Proof by induction 8C

1 $n=1$; LHS $=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^1=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$

RHS $=\begin{pmatrix}1 & 2(1)\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$

As LHS = RHS, the matrix equation is true for $n=1$.

Assume that the matrix equation is true for $n=k$.

i.e. $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k=\begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}$

With $n=k+1$ the matrix equation becomes

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}1+0 & 2+2k\\0+0 & 0+1\end{pmatrix}.$$
$$=\begin{pmatrix}1 & 2(k+1)\\0 & 1\end{pmatrix}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$, by mathematical induction.

2 $n=1$; $\text{LHS}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^1=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

$\text{RHS}=\begin{pmatrix}2(1)+1 & -4(1)\\ 1 & -2(1)+1\end{pmatrix}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

As LHS = RHS, the matrix equation is true for $n=1$.

Assume that the matrix equation is true for $n=k$.

i.e. $\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}$.

With $n=k+1$ the matrix equation becomes

$$\begin{aligned}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^{k+1}&=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}\\
&=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}\\
&=\begin{pmatrix}6k+3-4k & -8k-4+4k\\ 3k-2k+1 & -4k+2k-1\end{pmatrix}\\
&=\begin{pmatrix}2k+3 & -4k-4\\ k+1 & -2k-1\end{pmatrix}\\
&=\begin{pmatrix}2(k+1)+1 & -4(k+1)\\ (k+1) & -2(k+1)+1\end{pmatrix}\end{aligned}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$ by mathematical induction.

3 $n=1$; LHS $=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^1=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$

RHS $=\begin{pmatrix}2^1 & 0\\2^1-1 & 1\end{pmatrix}=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^k=\begin{pmatrix}2^k & 0\\2^k-1 & 1\end{pmatrix}$

With $n=k+1$ the matrix equation becomes

$$\begin{aligned}\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^{k+1}&=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^k\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}\\&=\begin{pmatrix}2^k & 0\\2^k-1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}\\&=\begin{pmatrix}2(2^k)+0 & 0+0\\2(2^k)-2+1 & 0+1\end{pmatrix}\\&=\begin{pmatrix}2^{1(2^k)} & 0\\2^{1(2^k)}-1 & 1\end{pmatrix}\\&=\begin{pmatrix}2^{k+1} & 0\\2^{k+1}-1 & 1\end{pmatrix}\end{aligned}$$

Therefore the matrix equation is true when $n=k+1$.

If the matrix equation is true for $n=k$, then it is shown to be true for $n=k+1$. As the matrix equation is true for $n=1$, it is now also true for all $n\geqslant 1$ and $n\in\mathbb{Z}^+$ by mathematical induction.

4 a $n=1;\ \text{LHS}=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^1=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$

$$\text{RHS}=\begin{pmatrix}4(1)+1 & -8(1)\\ 2(1) & 1-4(1)\end{pmatrix}=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^k=\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^{k+1}=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^k\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$$
$$=\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$$
$$=\begin{pmatrix}20k+5-16k & -32-8+24k\\ 10k+2-8k & -16k-3+12k\end{pmatrix}$$
$$=\begin{pmatrix}4k+5 & -8k-8\\ 2k+2 & -4k-3\end{pmatrix}$$
$$=\begin{pmatrix}4(k+1)+1 & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b n = 6

5 a Basis: $n = 1$: LHS = RHS $=\begin{pmatrix}2 & 5\\ 0 & 1\end{pmatrix}$

Assumption: $\mathbf{M}^k=\begin{pmatrix}2k & 5(2k-1)\\ 0 & 1\end{pmatrix}$

Induction: With $n = k + 1$ the matrix equation becomes

$$\mathbf{M}^{k+1}=\mathbf{M}^k\begin{pmatrix}2 & 5\\ 0 & 1\end{pmatrix}=\begin{pmatrix}2k & 5(2k-1)\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 5\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}2k+1 & 5\times 2k+5(2k-1)\\ 0 & 1\end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

5 b $\begin{pmatrix} 2^{-n} & 5\left(2^{-n}-1\right) \\ 0 & 1 \end{pmatrix}$

Challenge

Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$

Assumption: $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix} k = \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix}$

Induction:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix} k+1 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^k \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k & \frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3} & 4k \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3k+1 & 3k+\frac{3k-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k}{3}-4k & 4k \times 4 \end{pmatrix} = \begin{pmatrix} 3k+1 & \frac{3k+1-1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1-4k+1}{3} & 4k+1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.