Exercise 9A

1 a $\mathbf{r}=\begin{pmatrix}6\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\-1\end{pmatrix}$

b $\mathbf{r}=\begin{pmatrix}2\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$

c $\mathbf{r}=\begin{pmatrix}-7\\6\\2\end{pmatrix}+\lambda=\begin{pmatrix}3\\1\\2\end{pmatrix}$

d $\mathbf{r}=\begin{pmatrix}2\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\1\end{pmatrix}$

e $\mathbf{r}=\begin{pmatrix}6\\-11\\2\end{pmatrix}+\lambda\begin{pmatrix}0\\5\\-2\end{pmatrix}$

2 a i $\overrightarrow{PQ}=(5-3)\mathbf{i}+(3-(-4))\mathbf{j}+(-1-2)\mathbf{k}$
$=2\mathbf{i}+7\mathbf{j}-3\mathbf{k}$

ii Since the line passes through P, an equation of the line is
$\mathbf{r}=(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}+7\mathbf{j}-3\mathbf{k})$

b i $\overrightarrow{PQ}=(4-2)\mathbf{i}+(-2-1)\mathbf{j}+(1-(-3))\mathbf{k}$
$=2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$

ii $\mathbf{r}=(2\mathbf{i}+\mathbf{j}-3\mathbf{k})+\lambda(2\mathbf{i}-3\mathbf{j}+4\mathbf{k})$

c i $\overrightarrow{PQ}=(-2-1)\mathbf{i}+(-3-(-2))\mathbf{j}+(2-4)\mathbf{k}$
$=-3\mathbf{i}-\mathbf{j}-2\mathbf{k}$

ii $\mathbf{r}=(\mathbf{i}-2\mathbf{j}+4\mathbf{k})+\lambda(-3\mathbf{i}-\mathbf{j}-2\mathbf{k})$

d i $\overrightarrow{PQ}=\begin{pmatrix}-2\\3\\1\end{pmatrix}-\begin{pmatrix}3\\-1\\4\end{pmatrix}=\begin{pmatrix}-5\\4\\-3\end{pmatrix}$

ii $\mathbf{r}=\begin{pmatrix}3\\-1\\4\end{pmatrix}+\lambda\begin{pmatrix}-5\\4\\-3\end{pmatrix}$

e i $\overrightarrow{PQ}=\begin{pmatrix}-2\\2\\4\end{pmatrix}-\begin{pmatrix}4\\-2\\3\end{pmatrix}=\begin{pmatrix}-6\\4\\1\end{pmatrix}$

2 e ii $\mathbf{r}=\begin{pmatrix}4\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}-6\\4\\1\end{pmatrix}$

3 Vector $\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ is in the direction of the z-axis.

The point $(4,-3,8)$ has position vector $\begin{pmatrix}4\\-3\\8\end{pmatrix}$

The equation of the line is

$$\mathbf{r}=\begin{pmatrix}4\\-3\\8\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$$

4 a i $\mathbf{a}=\begin{pmatrix}2\\1\\9\end{pmatrix},\mathbf{b}=\begin{pmatrix}4\\-1\\8\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}4\\-1\\8\end{pmatrix}-\begin{pmatrix}2\\1\\9\end{pmatrix}=\begin{pmatrix}2\\-2\\-1\end{pmatrix}$

Equation is

$$\mathbf{r}=\begin{pmatrix}2\\1\\9\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

4 a ii $\mathbf{a}=\begin{pmatrix}-3\\5\\0\end{pmatrix}, \mathbf{b}=\begin{pmatrix}7\\2\\2\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}7\\2\\2\end{pmatrix}-\begin{pmatrix}-3\\5\\0\end{pmatrix}=\begin{pmatrix}10\\-3\\2\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}-3\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}10\\-3\\2\end{pmatrix}$

iii $\mathbf{a}=\begin{pmatrix}1\\11\\-4\end{pmatrix}, \mathbf{b}=\begin{pmatrix}5\\9\\2\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}5\\9\\2\end{pmatrix}-\begin{pmatrix}1\\11\\-4\end{pmatrix}=\begin{pmatrix}4\\-2\\6\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}1\\11\\-4\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\6\end{pmatrix}$

iv $\mathbf{a}=\begin{pmatrix}-2\\-3\\-7\end{pmatrix}, \mathbf{b}=\begin{pmatrix}12\\4\\-3\end{pmatrix}$

$\mathbf{b}-\mathbf{a}=\begin{pmatrix}12\\4\\-3\end{pmatrix}-\begin{pmatrix}-2\\-3\\-7\end{pmatrix}=\begin{pmatrix}14\\7\\4\end{pmatrix}$

