

**Proof by induction ME 8**

1 Let  $f(n) = 9^n - 1$ , where  $n \in \mathbb{Z}^+$ .

$$\therefore f(1) = 9^1 - 1 = 8, \text{ which is divisible by } 8.$$

$$\therefore f(n) \text{ is divisible by } 8 \text{ when } n = 1.$$

Assume that for  $n = k$ ,

$$f(k) = 9^k - 1 \text{ is divisible by } 8 \text{ for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \therefore f(k+1) &= 9^{k+1} - 1 \\ &= 9^k \cdot 9^1 - 1 \\ &= 9(9^k) - 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [9(9^k) - 1] - [9^k - 1] \\ &= 9(9^k) - 1 - 9^k + 1 \\ &= 8(9^k) \end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(9^k)$$

As both  $f(k)$  and  $8(9^k)$  are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore  $f(n)$  is divisible by 8 when  $n = k + 1$ .

If  $f(n)$  is divisible by 8 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 8 when  $n = k + 1$ . As  $f(n)$  is divisible by 8 when  $n = 1$ ,  $f(n)$  is also divisible by 8 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

$$2 \text{ a } \mathbf{B}^2 = \mathbf{B}\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$$

$$\text{b As } \mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix} \text{ and } \mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}, \text{ we suggest that } \mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}.$$

$$2 \text{ c } n=1; \text{ LHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

As LHS=RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{i.e. } \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix} \end{aligned}$$

Therefore the matrix is true when  $n = k+1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

$$3 \text{ Basis: } n = 1: \text{ LHS} = 3 \times 1 + 4 = 7; \text{ RHS} = \frac{1}{2} \times 1(3 \times 1 + 11) = 7$$

Assumption:

$$\sum_{r=1}^k (3r + 4) = \frac{1}{2} k(3k + 11)$$

Induction:

$$\begin{aligned} \sum_{r=1}^{k+1} (3r + 4) &= \sum_{r=1}^k (3r + 4) + 3(k + 1) + 4 \\ &= \frac{1}{2} k(3k + 11) + 3(k + 1) + 4 = \frac{1}{2} (3k^2 + 17k + 14) \\ &= \frac{1}{2} (k + 1)(3(k + 1) + 11) \end{aligned}$$

So if the statement holds for  $n = k$ , it holds for  $n = k + 1$ .

Conclusion: The statement holds for all  $n \in \mathbb{Z}^+$ .

$$\begin{aligned}
 \mathbf{4 \ a} \quad n=1; \text{LHS} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
 &= \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}
 \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{i.e. } \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}.$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned}
 \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
 &= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
 &= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix} \\
 &= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix} \\
 &= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix}
 \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

$$\begin{aligned}
 \mathbf{b} \quad \det(\mathbf{A}^n) &= (8n+1)(1-8n) - (-64n^2) \\
 &= 8n - 64n^2 + 1 - 8n + 64n^2 \\
 &= 1
 \end{aligned}$$

$$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1} \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

$$\text{So } \mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

$$\begin{aligned}
 \mathbf{5 \ a} \quad f(n+1) &= 5^{2(n+1)-1} + 1 \\
 &= 5^{2n+2-1} + 1 \\
 &= 5^{2n-1} \cdot 5^2 + 1 \\
 &= 24(5^{2n-1}) + 1
 \end{aligned}$$

$$\begin{aligned}
 \therefore f(n+1) - f(n) &= [25(5^{2n-1}) + 1] - [5^{2n-1} + 1] \\
 &= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1 \\
 &= 24(5^{2n-1})
 \end{aligned}$$

Therefore,  $\mu = 24$ .

$$\mathbf{b} \quad f(n) = 5^{2n-1} + 1, \text{ where } n \in \mathbb{Z}^+.$$

$$\therefore f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6, \text{ which is divisible by 6.}$$

$$\therefore f(n) \text{ is divisible by 6 when } n = 1.$$

Assume that for  $n = k$ ,

$$f(k) = 5^{2k-1} + 1 \text{ is divisible by 6 for } k \in \mathbb{Z}^+.$$

$$\text{Using (a) } f(k+1) - f(k) = 24(5^{2k-1})$$

$$\therefore f(k+1) = f(k) + 24(5^{2k-1})$$

As both  $f(k)$  and  $24(5^{2k-1})$  are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore  $f(n)$  is divisible by 6 when  $n = k + 1$ .

If  $f(n)$  is divisible by 6 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 6 when  $n = k + 1$ . As  $f(n)$  is divisible by 6 when  $n = 1$ ,  $f(n)$  is also divisible by 6 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

6 Let  $f(n) = 7^n + 4^n + 1$ , where  $n \in \mathbb{Z}^+$ .

$$\therefore f(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12, \text{ which is divisible by 6.}$$

$$\therefore f(n) \text{ is divisible by 6 when } n = 1.$$

Assume that for  $n = k$ ,

$$f(k) = 7^k + 4^k + 1 \text{ is divisible by 6 for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \therefore f(k+1) &= 7^{k+1} + 4^{k+1} + 1 \\ &= 7^k \cdot 7^1 + 4^k \cdot 4^1 + 1 \\ &= 7(7^k) + 4(4^k) + 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1] \\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1 \\ &= 6(7^k) + 3(4^k) \end{aligned}$$

Where both  $6(7^k)$  and  $3(4^k)$  are divisible by 6 since 4 is even.

