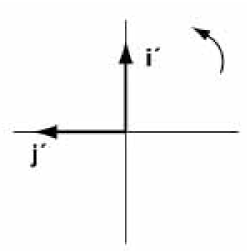
Exercise 7F

1 **a** $(1,0) \to (0,1)$
$(0,1) \to (-1,0)$



R represents a rotation of 90° anticlockwise about (0, 0)

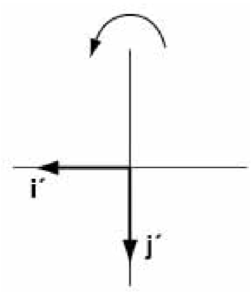
b det $\mathbf{R} = 0-(-1) = 1$

$$\therefore \mathbf{R}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

c $\mathbf{R}^{-1}$ represents a rotation of –90° anticlockwise about (0, 0)

(or … 90° clockwise … or … 270° anticlockwise…)

2 **a** **i** $(1,0) \to (-1,0)$
$(0,1) \to (0,-1)$



S represents a rotation of 180° about (0, 0)

ii $\mathbf{S}^2$ will be rotation of 180 + 180 = 360° about (0, 0) $\therefore \mathbf{S}^2 = \mathbf{I}$

or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

iii $\mathbf{S}^{-1} = \mathbf{S} =$ rotation of 180° about (0, 0)

b **i** $(1,0) \to (0,-1)$
$(0,1) \to (-1,0)$

T represents a reflection in the line $y = -x$

ii $\mathbf{T}^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

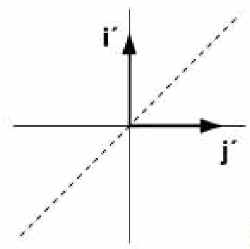
iii $\mathbf{T}^{-1} = \mathbf{T} =$ reflection in the line $y = -x$

c det $\mathbf{S} = 1-0 = 1$
det $\mathbf{T} = 0-1 = -1$

For both **S** and **T**, area is unaltered

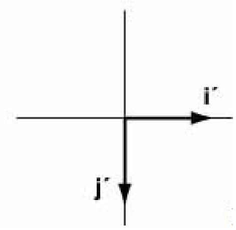
T represents a reflection and $\therefore$ has a negative determinant. Orientation is reversed

3 a $\mathbf{A}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$



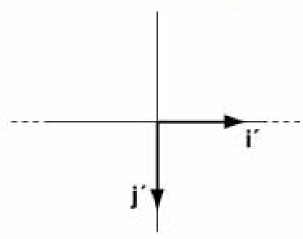
Reflection in $y=x$

$\mathbf{B}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$



Rotation of 270° (about (0, 0))

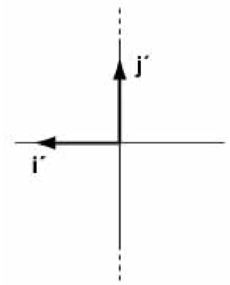
$\mathbf{C}=\mathbf{BA}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$



C represents a reflection in the line $y=0$ (or the x-axis)

b $\mathbf{C}^{-1}=\mathbf{C}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is a reflection in the line $y=0$

c $\mathbf{D}=\mathbf{AB}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$



D represents a reflection in the line $x=0$ (or the y-axis)

d $\mathbf{D}^{-1}=\mathbf{D}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ is a reflection in the line $x=0$

4 a $\mathbf{A}^{-1}=\frac{1}{4}\begin{pmatrix}-2 & -2\\ 3 & 1\end{pmatrix}$

$$\frac{1}{4}\begin{pmatrix}-2 & -2\\ 3 & 1\end{pmatrix}\begin{pmatrix}5\\ 8\end{pmatrix}=\frac{1}{4}\begin{pmatrix}-26\\ 23\end{pmatrix}=\begin{pmatrix}-\frac{13}{2}\\ \frac{23}{4}\end{pmatrix}$$

So coordinates of P are $\left(-\frac{13}{2},\frac{23}{4}\right)$

4 b

$$\mathbf{UA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{U} = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 3 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -3 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

5 a Enlargement, scale factor 4, centre (0, 0)

b $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

c $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 9 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ \frac{7}{4} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$

So coordinates of T are $\left(1, \frac{3}{2}\right), \left(\frac{9}{4}, \frac{7}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)$

6 a $\det \mathbf{M} = ab$

$$\Rightarrow \mathbf{M}^{-1} = \frac{1}{ab}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$$

6 b $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} -\frac{6}{a} \\ \frac{8}{b} \end{pmatrix}$

So coordinates of D are $\left(-\frac{6}{a}, \frac{8}{b}\right)$

7 a Rotation of 330° anticlockwise about (0, 0)

b $\mathbf{R}^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1-2\sqrt{3}}{2} \\ \frac{2+\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}-6-2-\sqrt{3}}{4} \\ \frac{1-2\sqrt{3}+2\sqrt{3}+3}{4} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So $p = -2$, $q = 1$

8 $\mathbf{AB} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -3 & -1 \end{pmatrix}$

$\Rightarrow (\mathbf{AB})^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{3}{2} & 2 \end{pmatrix}$

9 $\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}$

$\frac{1}{2}\begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{a}{2} - b \\ -2a - 3b \end{pmatrix}$

So coordinates of P are $\left(-\frac{a}{2} - b, -2a - 3b\right)$

10 $\mathbf{M}^{-1}=\begin{pmatrix}2 & \frac{3}{2} & -1\\ -1 & -1 & 1\\ -1 & -\frac{3}{2} & 1\end{pmatrix}$

$$\begin{pmatrix}2 & \frac{3}{2} & -1\\ -1 & -1 & 1\\ -1 & -\frac{3}{2} & 1\end{pmatrix}\begin{pmatrix}3\\ 2\\ -1\end{pmatrix}=\begin{pmatrix}10\\ -6\\ -7\end{pmatrix}$$

So $a=10,\ b=-6,\ c=-7$

11 a $\mathbf{PQ}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

Reflection in $y=x$

b $\mathbf{M}^{-1}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

Reflection in $y=x$

c $\mathbf{QP}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

Reflection in $y=-x$

d $\mathbf{N}=\mathbf{QP}\Rightarrow\mathbf{Q}^{-1}\mathbf{N}=\mathbf{P}\Rightarrow\mathbf{P}^{-1}\mathbf{Q}^{-1}\mathbf{N}=1\Rightarrow\mathbf{P}^{-1}\mathbf{Q}^{-1}=\mathbf{N}^{-1}$

$\mathbf{N}^{-1}=\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$; reflection in $y=-x$

12 $\mathbf{BA}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$

$$\Rightarrow\mathbf{B}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\mathbf{A}^{-1}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}-0.1 & 0.3 & 0.7\\ 0.3 & 0.1 & -0.1\\ -0.2 & 0.6 & 0.4\end{pmatrix}$$

$$=\begin{pmatrix}-0.1 & 0.3 & 0.7\\ 0.3 & 0.1 & -0.1\\ 0.2 & -0.6 & -0.4\end{pmatrix}$$