Equation is

$\mathbf{r}=\begin{pmatrix}-2\\-3\\-7\end{pmatrix}+\lambda\begin{pmatrix}14\\7\\4\end{pmatrix}$

4 b i $\dfrac{x-2}{2}=\dfrac{y-1}{-2}=\dfrac{z-9}{-1}$

ii $\dfrac{x+3}{10}=\dfrac{y-5}{-3}=\dfrac{z}{2}$

4 b iii $\dfrac{x-1}{4}=\dfrac{y-11}{-2}=\dfrac{z+4}{6}$

iv $\dfrac{x+2}{14}=\dfrac{y+3}{7}=\dfrac{z+7}{4}$

5 a $\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-4\\-9\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $2+\lambda=1\Rightarrow\lambda=-1$

j component: $-3+(-1)(-4)=p\Rightarrow p=1$

k component: $1+(-1)(-9)=q\Rightarrow q=10$

b $\begin{pmatrix}-4\\6\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-8\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $-4+2\lambda=1\Rightarrow\lambda=\dfrac{5}{2}$

j component: $6+\left(\dfrac{5}{2}\right)(-5)=p\Rightarrow p=-\dfrac{13}{2}$

$=-6\dfrac{1}{2}$

k component: $-1+\left(\dfrac{5}{2}\right)(-8)=q\Rightarrow q=-21$

c $\begin{pmatrix}16\\-9\\-10\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$

i component: $16+3\lambda=1\Rightarrow\lambda=-5$

j component: $-9+(-5)(2)=p\Rightarrow p=-19$

k component: $-10+(-5)(1)=q\Rightarrow q=-15$

6 Direction of l_1: $\begin{pmatrix}-1\\2\\4\end{pmatrix}$, direction of l_2: $\begin{pmatrix}2\\-4\\-8\end{pmatrix}$

$=-2\begin{pmatrix}-1\\2\\4\end{pmatrix}$, so parallel

7 $l_1 : \mathbf{r} = \begin{pmatrix} 3+2\lambda \\ 2-3\lambda \\ -1+4\lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

So the direction vector of l_1 is $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

l_2 has direction vector

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

Both direction vectors are the same, so line l_1 is parallel to l_2

8 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ -6 \end{pmatrix}$$

So $\overrightarrow{AC} = 2\overrightarrow{AB}$
So AC is parallel to AB and there is a common point A in both.
Therefore A, B and C are collinear.

9 $\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} 10 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 2 \end{pmatrix}$$

So not collinear

10 Since P,Q and R are collinear, the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel.

$$\overrightarrow{PQ} = \begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} a-2 \\ 5 \\ -3 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 3 \\ 10 \\ b \end{pmatrix} - \begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-a \\ 5 \\ b-1 \end{pmatrix}$$

$\therefore\ a-2 = 3-a \Rightarrow a = \frac{5}{2}$

$\therefore\ -3 = b-1 \Rightarrow b = -2$

So not collinear

11 Let $\mathbf{a}$ denote the position vector of A

$\therefore\ \mathbf{a} = (8\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + (-2)(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$

$= 2\mathbf{i} - 7\mathbf{j} + 16\mathbf{k}$

A line parallel to l_2 has the same direction vector: $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

So the equation of a line through A and parallel to l_2 is

$\mathbf{r} = (2\mathbf{i} - 7\mathbf{j} + 16\mathbf{k}) + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

12 a $(10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + b\mathbf{k}) = 4\mathbf{i} + a\mathbf{j}$

$\mathbf{i}$ component: $10 + \lambda = 4 \Rightarrow \lambda = -6$

$\mathbf{j}$ component: $8 + (-6)(-1) = a \Rightarrow a = 14$

$\mathbf{k}$ component: $-12 + (-6)(b) = 0 \Rightarrow b = -2$

b Substituting $\lambda = -1$:

$\mathbf{r} = (10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + (-1)(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= 9\mathbf{i} + 9\mathbf{j} - 10\mathbf{k}$

Therefore, X has coordinates $(9, 9, -10)$

13 Find the coordinates of A and B:

$$\lambda = 5:\ \mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix}$$

$$\lambda = 2:\ \mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$$

$$\therefore\ \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 6 \end{pmatrix}$$

$$\therefore\ |\overrightarrow{AB}| = \sqrt{(-3)^2 + (-6)^2 + (6)^2} = 9$$

14 Find the coordinates of C and A:

$\lambda = 4:\ \mathbf{c} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + 4\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}9\\2\\-1\end{pmatrix}$

$\lambda = 3:\ \mathbf{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + 3\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}7\\1\\0\end{pmatrix}$

