Complex numbers 1C

1 a $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

b $(\cos 3\theta + i\sin 3\theta)^4 = \cos 12\theta + i\sin 12\theta$

c

$$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

d

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8 = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

e

$$\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5 = \cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5} = \cos 2\pi + i\sin 2\pi$$
$$= 1$$

f

$$\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15} = \left(\cos\frac{-\pi}{10} + i\sin\frac{-\pi}{10}\right)^{15} = \cos(-\frac{3\pi}{2}) + i\sin(-\frac{3\pi}{2})$$
$$= i$$

2 a $$\frac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos\theta + i\sin\theta)^4} = \cos\theta + i\sin\theta = e^{i\theta}$$

b $$\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3} = \frac{(\cos\theta + i\sin\theta)^{14}}{(\cos\theta + i\sin\theta)^{12}} = (\cos\theta + i\sin\theta)^2 = e^{2i\theta}$$

c $$\frac{1}{(\cos 2\theta + i\sin 2\theta)^3} = \frac{1}{(\cos\theta + i\sin\theta)^6} = \frac{1}{e^{6i\theta}} = e^{-6i\theta}$$

d $$\frac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^9} = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{e^{i\theta}} = e^{-i\theta}$$

e $$\frac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos(-3\theta) + i\sin(-3\theta))^2} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos-\theta + i\sin-\theta)^6} = \frac{e^{5i\theta}}{e^{-6i\theta}} = e^{11i\theta}$$

2 f $$\frac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3} = \frac{\cos(-\theta)+i\sin(-\theta)}{(\cos(-2\theta)+i\sin(-2\theta))^3} = \frac{e^{-i\theta}}{(\cos-\theta+i\sin-\theta)^6} = \frac{e^{-i\theta}}{e^{-6i\theta}} = e^{5i\theta}$$

3 a

$$\frac{\left(\cos\frac{7\pi}{13}+i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13}-i\sin\frac{4\pi}{13}\right)^6} = \frac{\cos\frac{28\pi}{13}+i\sin\frac{28\pi}{13}}{\cos\frac{-24\pi}{13}+i\sin\frac{-24\pi}{13}} = \frac{e^{\frac{28\pi i}{13}}}{e^{\frac{-24\pi i}{13}}} = e^{\frac{52\pi i}{13}}$$

$$= e^{4\pi i} = 1$$

b Recall the identity $\sin(\pi + x) = -\sin(\pi - x)$ then

$$\frac{\left(\cos\frac{3\pi}{7}-i\sin\frac{11\pi}{7}\right)^3}{\left(\cos\frac{15\pi}{7}+i\sin\frac{\pi}{7}\right)^2} = \frac{\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)^2} = \frac{\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}}{\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}} = \frac{e^{\frac{9\pi i}{7}}}{e^{\frac{2\pi i}{7}}} = e^{\frac{7\pi i}{7}}$$

$$= e^{\pi i} = -1$$

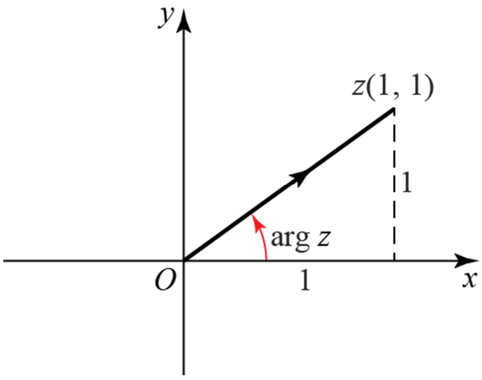
c

$$\frac{\left(\cos\frac{4\pi}{3}-i\sin\frac{2\pi}{3}\right)^7}{\left(\cos\frac{10\pi}{3}+i\sin\frac{4\pi}{3}\right)^4} = \frac{\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)^7}{\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)^4} = \frac{\cos\frac{28\pi}{3}+i\sin\frac{28\pi}{3}}{\cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}} = \frac{e^{\frac{28\pi i}{3}}}{e^{\frac{16\pi i}{3}}} = e^{\frac{12\pi i}{3}}$$

$$= e^{4\pi i} = 1$$

4 a $(1+\text{i})^5$

If $z = 1+\text{i}$, then



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

So, $1+\text{i} = \sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$$\therefore (1+\text{i})^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)\right]^5$$

$$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + \text{i}\sin\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + \text{i}\sin\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\text{i}\right)$$

