Proof by induction 8B

1 a Let $f(n) = 8^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 8^1 - 1 = 7$, which is divisible by 7.

$\therefore f(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 1$ is divisible by 7 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 8^{k+1} - 1\\ &= 8^k.8^1 - 1\\ &= 8(8^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [8(8^k) - 1] - [8^k - 1]\\ &= 8(8^k) - 1 - 8^k + 1\\ &= 7(8^k)\end{aligned}$$

$\therefore f(k+1) = f(k) + 7(8^k)$

As both $f(k)$ and $7(8^k)$ are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore $f(n)$ is divisible by 7 when $n = 1$.

If $f(n)$ is divisible by 7 when $n = k$, then it has been shown that $f(n)$ is also divisible by 7 when $n = k + 1$. As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 b Let $f(n) = 3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 3^{2(1)} - 1 = 9 - 1 = 8$, which is divisible by 8.

$\therefore$ $f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k} - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 3^{2(k+1)} - 1\\ &= 3^{2k+2} - 1\\ &= 3^{2k}.3^2 - 1\\ &= 9(3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [9(3^{2k}) - 1] - [3^{2k} - 1]\\ &= 9(3^{2k}) - 1 - 3^{2k} + 1\\ &= 8(3^{2k})\end{aligned}$$

$\therefore f(k+1) = f(k) + 8(3^{2k})$

As both $f(k)$ and $8(3^{2k})$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n = k + 1$.

If $f(n)$ is divisible by 8 when $n = k$, then it has been shown that $f(n)$ is also divisible by 8 when $n = k + 1$. As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 c Let $\mathrm{f}(n) = 5^n + 9^n + 2$, where $n \in \mathbb{Z}^+$.

$\therefore\ \mathrm{f}(1) = 5^1 + 9^1 + 2 = 5 + 9 + 2 = 16$, which is divisible by 4.

$\therefore \mathrm{f}(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$\mathrm{f}(k) = 5^k + 9^k + 2$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \mathrm{f}(k+1) &= 5^{k+1} + 9^{k+1} + 2\\ &= 5^k.5^1 + 9^k.9^1 + 2\\ &= 5(5^k) + 9(9^k) + 2\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{f}(k+1) - \mathrm{f}(k) &= [5(5^k) + 9(9^k) + 2] - [5^k + 9^k + 2]\\ &= 5(5^k) + 9(9^k) + 2 - 5^k - 9^k - 2\\ &= 4(5^k) + 8(9^k)\\ &= 4[5^k + 2(9)^k]\end{aligned}$$

$$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 4[5^k + 2(9)^k]$$

As both f(k) and $4[5^k + 2(9)^k]$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore f(n) is divisible by 4 when $n = k + 1$.

If f(n) is indivisible by 4 when $n = k$, then it has been shown that f(n) is also divisible by 4 when $n = k + 1$. As f(n) is divisible by 4 when $n = 1$, f(n) is also divisible by 4 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 d Let $\mathrm{f}(n) = 2^{4n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ \mathrm{f}(1) = 2^{4(1)} - 1 = 16 - 1 = 15,$ which is divisible by 15.

$\therefore \mathrm{f}(n) =$ is divisible by 15 when $n = 1$.

Assume that for $n = k$,

$\mathrm{f}(k) = 2^{4k} - 1$ is divisible by 15 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \mathrm{f}(k+1) &= 2^{4(k+1)} - 1\\ &= 2^{4k+4} - 1\\ &= 2^{4k}.2^4 - 1\\ &= 16(2^{4k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{f}(k+1) - \mathrm{f}(k) &= [16(2^{4k}) - 1] - [2^{4k} - 1]\\ &= 16(2^{4k}) - 1 - 2^{4k} + 1\\ &= 15(2^{4k})\end{aligned}$$

$$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 15(2^{4k})$$

As both $\mathrm{f}(k)$ and $15\,(2^{4k})$ are divisible by 15 then the sum of these two terms must also be divisible by 15. Therefore $\mathrm{f}(n)$ is divisible by 15 when $n = k + 1$.

If $\mathrm{f}(n)$ is divisible by 15 when $n = k$, then it has been shown that $\mathrm{f}(n)$ is also divisible by 15 when $n = k + 1$. As $\mathrm{f}(n)$ is divisible by 15 when $n = 1$, $\mathrm{f}(n)$ is also divisible by 15 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 e Let $\mathrm{f}(n) = 3^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ \mathrm{f}(1) = 3^{2(1)-1} + 1 = 3 + 1 = 4$, which is divisible by 4.