$\therefore\ \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix}9\\2\\-1\end{pmatrix} - \begin{pmatrix}7\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\therefore\ \mathbf{b} = \mathbf{c} + \overrightarrow{AC} = \begin{pmatrix}9\\2\\-1\end{pmatrix} + \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}11\\3\\-2\end{pmatrix}$

So B has position vector $\begin{pmatrix}11\\3\\-2\end{pmatrix}$

15 A vector equation for l is

$\mathbf{r} = \begin{pmatrix}5\\-1\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix}$

Let $\mathbf{c} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$

$\mathbf{c} - \mathbf{r} = \begin{pmatrix}4\\-1\\2\end{pmatrix} - \begin{pmatrix}5\\-1\\6\end{pmatrix} - \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}-1-\lambda\\-3\lambda\\-4+2\lambda\end{pmatrix}$

Since it is given that C intersects l at two distinct points, the equation $|\mathbf{c} - \mathbf{r}| = 3\sqrt{5}$ has two solutions.

$\therefore \sqrt{(-1-\lambda)^2 + (-3\lambda)^2 + (-4+2\lambda)^2} = 3\sqrt{5}$

$\Rightarrow \sqrt{14\lambda^2 - 14\lambda + 17} = 3\sqrt{5}$

$\Rightarrow 14\lambda^2 - 14\lambda + 17 = 45$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda = -1$ or $\lambda = 2$

Therefore, substituting these values of λ into the vector equation, the coordinates are $(4, -4, 8)$ and $(7, 5, 2)$

16 a $\lambda = 2:\ \mathbf{r} = \begin{pmatrix}-4\\6\\5\end{pmatrix} + 2\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\4\\7\end{pmatrix}$

$\lambda = 5:\ \mathbf{r} = \begin{pmatrix}-4\\6\\5\end{pmatrix} + 5\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\10\end{pmatrix}$

$\therefore\ A\begin{pmatrix}-2\\4\\7\end{pmatrix}$ and $B\begin{pmatrix}1\\1\\10\end{pmatrix}$

b Since l_2 is parallel to l_1, they have the same direction vector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$. Therefore a vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}0\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$

c $|\overrightarrow{AB}| = \sqrt{(1-(-2))^2 + (1-4)^2 + (10-7)^2}$

$= \sqrt{27}$

$\therefore\ |\overrightarrow{AC}| = |\overrightarrow{AD}| = \sqrt{27}$

$\begin{pmatrix}0\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}-2\\4\\7\end{pmatrix} = \begin{pmatrix}2+\lambda\\-2-\lambda\\-4+\lambda\end{pmatrix}$

$\therefore \sqrt{(2+\lambda)^2 + (-2-\lambda)^2 + (-4+\lambda)^2} = \sqrt{27}$

$\Rightarrow \sqrt{3\lambda^2 + 24} = \sqrt{27}$

$\Rightarrow 3\lambda^2 + 24 = 27$

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\therefore\ C\begin{pmatrix}1\\1\\4\end{pmatrix}$ and $D\begin{pmatrix}-1\\3\\2\end{pmatrix}$

$\Rightarrow \overrightarrow{CD} = \begin{pmatrix}-2\\2\\-2\end{pmatrix}$

$\begin{pmatrix}0\\2\\3\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-2\\2\\-2\end{pmatrix} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$

i.e. P is the midpoint of CD

17 a A vector equation for the tightrope is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 20 \\ 15 \\ 0 \end{pmatrix}$$

Let $B(14,1,0)$ and $C(6,17,0)$

$$\therefore \overrightarrow{BA} = \begin{pmatrix} -12+20\lambda \\ 2+15\lambda \\ 8 \end{pmatrix}, \overrightarrow{CA} = \begin{pmatrix} -4+20\lambda \\ -14+15\lambda \\ 8 \end{pmatrix}$$

$$|\overrightarrow{BA}| = \sqrt{(20\lambda-12)^2+(15\lambda+2)^2+8^2} = 12$$

$$\Rightarrow 625\lambda^2 - 420\lambda + 68 = 0$$

$$\Rightarrow \lambda = \frac{2}{5} \quad \text{or} \quad \lambda = \frac{34}{125}$$

$$|\overrightarrow{CA}| = \sqrt{(20\lambda-4)^2+(15\lambda-14)^2+8^2} = 12$$

$$\Rightarrow 625\lambda^2 - 580\lambda + 132 = 0$$

$$\Rightarrow \lambda = \frac{2}{5} \quad \text{or} \quad \lambda = \frac{66}{125}$$

$$\therefore \lambda = \frac{2}{5}$$

Therefore the position vector of A is

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 20 \\ 15 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 8 \end{pmatrix}$$

$\therefore A(10,9,8)$

b Tightrope will bow in middle due to acrobat's weight.