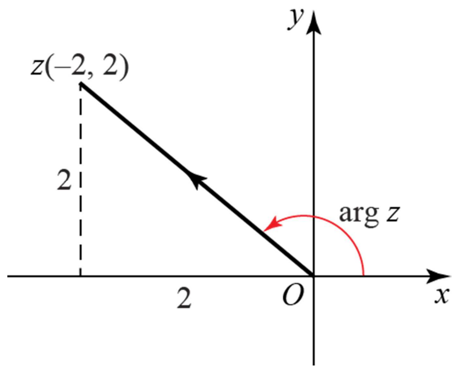
$$= -4 - 4\text{i}$$

Therefore, $(1+\text{i})^5 = -4-4\text{i}$

$(\sqrt{2}) = \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}$
$= 4\sqrt{2}$

4 b $(-2+2\text{i})^8$

If $z=-2+\text{i}$, then



$r=\sqrt{(-2)^2+2^2}=\sqrt{4+4}=\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$

$\theta=\arg z=\pi-\tan^{-1}\left(\frac{2}{2}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

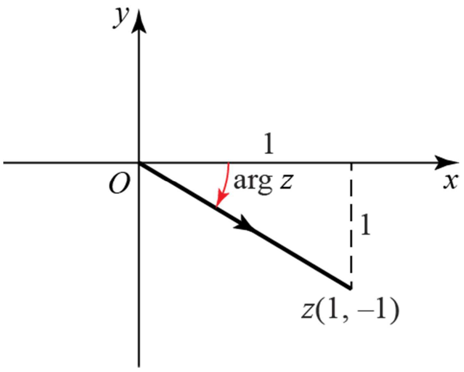
So, $-2+2\text{i}=2\sqrt{2}\left(\cos\frac{3\pi}{4}+\text{i}\sin\frac{3\pi}{4}\right)$

$$\begin{aligned}\therefore\ (-2+2\text{i})^8&=\left[2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+\text{i}\sin\left(\frac{3\pi}{4}\right)\right)\right]^8\\&=(2\sqrt{2})^8\left(\cos\left(\frac{24\pi}{4}\right)+\text{i}\sin\left(\frac{24\pi}{4}\right)\right)\\&=(256)(16)\ (\cos 6\pi+\text{i}\sin 6\pi)\\&=4096(1+\text{i}(0))\\&=4096\end{aligned}$$

Therefore, $(-2+2\text{i})^8=4096$

4 c $(1-\text{i})^6$

If $z = 1-\text{i}$, then



$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$$

So, $1-\text{i} = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\text{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$$\therefore\ (1-\text{i})^6 = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\text{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^6$$

$$= (\sqrt{2})^6\left(\cos\left(-\frac{6\pi}{4}\right)+\text{i}\sin\left(-\frac{6\pi}{4}\right)\right)$$

$$= 8\left(\cos\left(-\frac{3\pi}{2}\right)+\text{i}\sin\left(-\frac{3\pi}{2}\right)\right)$$