$\therefore \mathrm{f}(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$\mathrm{f}(k) = 3^{2k-1} + 1$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \mathrm{f}(k+1) &= 3^{2(k+1)-1} + 1 \\ &= 3^{2k+2-1} + 1 \\ &= 3^{2k-1}.3^2 + 1 \\ &= 9(3^{2k-1}) + 1\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{f}(k+1) - \mathrm{f}(k) &= \left[9(3^{2k-1}) + 1\right] - \left[3^{2k-1} + 1\right] \\ &= 9(3^{2k-1}) + 1 - 3^{2k-1} + 1 \\ &= 8(3^{2k-1})\end{aligned}$$

$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 8(3^{2k-1})$

As both f(k) and $8(3^{2k-1})$ are divisible by 4 then the sum of these two terms must also by divisible by 4. Therefore f(n) is divisible by 4 when $n = k + 1$.

If f(n) is divisible by 4 when $n = k$, then it has been shown that f(n) is also divisible by 4 when $n = k + 1$. As f(n) is divisible by 4 when $n = 1$, f(n) is also divisible by 8 for $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 f Let $f(n) = n^3 + 6n^2 + 8n$, where $n \geqslant 1$ and $n \in \mathbb{Z}^+$.

$\therefore f(1) = 1 + 6 + 8 = 15$, which is divisible by 3.

$\therefore f(n)$ is divisible by 3 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 6k^2 + 8k$ is divisible by 3 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= (k+1)^3 + 6(k+1)^2 + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6(k^2 + 2k + 1) + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 8k + 8\\ &= k^3 + 9k^2 + 23k + 15\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [k^3 + 9k^2 + 23k + 15] - [k^3 + 6k^2 + 8k]\\ &= 3k^2 + 15k + 15\\ &= 3(k^2 + 5k + 5)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 3(k^2 + 5k + 5)$$

As both f(k) and $3\left(k^2 + 5k + 5\right)$ are divisible by 3 then the sum of these two terms must also be divisible by 3.

Therefore f(n) is divisible by 3 when $n = k + 1$.

If f(n) is divisible by 3 when $n = k$, then it has been shown that f(n) is also divisible by 3 when $n = k + 1$. As f(n) is divisible by 3 when $n = 1, f(n)$ is also divisible by 3 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 g Let $f(n) = n^3 + 5n$, where $n \geqslant 1$ and $n \in \mathbb{Z}^+$.

$\therefore f(1) = 1 + 5 = 6$, which is divisible by 6.

$\therefore f(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 5k$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= (k+1)^3 + 5(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 5(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5\\ &= k^3 + 3k^2 + 8k + 6\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [k^3 + 3k^2 + 8k + 6] - [k^3 + 5k]\\ &= 3k^2 + 3k + 6\\ &= 3k(k+1) + 6\\ &= 3(2m) + 6\\ &= 6m + 6\\ &= 6(m+1)\end{aligned}$$

Let $k(k+1) = 2m, m \in \mathbb{Z}^+$, as the product of two consecutive integers must be even.

$\therefore f(k+1) = f(k) + 6(m+1)$.

As both $f(k)$ and $6(m+1)$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore $f(n)$ is divisible by 6 when $n = k + 1$.

If $f(n)$ is divisible by 6 when $n = k$, then it has been shown that $f(n)$ is also divisible by 6 when $n = k + 1$. As $f(n)$ is divisible by 6 when $n = 1, f(n)$ is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

1 h Let $\mathrm{f}(n) = 2^n.3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore \mathrm{f}(1) = 2^1.3^{2(1)} - 1 = 2(9) - 1 = 18 - 1 = 17$, which is divisible by 17.

$\therefore \mathrm{f}(n)$ is divisible by 17 when $n = 1$.

Assume that for $n = k$,

$\mathrm{f}(k) = 2^k.3^{2k} - 1$ is divisible by 17 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \mathrm{f}(k+1) &= 2^{k+1}.3^{2(k+1)} - 1\\ &= 2^k(2)^1(3)^{2k}(3)^2 - 1\\ &= 2^k(2)^1(3)^{2k}(9) - 1\\ &= 18(2^k.3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{f}(k+1) - \mathrm{f}(k) &= \left[18(2^k.3^{2k}) - 1\right] - \left[2^k.3^{2k} - 1\right]\\ &= 18(2^k.3^{2k}) - 1 - 2^k.3^{2k} + 1\\ &= 17(2^k.3^{2k})\end{aligned}$$

$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 17(2^k.3^{2k})$

As both $\mathrm{f}(k)$ and $17(2^k.3^{2k})$ are divisible by 17 then the sum of these two terms must also be divisible by 17.

Therefore $\mathrm{f}(n)$ is divisible by 17 when $n = k + 1$.

