Exercise 9E

1 a The line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

Use column vector form for clarity. Put the two equations equal and compare x, y and z components. Then solve simultaneous equations.

and the line l_1 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

These lines meet when

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

i.e. $1 + \lambda = -1 + \mu$ **(1)**

$3 - \lambda = -3 + \mu$ **(2)**

$5\lambda = 2 + 2\mu$ **(3)**

Add equations **(1)** and **(2)**

$4 = -4 + 2\mu$

$\therefore 2\mu = 8$

i.e: $\mu = 4$

Substitute into equation **(1)**

$\therefore 1 + \lambda = -1 + 4$

i.e: $\lambda = 2$

Substitute $\lambda = 2$ into equation for line l_1

$\therefore (x, y, z) = (3, 1, 10)$

Substitute $\mu = 4$ into equation for line l_2

$\therefore (x, y, z) = (3, 1, 10)$

So the two lines do meet at the point (3, 1, 10)

1 b l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

These lines meet when $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

i.e. $3+\lambda = 4-\mu$ **(1)**

$2+\lambda = 3+\mu$ **(2)**

$1+2\lambda = -\mu$ **(3)**

Add equations **(1)** and **(2)**

$\therefore 5+2\lambda = 7$

i.e. $\lambda = 1$

Substitute into equation **(1)**

$\therefore 3+1 = 4-\mu$

i.e: $\mu = 0$

Substitute $\lambda = 1$ into equation for line l_1:

$\therefore (x, y, z) = (4, 3, 3)$

Substitute $\mu = 0$ into line l_2:

$\therefore (x, y, z) = (4, 3, 0)$

This is a contradiction and the lines do not meet.

[N.B. $\lambda = 1$ and $\mu = 0$ do not satisfy equation **(3)** above.]

1 c l_1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and

l_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

l_1 meets l_2 when $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

i.e. $1 + 2\lambda = 1 + \mu$ **(1)**

$3 + 3\lambda = 2\frac{1}{2} + \mu$ **(2)**

$5 + \lambda = 2\frac{1}{2} - 2\mu$ **(3)**

Substitute equation **(1)** from equation **(2)**

$\therefore 2 + \lambda = 1\frac{1}{2}$

i.e. $\lambda = -\frac{1}{2}$

Substitute into equation **(1)**

$\therefore 1 - 1 = 1 + \mu$

i.e. $\mu = -1$

Substitute $\lambda = -\frac{1}{2}$ into equation for line l_1:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

Substitute $\mu = -1$ into equation for line l_2:

$\therefore (x, y, z) = \left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

So the two lines do meet at the point $\left(0, 1\frac{1}{2}, 4\frac{1}{2}\right)$

2 $$\begin{pmatrix} -6+\lambda \\ -\lambda \\ 11+\lambda \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ -2+\mu \\ 9-3\mu \end{pmatrix}$$

$$\therefore \begin{pmatrix} \lambda-2\mu \\ -\lambda-\mu \\ \lambda+3\mu \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -2 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = 4,\ \mu = -2$

Check **k** component: $4+3(-2) = -2$

So the equations are consistent.

Therefore l_1 and l_2 meet.

Substituting the value of λ into l_1,

$$\begin{pmatrix} -6 \\ 0 \\ 11 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 15 \end{pmatrix}$$

3 $$\begin{pmatrix} 3+2\lambda \\ 1+2\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} 5+2\mu \\ 4+\mu \\ -\mu \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2\lambda-2\mu \\ 2\lambda-\mu \\ 3\lambda+\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = 2,\ \mu = 1$

Check **k** component: $3(2)+1 = 7 \neq 2$

So the equations are inconsistent. Therefore l_1 and l_2 do not intersect.

4 a The line meets the plane when

$$[(1-2\lambda)\mathbf{i}+(1+\lambda)\mathbf{j}+(1-4\lambda)\mathbf{k}]\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=16$$
$$\text{i.e. } 3(1-2\lambda)-4(1+\lambda)+2(1-4\lambda)=16$$
$$\therefore\ 3-6\lambda-4-4\lambda+2-8\lambda=16$$
$$\therefore 1-18\lambda=16$$
$$\text{i.e. } -18\lambda=15$$
$$\therefore \lambda=-\frac{15}{18}$$
$$\text{i.e. } \lambda=\frac{-5}{6}$$

Assume that the line meets the plane and perform the scalar product. Solve the resulting equation to find the value of λ. If there is no value for λ, then the line does not meet the plane.

Substitute into the equation of the line

$$\therefore (x,y,z)=\left(1+\frac{10}{6},1-\frac{5}{6},1+\frac{20}{6}\right)$$
$$=\left(2\frac{2}{3},\frac{1}{6},4\frac{1}{3}\right)$$

b The line meets the plane when

$$[\mathbf{i}+(1+2\lambda)\mathbf{j}+(1-2\lambda)\mathbf{k}]\cdot(3\mathbf{i}-\mathbf{j}-6\mathbf{k})=1$$
$$\text{i.e. } 3-(1+2\lambda)-6(1-2\lambda)=1$$
$$\text{i.e. } 3-1-2\lambda-6+12\lambda=1$$
$$\therefore 10\lambda-4=1$$
$$\therefore \lambda=\frac{1}{2}$$

Substitute into the equation of the line

$$\therefore (x,y,z)=(1,\ 1+1,\ 1-1)$$
$$=(1,\ 2,\ 0)$$

5 a If l meets the plane, there exists a λ such that

$$\left(\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=1$$

$$\Rightarrow \begin{pmatrix}2\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=1$$

$$\Rightarrow 2+3+4+\lambda(1+1-2)=1$$

$\Rightarrow 9=1$. This is a contradiction.

