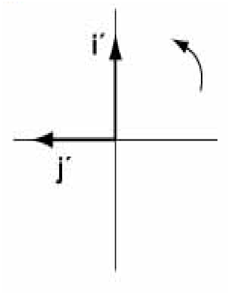
Mixed exercise 7

1 a



$$\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b $\mathbf{AB} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{A} = \mathbf{YB}^{-1}$.

$$\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \det \mathbf{B} = 3 - 4 = -1$$

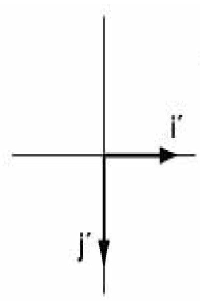
$$\therefore \mathbf{B}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\therefore \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Identity matrix); four successive anticlockwise rotations of 90° about (0, 0).

2 Reflection in x-axis



$$\Rightarrow \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Enlargement S. F. 2 centre (0, 0) $\Rightarrow \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

a $\mathbf{C} = \mathbf{ER} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection in the x-axis and enlargement scale factor 2, centre (0, 0).

$$= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

b $\mathbf{C}^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Reflection in the x-axis and enlargement scale factor $\frac{1}{2}$. Centre (0, 0)

3 a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = P \Rightarrow M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} P$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

b $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} \Rightarrow -3k = 9$

$\Rightarrow k = -3$

c $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} -3x \\ -3mx \end{pmatrix}$

The point $(-3x, -3mx)$ lies on the line $y = mx$ for $m = 1$ and $m = -1$, so the line is invariant under T for these two values of m.

4 a $\det \mathbf{M} = \left(2\sqrt{3}\right)^2 + 4 = 16$

$\Rightarrow$ scale factor $= 4$

b $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} R$

$\Rightarrow R = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$= \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$

So angle is 30° anti-clockwise.

c $\mathbf{M}^{-1} = \frac{1}{16}\begin{pmatrix} 2\sqrt{3} & 2 \\ -2 & 2\sqrt{3} \end{pmatrix}$

$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$

$= \frac{1}{16}\begin{pmatrix} 2\sqrt{3} & 2 \\ -2 & 2\sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}a}{8} + \frac{b}{8} \\ -\frac{a}{8} + \frac{\sqrt{3}b}{8} \end{pmatrix}$

So coordinates of P are $\left(\frac{\sqrt{3}a}{8} + \frac{b}{8}, -\frac{a}{8} + \frac{\sqrt{3}b}{8}\right)$

5 a **A** represents a reflection in the line $y = x$;
B represents a rotation through 270° anticlockwise about (0, 0)

b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}$

So coordinates of image are $(-p, q)$

6 a Area of **T** $= \frac{1}{2}|k-6| \times 5 = \frac{5}{2}|k-6|$

$\det \mathbf{M} = 8 - 3 = 5$

$\Rightarrow 5 \times \frac{5}{2}|k-6| = 110$

$\Rightarrow |k-6| = 8.8$

$\Rightarrow k = 14.8 \text{ or } -2.8$

b $\begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ -\frac{1}{3}x \end{pmatrix} = \begin{pmatrix} -5x \\ \frac{5}{3}x \end{pmatrix}$; $(-5x) + 3\left(\frac{5}{3}\right) = 0$

so the point satisfies the equation of the original line.

7 a $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}$

b $\det \mathbf{P} = -12$

So area of $T = 60 \div 12 = 5$

8 a $\mathbf{A} = \begin{pmatrix} \cos 150^\circ & -\sin 150^\circ & 0 \\ \sin 150^\circ & \cos 150^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So **A** represents a 150° anti-clockwise rotation about the z-axis.

b

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ -a \end{pmatrix} = \begin{pmatrix} -\frac{a\sqrt{3}}{2} - \frac{b}{2} \\ \frac{a}{2} - \frac{b\sqrt{3}}{2} \\ -a \end{pmatrix}$$

So coordinates of image are $\left(-\frac{a\sqrt{3}}{2} - \frac{b}{2}, \frac{a}{2} - \frac{b\sqrt{3}}{2}, -a\right)$

9 a $\mathbf{P}^{-1}=\frac{1}{a^2}\begin{pmatrix}a & 0\\0 & a\end{pmatrix}=\begin{pmatrix}\frac{1}{a} & 0\\0 & \frac{1}{a}\end{pmatrix}$

b $\mathbf{P}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\7\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{P}^{-1}\begin{pmatrix}4\\7\end{pmatrix}=\begin{pmatrix}\frac{1}{a} & 0\\0 & \frac{1}{a}\end{pmatrix}\begin{pmatrix}4\\7\end{pmatrix}=\begin{pmatrix}\frac{4}{a}\\\frac{7}{a}\end{pmatrix}$

So coordinates of **A** are $\left(\frac{4}{a},\frac{7}{a}\right)$

10 The matrix representing a 90° rotation about the origin is $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$.

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{P}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}-1 & 2\\-5 & 8\end{pmatrix}=\begin{pmatrix}5 & -8\\-1 & 2\end{pmatrix}$

$\begin{pmatrix}5 & -8\\-1 & 2\end{pmatrix}^{-1}=\frac{1}{2}\begin{pmatrix}2 & 8\\1 & 5\end{pmatrix}=\begin{pmatrix}1 & 4\\\frac{1}{2} & \frac{5}{2}\end{pmatrix}$

11 a $\mathbf{P}=\mathbf{AB}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4 & -1\\3 & -2\end{pmatrix}=\begin{pmatrix}-4 & 1\\-3 & 2\end{pmatrix}$

b $\det\mathbf{P}=-8--3=-5$

So area of $T=35\div 5=7$

c $\mathbf{Q}=\mathbf{P}^{-1}=\frac{1}{-5}\begin{pmatrix}2 & -1\\3 & -4\end{pmatrix}=-\begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\-\frac{3}{5} & \frac{4}{5}\end{pmatrix}$

12 $\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \end{pmatrix}$$

$\Rightarrow a = 0, b = -\sqrt{2}, c = 0$

13 a $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{19}{6} \\ \frac{11}{6} \end{pmatrix}$$

So coordinates of **P** are $\left(-\frac{1}{6}, -\frac{19}{6}, \frac{11}{6}\right)$

13 b $\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{3} & \frac{1}{6} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{7}{3} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -8 & 1 & 4 \\ -4 & -1 & 2 \\ -14 & 1 & 4 \end{pmatrix}$$

Challenge

1 $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

2 a Let the point be $P = (a, b)$: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0+b \\ 0+b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix}$

So P' is (b, b); its x- and y-coordinates are equal, so it is on $y = x$

b $\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$

c If c does not equal 0, then the line $ax + by = c$ does not go through the origin. Hence the origin cannot be mapped to itself, and the transformation is not linear.