Exercise 9B

1 a Let

$\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = (4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j})$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b Let

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \mathbf{b}$

$= -\mathbf{i} - 2\mathbf{j} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$

and

$\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

$= -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\therefore \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda'(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c Let

$\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{b}$

$= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k})$

$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

and $\mathbf{c} = 4\mathbf{i} + \mathbf{k} - (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\therefore \mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Let

$\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{b}$

$= -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{j} + 2\mathbf{k}$

and

$\mathbf{c} = 4\mathbf{j} + 4\mathbf{k} - (-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\therefore \mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

2 The plane has normal vector $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ so a cartesian equation of the plane takes takes the form

$-x + 3y + 2z = d$

The plane contains the point $(4, -2, 6)$ so

$-(4) + 3(-2) + 2(6) = d \Rightarrow d = 2$

$\therefore\ -x + 3y + 2z = 2$

3 a $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ k \end{pmatrix}$

i component: $2 + 3\lambda + \mu$

$= 7 \Rightarrow 3\lambda + \mu = 5$

j component: $-1 + 2\lambda - \mu$

$= -1 \Rightarrow 2\lambda - \mu = 0$

Solving these simultaneous equations:

$\lambda = 1,\ \mu = 2$

k component: $3 + 1(-2) + 2(3)$

$= k \Rightarrow k = 7$

b $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ k \\ 11 \end{pmatrix}$

i component: $2 + 3\lambda + \mu = 1$

$\Rightarrow 3\lambda + \mu = -1$

k component: $3 - 2\lambda + 3\mu = 11$

$\Rightarrow -2\lambda + 3\mu = 8$

Solving these simultaneous equations:

$\lambda = -1,\ \mu = 2$

j component: $-1 - 2 + 2(-1) = k \Rightarrow k = -5$

c $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} k \\ -4 \\ 10 \end{pmatrix}$

j component: $-1 + 2\lambda - \mu = -4$

$\Rightarrow 2\lambda - \mu = -3$

k component: $3 - 2\lambda + 3\mu = 10$

$\Rightarrow -2\lambda + 3\mu = 7$

Solving these simultaneous equations:

$\lambda = -\frac{1}{2},\ \mu = 2$

i component: $2 - \frac{1}{2}(3) + 2(1) = k \Rightarrow k = \frac{5}{2}$

3 d $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ k \\ -k \end{pmatrix}$

j component: $-1 + 2\lambda - \mu = k$

$\Rightarrow 2\lambda - \mu = k + 1$

k component: $3 - 2\lambda + 3\mu = -k$

$\Rightarrow -2\lambda + 3\mu = -k - 3$

Adding these two equations together:

$2\mu = -2 \Rightarrow \mu = -1$

i component: $2+3\lambda+(-1) = 10$

$\Rightarrow \lambda = 3$

j component: $-1 + 3(2) + (-1)(-1) = k$

$\Rightarrow k = 6$

4 a i $2(1) - 3(2) + 5(1) = 1$

ii $2(2) - 3(-4) + 5(-3) = 1$

b $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

5 a l is normal to the plane and passes through $(2,3,-2)$ so an equation for l is

$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$

$\Rightarrow \frac{x-2}{5} = \frac{y-3}{-3} = \frac{z+2}{-4}$

6 $x = 0,\ 0 \le y, z \le 3$

$x = 3,\ 0 \le y, z \le 3$

$y = 0,\ 0 \le x, z \le 3$

$y = 3,\ 0 \le x, z \le 3$

$z = 0,\ 0 \le x, y \le 3$

$z = 3,\ 0 \le x, y \le 3$

7 Find the plane containing $A(2,2,3)$ $B(1,5,3)$ and $C(4,3,-1)$, then show that $D(3,6,-1)$ lies in this plane.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

So a vector equation of the plane is

$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

If D also lies in this plane then there exist λ, μ such that

$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -1 \end{pmatrix}$

k component: $3 - 4\mu = -1 \Rightarrow \mu = 1$

j component: $2 + 3\lambda + 1(1) = 6 \Rightarrow \lambda = 1$

Check **i** component: $2 + 1(-1) + 1(2) = 3$

Therefore the points are coplanar.

8 Find the plane containing $A(2,3,4)$ $B(2,-1,3)$ and $C(5,3,-2)$, then show that $D(-1,-9,8)$ lies in this plane.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

So a vector equation of the plane is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$$

If D also lies in this plane then there exist λ, μ such that

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 8 \end{pmatrix}$$

i component: $2 + 3\mu = -1 \Rightarrow \mu = -1$

j component: $3 - 4\lambda = -9 \Rightarrow \lambda = 3$

Check **k** component: $4 + 3(-1) - (-6) = 7 \neq 8$

So the equations are not consistent.

Therefore the points are not coplanar.

9 a $$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix}$$

b $$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ -1 \end{pmatrix}$$

i component: $3 - 2\lambda + 4\mu = 1 \Rightarrow \lambda - 2\mu = 1$

j component: $-2 + 3\lambda + 2\mu = -7 \Rightarrow 3\lambda + 2\mu = -5$

k component: $1 + 5\lambda - 3\mu = -1 \Rightarrow 5\lambda - 3\mu = -2$

$\lambda = \mu = -1$ satisfies all of the equations, so $B(1,-7,-1)$ lies on the plane.

c $A(9,5,0)$ and $B(1,-7,-1)$

So an equation for the line through A and B is $\mathbf{r} = \begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix}$

d $|\overrightarrow{OA}| = |\overrightarrow{OC}|$

$|\overrightarrow{OA}| = \sqrt{9^2 + 5^2} = \sqrt{106}$

$\therefore |\overrightarrow{OC}| = \sqrt{(9-8\lambda)^2 + (5-12\lambda)^2 + \lambda^2} = \sqrt{106}$

$\Rightarrow 209\lambda^2 - 264\lambda + 106 = 106$

$\Rightarrow \lambda(19\lambda - 24) = 0$

$\lambda \neq 0$ (otherwise that would give A) so $\lambda = \dfrac{24}{19}$

Therefore C has position vector

$$\begin{pmatrix} 9 \\ 5 \\ 0 \end{pmatrix} + \frac{24}{19} \begin{pmatrix} -8 \\ -12 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{21}{19} \\ -\frac{193}{19} \\ -\frac{24}{19} \end{pmatrix}$$

Challenge

A: $2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ lies on plane; $\lambda = -2, \mu = 1$

$\overrightarrow{AB} = 5\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}$

B: $7\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ lies on plane; $\lambda = 1, \mu = 2$

So line lies entirely within plane.