Exercise 7E

1 a $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

f $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

2 a Reflection in the plane $z = 0$

b Rotation anticlockwise 90° about y-axis

c Rotation anticlockwise 135° about z-axis

3 a Rotation anticlockwise 90° about x-axis

b $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$

So coordinates of A′ are (3, -4, -1)

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -a \\ 2a-1 \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a \\ 1-2a \\ -a \end{pmatrix} = \begin{pmatrix} a \\ a-5 \\ -a \end{pmatrix}$

$\Rightarrow 1-2a = a-5 \Rightarrow 3a = 6 \Rightarrow a = 2$

4 a $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-3+\sqrt{3}}{2} \\ \frac{3\sqrt{3}+1}{2} \\ 0 \end{pmatrix}$

So coordinates of Q' are

$\left(\frac{-3+\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}, 0\right)$

c $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} \frac{-k}{2} \\ \frac{\sqrt{3}k}{2} \\ k \end{pmatrix}$

So coordinates of R' are

$\left(\frac{-k}{2}, \frac{\sqrt{3}k}{2}, k\right)$

5 a $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ b \\ c \end{pmatrix}$

So coordinates of P′ are (-a, b, c)

c $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}$

So coordinates of P'' are (−a, −b, c)

6 a Since $\mathbf{M} = \begin{pmatrix} \cos 210^\circ & 0 & \sin 210^\circ \\ 0 & 1 & 0 \\ -\sin 210^\circ & 0 & \cos 210^\circ \end{pmatrix}$ and

$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ represents an

anti-clockwise rotation about the y-axis,

$\mathbf{M}$ represents an anti-clockwise roation of

210° about the y-axis.

b $\begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{k\sqrt{3}}{2} \\ -k \\ \frac{k}{2} \end{pmatrix}$

So coordinates of M' are $\left(-\frac{k\sqrt{3}}{2}, -k, \frac{k}{2}\right)$.

7 a $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

b $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So coordinates of T' are

$(\sqrt{2}, 0, 0), (\sqrt{2}, 1, 0)$

$(0, 2, 3\sqrt{2}), (0, 0, 0)$

c Volume of a tetrahedron:

$= \frac{1}{3} \times$ base area $\times$ height

To find the volume of T':

Area of base on plane z

$= 0 = \frac{1}{2} \times \sqrt{2} \times 1 = \frac{1}{2}\sqrt{2}$

So volume of T'

$= \frac{1}{3} \times \frac{1}{2}\sqrt{2} \times 3\sqrt{2} = 1$

So volume of $T = 1$ as the determinant of the transformation matrix $= 1$.

Challenge

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$