Therefore  $f(n)$  is divisible by 6 when  $n = k + 1$ .

If  $f(n)$  is divisible by 6 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 6 when  $n = k + 1$ . As  $f(n)$  is divisible by 6 when  $n = 1$ ,  $f(n)$  is also divisible by 6 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

7 Basis:  $n = 1$ : LHS =  $1 \times 5 = 5$ ; RHS =  $\frac{1}{6} \times 1 \times 2 \times (2 + 13) = 5$

Assumption:

$$\sum_{r=1}^k r(r+4) = \frac{1}{6} k(k+1)(2k+13)$$

Induction:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+4) &= \sum_{r=1}^k r(r+4) + (k+1)(k+5) \\ &= \frac{1}{6} k(k+1)(2k+13) + (k+1)(k+5) \\ &= \frac{1}{6} (2k^3 + 21k^2 + 49k + 30) = \frac{1}{6} (k+1)(k+2)(2(k+1) + 13) \end{aligned}$$

So if the statement holds for  $n = k$ , it holds for  $n = k + 1$ .

Conclusion: The statement holds for all  $n \in \mathbb{Z}^+$ .

8 a Basis:  $n = 1$ : LHS =  $1 + 4 = 5$ ; RHS =  $\frac{1}{3} \times 1 \times 3 \times 5 = 5$

Assumption:

$$\sum_{r=1}^{2k} r^2 = \frac{1}{3} k(2k+1)(4k+1)$$

Induction:

$$\begin{aligned}
\sum_{r=1}^{2(k+1)} r^2 &= \sum_{r=1}^{2k} r^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2 \\
&= \frac{1}{3}(8k^3 + 30k^2 + 37k + 15) \\
&= \frac{1}{3}(k+1)(2(k+1)+1)(4(k+1)+1)
\end{aligned}$$

So if the statement holds for  $n = k$ , it holds for  $n = k + 1$ .

Conclusion: The statement holds for all  $n \in \mathbb{Z}^+$ .

**b** Using **a** and the formula for  $\sum_{r=1}^n r^2$

$$\begin{aligned}
\frac{1}{6} \times 2n(2n+1)(4n+1) &= \frac{1}{6}kn(n+1)(2n+1) \\
2n(2n+1) + (4n+1) &= kn(n+1)(2n+1) \\
16n^3 + 12n^3 + 12n^2 + 2n &= k(2n^3 + 3n^2) \\
\Rightarrow k &= \frac{2n(8n^2 + 6n + 1)}{n(2n^2 + 3n + 1)} = \frac{2(2n+1)(4n+1)}{(2n+1)(n+1)} = \frac{8n+2}{n+1} \\
\Rightarrow kn + k &= 8n + 2 \Rightarrow n(k-8) = 2-k \Rightarrow n = \frac{2-k}{k-8}
\end{aligned}$$

**9 a** Basis:  $n = 1$ : LHS = RHS =  $\begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = c^k \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix}$$

Induction:

$$\begin{aligned} \mathbf{M}^{k+1} &= \mathbf{M}^k \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix} \\ &= c^k \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{c} \\ 0 & 1 \end{pmatrix} \\ &= c^{k+1} \begin{pmatrix} 2^{k+1} & \frac{2^k + 2^k - 1}{c} \\ 0 & 1 \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^{k+1} & \frac{2^{k+1} - 1}{c} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So if the statement holds for  $n = k$ , it holds for  $n = k + 1$ .

Conclusion: The statement holds for all  $n \in \mathbb{Z}^+$ .

- b** Consider  $n = 1$ :  $\det \mathbf{M} = 50 \Rightarrow 2c^2 = 50$   
So  $c = 5$ , since  $c$  is +ve.

**Challenge**

**a** Basis:  $n = 1$ :  $\text{LHS} = \text{RHS} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

Induction:

$$\begin{aligned} \mathbf{M}^{k+1} &= \mathbf{M}^k \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos k\theta \cos \theta - \sin k\theta \sin \theta & -\cos k\theta \sin \theta - \sin k\theta \cos \theta \\ \sin k\theta \cos \theta + \cos k\theta \sin \theta & -\sin k\theta \sin \theta + \cos k\theta \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos((k+1)\theta) & -\sin((k+1)\theta) \\ \sin((k+1)\theta) & \cos((k+1)\theta) \end{pmatrix} \end{aligned}$$

So if the statement holds for  $n = k$ , it holds for  $n = k + 1$ .

Conclusion: The statement holds for all  $n \in \mathbb{Z}^+$ .

**b** The matrix  $\mathbf{M}$  represents a rotation through angle  $\theta$ , and so  $\mathbf{M}^n$  represents a rotation through angle  $n\theta$ .