Exercise 7C

1 a $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

c $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

d $\begin{pmatrix} 5 & 0 \\ 0 & 0.5 \end{pmatrix}$

2 a 4
b 3
c 4
d 2.5

3 a (0, 0)
b (0,0), (3,0), (3,4), (0,4)
(0,0), (3,0), (3,4), (0,4)
So area $= 3 \times 4 = 12$

4 a $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$

c $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

5 $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} ax \\ kax \end{pmatrix}$; $kax = k(ax)$, so this point lies on $y = kx$ and the line is invariant.

6 a Stretch parallel to the x-axis, scale factor 2 and stretch parallel to the y-axis, scale factor −3
b $\det \mathbf{M} = -6$ so the area has been multiplied by 6. So $k = 4$.

7 a $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$

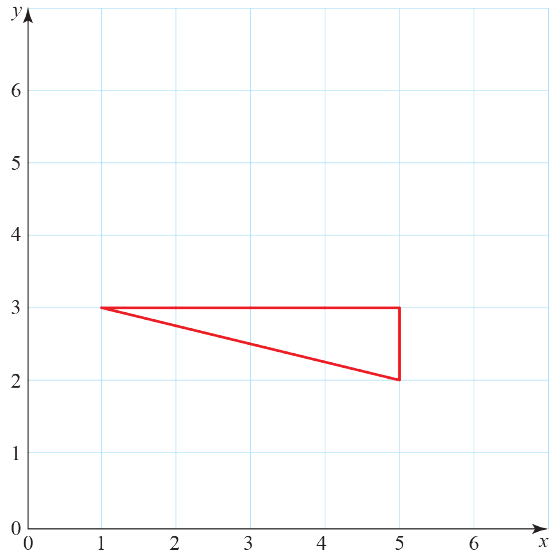
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 9 \end{pmatrix}$

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \end{pmatrix}$

So vertices are (3,9), (15,9), (15,6)

7 b Area of original triangle $= \frac{1}{2} \times 4 \times 1 = 2$

$\det \mathbf{M} = 9$ so new area $= 18$



8 a $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

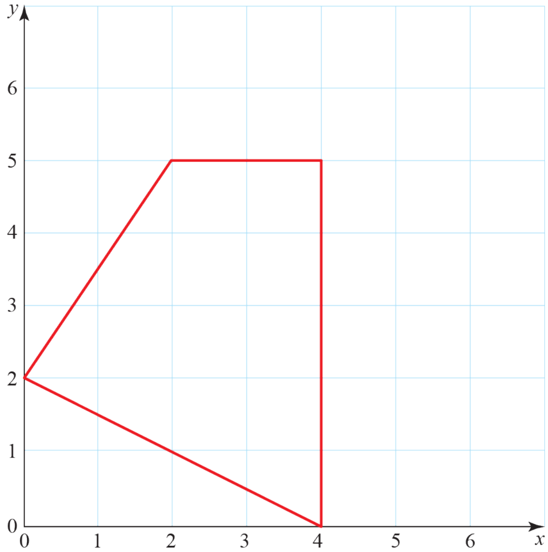
$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -15 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -15 \end{pmatrix}$

So vertices are (4,0), (8,0), (8,-15), (4,-15)

8 b Area of original rectangle $= 2 \times 5 = 10$

det $\mathbf{M} = -6$ so new area = 60



9 a Enlargement, centre (0, 0), scale factor $2\sqrt{5}$

b Area of original triangle

$= \left|\frac{1}{2}\times(4-a)\times 2\right| = |4-a|$

det $\mathbf{M} = 20 \Rightarrow |4-a| = 3$

$\Rightarrow a = 7$ or 1

10 a $\mathbf{M}^2 = \begin{pmatrix} p & 1 \\ p & q \end{pmatrix}\begin{pmatrix} p & 1 \\ p & q \end{pmatrix}$

$= \begin{pmatrix} p^2+p & p+q \\ p^2+pq & p+q^2 \end{pmatrix}$

b $\mathbf{M}^2 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \Rightarrow p^2 + p = 6$

$\Rightarrow (p+3)(p-2) = 0$

$\Rightarrow p = -3$ or 2

But also $p + q = 0$ so p must be -3 because $q > 0$

So $p = -3, q = 3$

11 a $\mathbf{M} = \begin{pmatrix} 5 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$

b Stretch parallel to the x-axis, scale factor 8;
and stretch parallel to the y-axis, scale factor −8. Or enlargement scale factor 8 and
centre (0, 0) and reflection in the x-axis.

c det $\mathbf{M} = -6$

so original area $= 320 \div 64 = 5$

$\Rightarrow |2(k-1)| = 5$

$\Rightarrow k = -\frac{3}{2}$ or $\frac{7}{2}$

12 det $\mathbf{M} = -1 \times -1 - (-\sqrt{2} \times \sqrt{2})$

$= 1 - -2 = 3$

So area of $P' = 12 \times 3 = 36$

13 det $\mathbf{M} = 8 + k$

Area of $T = \frac{1}{2}(4-k)(k-1)$

$\Rightarrow \frac{1}{2}(4-k)(k-1)(k+8) = 10$

$\Rightarrow (4-k)(k-1)(k+8) = 20$

$\Rightarrow (k^2 - 5k + 4)(k+8) = -20$

$\Rightarrow k^3 + 3k^2 - 36k + 52 = 0$

$\Rightarrow (k-2)(k^2 + 5k - 26) = 0$

$\Rightarrow k = 2$ or $k = \frac{-5 \pm \sqrt{129}}{2}$

$\Rightarrow k = 2$

for integer values of k.

14 a $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 7\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

So vertices of T' are

$(0,0), (0, 7\sqrt{2}), \left(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

14 c $\det \mathbf{M} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

d Area of $T' = \frac{1}{2} \times 7\sqrt{2} \times \frac{5}{\sqrt{2}} = 17.5$

So area of $T = 17.5$

Challenge

a 0

b For any x, y, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x \\ 0 \end{pmatrix}$

The y-coordinate is 0, so all points (x, y) map onto the x-axis.