Therefore the line does not intersect the plane.

b The line is parallel to the plane and not contained within the plane.

6 a Since the lines are perpendicular, the scalar product of their direction vectors is 0.

$$\therefore \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ p \\ p \end{pmatrix} = -3 - p + 2p = 0 \Rightarrow p = 3$$

b $\begin{pmatrix} 5+3\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix} = \begin{pmatrix} -\mu \\ 11+3\mu \\ 3+3\mu \end{pmatrix}$

$$\therefore \begin{pmatrix} 3\lambda+\mu \\ -\lambda-3\mu \\ 2\lambda-3\mu \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = -1, \ \mu = -2$

Check **k** component: $2(-1) - 3(-2) = 4$

So the equations are consistent

Therefore the lines intersect

Subsituting $\lambda = -1$ into the first line, the point of intersection is

$$\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \text{ so } (2, 5, -3)$$

7 a $\begin{pmatrix} 5-\lambda \\ 2+\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} 4+\mu \\ 1 \\ 1-\mu \end{pmatrix}$

$$\therefore \begin{pmatrix} -\lambda-\mu \\ \lambda \\ 2\lambda+\mu \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

i and **j** components $\Rightarrow \lambda = -1, \ \mu = 2$

Substituting $\lambda = -1$ into l_1,

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$

b Let $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1 - 2 = -3$$

$$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$$

$$|\mathbf{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\therefore \cos\theta = \left| \frac{-3}{\sqrt{6}\sqrt{2}} \right| = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

8 $\begin{pmatrix} -3\lambda \\ -1+5\lambda \\ 2+4\lambda \end{pmatrix} = \begin{pmatrix} \mu \\ 1-2\mu \\ -5+2\mu \end{pmatrix}$

$-3\lambda = \mu$ (1)
$-1+5\lambda = 1-2\mu$ (2)

$(2) + 2\times(1)$ gives $-1+5\lambda-6\lambda = 1-2\mu+2\mu$

$-1-\lambda = 1$
So $\lambda = -2$

Substituting in (1) gives $\mu = 6$

Check for consistency: $2+4\lambda = -6$ and $-5+2\mu = 7$
$2+4\lambda \neq -5+2\mu$, so equations are not consistent.

The direction of l_1 is $\begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$ and the direction of l_2 is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

The direction vectors are not scalar multiples of each other, so l_1 and l_2 are not parallel.
Hence l_1 and l_2 are skew.

9 a Since l_1 and l_2 are perpendicular, the scalar product of their direction vectors is 0

$\therefore \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ -1 \end{pmatrix} = -q+6-2 = 0 \Rightarrow q = 4$

b Since l_1 and l_2 intersect, there exist λ and μ such that

$\begin{pmatrix} 8-\lambda \\ 2+3\lambda \\ -12+2\lambda \end{pmatrix} = \begin{pmatrix} -4+4\mu \\ 10+2\mu \\ p-\mu \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -\lambda-4\mu \\ 3\lambda-2\mu \\ 2\lambda+\mu \end{pmatrix} = \begin{pmatrix} -12 \\ 8 \\ p+12 \end{pmatrix}$

i and **j** components $\Rightarrow$ $\lambda = 4$, $\mu = 2$

k component: $2(4)+2 = p+12 \Rightarrow p = -2$

c Substituting $\lambda = 4$ into l_1,

$\begin{pmatrix} 8 \\ 2 \\ -12 \end{pmatrix} + 4\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix}$

So $(4, 14, -4)$

9 d Let the intersection point be P.

$\therefore \overrightarrow{OB} = \overrightarrow{OP} - \overrightarrow{PA}$

$$\Rightarrow \overrightarrow{OB} = \begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -15 \\ -10 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \\ 6 \end{pmatrix}$$

10 a $k = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 12 - 6 - 4 = 2$

b Expanding the scalar product, $2x + 3y - z = 2$

10 c Find an equation for l: since it passes through $P(6, 4, 8)$ and is normal to the plane, an equation is

$$\mathbf{r} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$\therefore$ At N,

$$\left(\begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2$$

$$\Rightarrow \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2$$

$\Rightarrow 12 + 12 - 8 + \lambda(4 + 9 + 1) = 2$

$\Rightarrow \lambda = -1$

Substituting this into l,

$$\begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

$\therefore N(4, 1, 9)$

11 a $\dfrac{x-3}{5} = \dfrac{y+2}{3} = \dfrac{4-z}{1} = \lambda$

$x = 3 + 5\lambda,\ y = -2 + 3\lambda,\ z = 4 - \lambda$

$4(3 + 5\lambda) + 3(-2 + 3\lambda) - 2(4 - \lambda) = -10$

$\Rightarrow \lambda = -\dfrac{8}{31}$

$\therefore x = \dfrac{53}{31},\ y = -\dfrac{86}{31},\ z = \dfrac{132}{31}$

$\therefore P\left(\dfrac{53}{31}, -\dfrac{86}{31}, \dfrac{132}{31}\right)$

11 b $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

$$\mathbf{b} \cdot \mathbf{n} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 20 + 9 + 2 = 31$$

$$|\mathbf{b}| = \sqrt{5^2 + 3^2 + (-1)^2} = \sqrt{35}$$

$$|\mathbf{n}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

$$\therefore \sin\theta = \frac{31}{\sqrt{35}\sqrt{29}}$$

$$\Rightarrow \theta = 76.7^o \quad \text{(1d.p.)}$$