1 & -1 & 1+2a \\ -1 & 13 & 11 \\ -4-5a & -8 & 3a-4 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} a+1 & 1 & 1+2a \\ 1 & 13 & -11 \\ -4-5a & 8 & 3a-4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} a+1 & 1 & -4-5a \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{13a+12}\begin{pmatrix} a+1 & 1 & -4-5a \\ 1 & 13 & 8 \\ 1+2a & -11 & 3a-4 \end{pmatrix}$$

11 Write the system of equations using matrices:

$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}$

so $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$

Find the inverse of the left-hand matrix:

$$\det\begin{vmatrix} 1 & 1 & 1 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{vmatrix} = 26$$

So $\mathbf{A}$ is non-singular and has an inverse.

There is a unique solution to the set of equations.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -4 & 2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -7 & 9 \\ -4 & -5 & -1 \\ 6 & 1 & -5 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 10 & 7 & 9 \\ 4 & -5 & 1 \\ 6 & -1 & -5 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{26}\begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix} 10 & 4 & 6 \\ 7 & -5 & -1 \\ 9 & 1 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix} 52 \\ 52 \\ 52 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

The planes intersect at a single point, (2, 2, 2)

12 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.06 & 1.04 & 1.03 \end{pmatrix}\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} 2500 \\ 300 \\ 2610 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\det\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.06 & 1.04 & 1.03 \end{vmatrix} = -0.01$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & -1 \\ 1.04 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1.06 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1.06 & 1.04 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1.04 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.06 & 1.03 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.06 & 1.04 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1.04 & 2.09 & 1.04 \\ -0.01 & -0.03 & -0.02 \\ -1 & -2 & -1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1.04 & -2.09 & 1.04 \\ 0.01 & -0.03 & 0.02 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}$$

$$\text{Inverse} = -\frac{1}{0.01}\begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} \mathbf{H} \\ \mathbf{D} \\ \mathbf{W} \end{pmatrix}$$

$$= -\frac{1}{0.01}\begin{pmatrix} 1.04 & 0.01 & -1 \\ -2.09 & -0.03 & 2 \\ 1.04 & 0.02 & -1 \end{pmatrix}\begin{pmatrix} 2500 \\ 300 \\ 2610 \end{pmatrix}$$

$$= -\frac{1}{0.01}\begin{pmatrix} -7 \\ -14 \\ -4 \end{pmatrix} = \begin{pmatrix} 700 \\ 1400 \\ 400 \end{pmatrix}$$

700 Hampshire, 1400 Dorset horn, 400 Wiltshire horn

13 a Write the system of equations using matrices:

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ a & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ b \end{pmatrix}$$

$$\det\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ a & 1 & 4 \end{vmatrix} = 5a + 15$$

$5a + 15 = 0$

$5a = -15 \Rightarrow a = -3$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 11 & 11 & 11 \\ 5 & 5 & 5 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 2 & -2 & 2 \\ -11 & 11 & -11 \\ 5 & -5 & 5 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{5a+15}\begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{5a+15}\begin{pmatrix} 2 & -11 & 5 \\ -2 & 11 & -5 \\ 2 & -11 & 5 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \\ b \end{pmatrix}$$

Consider the last row to find b:

$$\frac{1}{5a+15}\left(2\times 6 + (-11)\times 7 + 5\times b\right) = 0$$

$$= \frac{1}{5a+15}(5b - 65) = 0$$

$5b = 65 \Rightarrow b = 13$

13 b The three planes form a sheaf.

Challenge

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} h & j \\ k & l \end{pmatrix}$

det $\mathbf{A} = ad - bc$,

det $\mathbf{B} = hl - jk$

$$\det \mathbf{A} \det \mathbf{B} = (ad - bc)(hl - jk)$$
$$= adhl - adjk - bchl - bcjk$$

$$\mathbf{AB} = \begin{pmatrix} ah + bk & aj + bl \\ ch + dk & cj + dl \end{pmatrix}$$

$$\det(\mathbf{AB}) = (ah + bk)(cj + dl) - (aj + bl)(ch + dk)$$
$$= adhl - adjk - bchl - bcjk$$
$$= \det \mathbf{A} \det \mathbf{B}$$