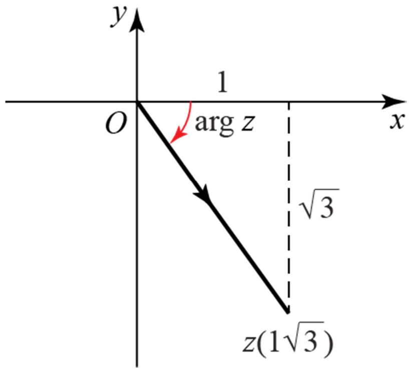
$$= 8(0+\text{i})$$

$$= 8\text{i}$$

Therefore, $(1-\text{i})^6 = 8\text{i}$

4 d $(1-\sqrt{3}\,\mathrm{i})^6$

If $z = 1-\sqrt{3}\,\mathrm{i}$, then



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

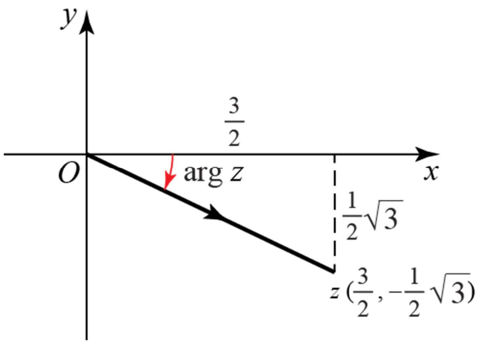
$$\text{So, } 1-\sqrt{3}\,\mathrm{i} = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$\begin{aligned}\therefore\ (1-\sqrt{3}\,\mathrm{i})^6 &= \left[2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\right]^6 \\ &= (2)^6\left(\cos\left(-\frac{6\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{3}\right)\right) \\ &= 64(\cos(-2\pi) + \mathrm{i}\sin(-2\pi)) \\ &= 64(1+\mathrm{i}(0)) \\ &= 64\end{aligned}$$

Therefore, $(1-\sqrt{3}\,\mathrm{i})^6 = 64$

4 e $\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9$

If $z=\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$, then



$$r=\sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{1}{2}\sqrt{3}\right)^2}=\sqrt{\frac{9}{4}+\frac{3}{4}}=\sqrt{\frac{12}{4}}=\sqrt{3}$$

$$\theta=\arg z=-\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right)=-\tan^{-1}\frac{\sqrt{3}}{3}=-\frac{\pi}{6}$$

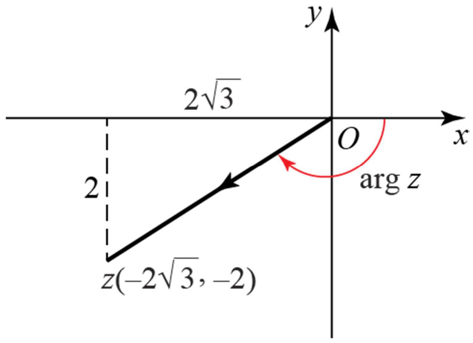
So, $\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}=\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$$\begin{aligned}
\therefore\ \left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9 &= \left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right]^9\\
&=(\sqrt{3})^9\left(\cos\left(-\frac{9\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{9\pi}{6}\right)\right)\\
&=81\sqrt{3}\left(\cos\left(-\frac{3\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right)\\
&=81\sqrt{3}(0+\mathrm{i})\\
&=81\sqrt{3}\,\mathrm{i}
\end{aligned}$$

Therefore, $\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9=81\sqrt{3}\,\mathrm{i}$

4 f $(-2\sqrt{3}-2\mathrm{i})^5$

If $z=-2\sqrt{3}-2\mathrm{i}$, then



$$r=\sqrt{(-2\sqrt{3})^2+(-2)^2}=\sqrt{12+4}=\sqrt{16}=4$$

$$\theta=\arg z=-\pi+\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=-\pi+\frac{\pi}{6}=-\frac{5\pi}{6}$$

$$\text{So, } -2\sqrt{3}-2\mathrm{i}=4\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$\therefore\ (-2\sqrt{3}-2\mathrm{i})^5=\left[4\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5$$

$$=4^5\left(\cos\left(-\frac{25\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{25\pi}{6}\right)\right)$$

$$=1024\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$$

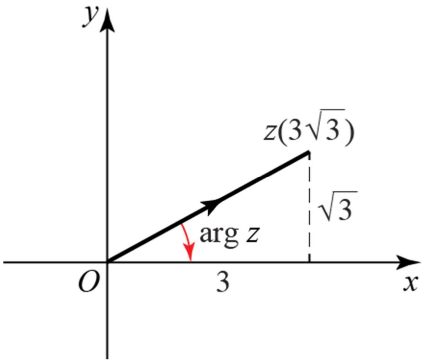
$$=1024\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)$$