If $\mathrm{f}(n)$ is divisible by 17 when $n = k$, then it has been shown that $\mathrm{f}(n)$ is also divisible by 17 when $n = k + 1$. As f(n) is divisible by 17 when $n = 1$, f(n) is also divisible by 17 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 a $\mathrm{f}(k + 1) = 13^{k+1} - 6^{k+1} = 13 \times 13^k - 6 \times 6^k$
$= 6(13^k - 6^k) + 7 \times 13^k = 6\mathrm{f}(k) + 7(13^k)$

b <u>Basis:</u> $n = 1$: f(1) = 13 − 6 = 7 is divisible by 7.
<u>Assumption:</u> f(k) is divisible by 7.
<u>Induction:</u> f(k + 1) = 6f(k) + 7(13^k) by part **a**.
So if the statement holds for $n = k$, it holds for $n = k + 1$.
<u>Conclusion:</u> The statement holds for all $n \in \mathbb{Z}^+$.

3 a $\mathrm{g}(k + 1) = 5^{2(k+1)} - 6(k + 1) + 8 = 25 \times 52k - 6k + 2$
$= 25(5^{2k} - 6\mathrm{k} + 8) + 144k - 198$
$= 25\mathrm{g}(k) + 9(16k - 22)$

b <u>Basis</u>: $n = 1$: g(1) = $5^2 - 6 + 8 = 27$ is divisible by 9.
<u>Assumption:</u> g(k) is divisible by 9.
<u>Induction:</u> g(k + 1) = 25 g(k) + 9(16k − 22) by part **a**.
So if the statement holds for $n = k$, it holds for $n = k + 1$.
<u>Conclusion:</u> The statement holds for all $n \in \mathbb{Z}^+$.

4 $f(n) = 8^n - 3^n$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 8^1 - 3^1 = 5,$ which is divisible by 5.

$\therefore f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 3^k$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 8^{k+1} - 3^{k+1}\\ &= 8^k.8^1 - 3^k.3^1\\ &= 8(8^k) - 3(3^k)\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - 3f(k) &= \left[8(8^k) - 3(3^k)\right] - 3\left[8^k - 3^k\right]\\ &= 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k)\\ &= 5(8^k)\end{aligned}$$

From (a), $f(k+1) = f(k) + 5(8^k)$

As both f(k) and $5(8^k)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore f(n) is divisible by 5 when $n = k+1$.

If f(n) is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also be divisible by 5 when $n = k+1$. As f(n) is divisible by 5 when $n = 1$, f(n) is also divisible by 5 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

5 $f(n) = 3^{2n+2} + 8n - 9$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 3^{2(1)+2} + 8(1) - 9$

$= 3^4 + 8 - 9 = 81 - 1 = 80$, which is divisible by 8.

$\therefore f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k+2} + 8k - 9$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} f(k+1) &= 3^{2(k+1)+2} + 8(k+1) - 9 \\ &= 3^{2k+2+2} + 8(k+1) - 9 \\ &= 3^{2k+2}.(3^2) + 8^k + 8 - 9 \\ &= 9(3^{2k+2}) + 8k - 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= \left[9(3^{2k+2}) + 8k - 1\right] - \left[3^{2k+2} + 8k - 9\right] \\ &= 9(3^{2k+2}) + 8k - 1 - 3^{2k+2} - 8k + 9 \\ &= 8(3^{2k+2}) + 8 \\ &= 8\left[3^{2k+2} + 1\right] \end{aligned}$$

$$\therefore f(k+1) = f(k) + 8\left[3^{2k+2} + 1\right]$$

As both f(k) and $8\left[3^{2k+2} + 1\right]$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f(n) is divisible by 8 when $n = k + 1$.

If f(n) is divisible by 8 when $n = k$, then it has been shown that f(n) is also divisible by 8 when $n = k + 1$. As f(n) is divisible by 8 when $n = 1$ f(n) is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

6 $f(n) = 2^{6n} + 3^{2n-2}$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 2^{6(1)} + 3^{2(1)-2} = 2^6 + 3^0 = 64 + 1 = 65$, which is divisible by 5.

$\therefore$ $f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$.

$f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 2^{6(k+1)} + 3^{2(k+1)-2}\\ &= 2^{6k+6} + 3^{2k+2-2}\\ &= 2^6(2^{6k}) + 3^2(3^{2k-2})\\ &= 64(2^{6k}) + 9(3^{2k-2})\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= \left[64(2^{6k}) + 9(3^{2k-2})\right] - \left[2^{6k} + 3^{2k-2}\right]\\ &= 64(2^{6k}) + 9(3^{2k-2}) - 2^{6k} - 3^{2k-2}\\ &= 63(2^{6k}) + 8(3^{2k-2})\\ &= 63(2^{6k}) + 63(3^{2k-2}) - 55(3^{2k-2})\\ &= 63\left[2^{6k} + 3^{2k-2}\right] - 55(3^{2k-2})\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) &= f(k) + 63\left[2^{6k} + 3^{2k-2}\right] - 55(3^{2k-2})\\ &= f(k) + 63f(k) - 55(3^{2k-2})\\ &= 64f(k) - 55(3^{2k-2})\end{aligned}$$

$$\therefore f(k+1) = 64f(k) - 55(3^{2k-2})$$

As both 64 $f(k)$ and $-55(3^{2k-2})$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.