$$=512\sqrt{3}-512\mathrm{i}$$

Therefore, $(-2\sqrt{3}-2\mathrm{i})^5=512\sqrt{3}-512\mathrm{i}$

5 $(3+\sqrt{3}\text{i})^5$

If $z = 3+\sqrt{3}\,\text{i}$, then



$$r = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

So, $3+\sqrt{3}\,\text{i} = 2\sqrt{3}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$

$$\therefore\ (3+\sqrt{3}\,\text{i})^5 = \left[2\sqrt{3}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)\right]^5$$

$$= (2\sqrt{3})^5\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$$

$$= 32(9\sqrt{3})\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)$$

$$= 288\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)$$

$$= -144\sqrt{3}\sqrt{3} + 144\sqrt{3}\,\text{i}$$

$$= -432 + 144\sqrt{3}\,\text{i}$$

Therefore, $(3+\sqrt{3}\,\text{i})^5 = -432 + 144\sqrt{3}\,\text{i}$

6

$$w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$w^4 = 2^4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4 = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 16\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= -8 + 8i\sqrt{3}$$

7

$$z=\sqrt{3}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$$

$$z^6=\left(\sqrt{3}\right)^6\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)^6=27\left(\cos\frac{-18\pi}{4}+i\sin\frac{-18\pi}{4}\right)$$

$$=27\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right)=-27i$$

8 a Let $z=1+i\sqrt{3}$ and $w=1-i\sqrt{3}$ first of all note that $|z|=|w|=\sqrt{1+3}=2$ so it follows that we can write $z=2e^{i\theta}$ and $w=2e^{i\varphi}$ with $\tan\theta=\sqrt{3}$ and $\tan\varphi=-\sqrt{3}$ so that $\theta=\frac{\pi}{3}$ and $\varphi=-\frac{\pi}{3}$ then it follows that

$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}}=\frac{z}{w}=\frac{2e^{\frac{i\pi}{3}}}{2e^{-\frac{i\pi}{3}}}=e^{\frac{i2\pi}{3}}$$

b We have

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n=e^{\frac{i2n\pi}{3}}=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$$

This is real precisely when $\frac{2n\pi}{3}=k\pi$ for some integer k so the smallest value of n for which this is real is $n=\frac{3}{2}$ and we have $\cos\frac{2\pi}{3}\frac{3}{2}=-1$ so this is not admissible, the next smallest value for which this is real is $n=3$ in which case $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^3=\cos 2\pi=1$ which is positive hence this is the smallest value.

9 We start by noting that we can write

$$a+bi=r\left(\cos\theta+i\sin\theta\right)=re^{i\theta}$$

$$a-bi=r(\cos\theta-i\sin\theta)=re^{-i\theta}$$

Then it follows that

$$\left(a+bi\right)^n+\left(a-bi\right)^n=r^ne^{in\theta}+r^ne^{-in\theta}=r^n\left(e^{in\theta}+e^{-in\theta}\right)$$

$$=r^n\left(\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta\right)=2r^n\cos n\theta$$

which is real.

Challenge

Given $n \in \mathbb{Z}^+$, we have:

$$(r(\cos\theta + \text{isin}))^{-n} = \frac{1}{\left(r\left(\cos\theta + \text{i}\sin\right)\right)^n} = \frac{1}{r^n\left(\cos n\theta + \text{i}\sin n\theta\right)}$$

by de Moivre's theorem for positive integer exponents.

$$= \frac{1}{r^n\left(\cos n\theta + \text{i}\sin n\theta\right)} \times \frac{\cos n\theta - \text{i}\sin n\theta}{\cos n\theta - \text{i}\sin n\theta}$$

$$= \frac{\cos n\theta - \text{i}\sin n\theta}{r^n\left(\cos^2 n\theta - \text{i}^2\sin^2 n\theta\right)} = \frac{\cos n\theta - \text{i}\sin n\theta}{r^n\left(\cos^2 n\theta + \sin^2 n\theta\right)}$$

$$= r^{-n}\left(\cos n\theta - \text{i}\sin n\theta\right) = r^{-n}\left(\cos\left(-n\theta\right) + \text{i}\sin\left(-n\theta\right)\right)$$