Complex numbers – mixed exercise 1

1 a We have $e^{i\theta} = \cos\theta + i\sin\theta$ hence

$$\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos-\theta + i\sin-\theta\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos\theta - \sin\theta\right) = \cos\theta$$

Where we have used the fact that

$\cos\theta = \cos-\theta$

$\sin\theta = -\sin-\theta$

b We have

$$\cos A\cos B = \frac{1}{2}\left(e^{iA} + e^{-iA}\right) \times \frac{1}{2}\left(e^{iB} + e^{-iB}\right) = \frac{1}{4}\left(e^{i(A+B)} + e^{i(A-B)} + e^{i(B-A)} + e^{-i(A+B)}\right)$$

$$= \frac{1}{4}\left(\cos(A+B) + i\sin(A+B) + \cos(A-B) + i\sin(A-B) + \cos(B-A) + i\sin(B-A) + \cos(-(A+B)) + i\sin(-(A+B))\right)$$

$$= \frac{1}{4}\left(2\cos(A+B) + 2\cos(A-B)\right) = \frac{\cos(A+B) + \cos(A-B)}{2}$$

2 We want to prove by induction that if $z = r(\cos\theta + i\sin\theta)$ then for all $n \in \mathbb{N}$ $z^n = r^n(\cos n\theta + i\sin n\theta)$ clearly the base case $n = 1$ is true, now assume the statement is true for $n = k$ then we have

$$z^{k+1} = z \times z^k = r(\cos\theta + i\sin\theta) \times r^k(\cos k\theta + i\sin k\theta) = r^{k+1}(\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$$

$$= r^{k+1}\left(\cos\theta\cos k\theta - \sin k\theta\sin\theta + i\left(\cos\theta\sin k\theta + \cos k\theta\sin\theta\right)\right)$$

$$= r^{k+1}\left(\cos(k+1)\theta + i\sin(k+1)\theta\right)$$

Hence the statement is true for $n = k + 1$, this completes the induction.

3 $$\frac{(\cos 3x + \mathrm{i}\sin 3x)^2}{\cos x - \mathrm{i}\sin x}$$

$$= \frac{(\cos 3x + \mathrm{i}\sin 3x)^2}{\cos(-x) + \mathrm{i}\sin(-x)}$$

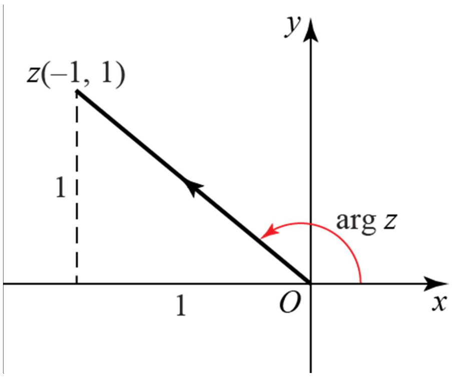
$$= \frac{\cos 6x + \mathrm{i}\sin 6x}{\cos(-x) + \mathrm{i}\sin(-x)}$$

$$= \cos(6x - -x) + \mathrm{i}\sin(6x - -x)$$

$$= \cos 7x + \mathrm{i}\sin 7x$$

4 a $(-1+\mathrm{i})^8$

If $z=-1+\mathrm{i}$, then



$r=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\theta=\arg z=\pi-\tan^{-1}\left(\frac{1}{1}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

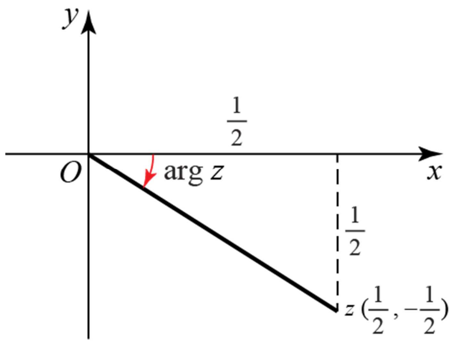
So, $-1+\mathrm{i}=\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$

$$\begin{aligned}\therefore(-1+\mathrm{i})^8&=\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)\right]^8\\&=(\sqrt{2})^8\left(\cos\frac{24\pi}{4}+\mathrm{i}\sin\frac{24\pi}{4}\right)\\&=16(\cos 6\pi+\mathrm{i}\sin 6\pi)\\&=16(1+\mathrm{i}(0))\end{aligned}$$

Therefore, $(-1+\mathrm{i})^8=16$

4 b $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=\left(\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}\right)^{-16}$

Let $z=\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}$, then



$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\theta=\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)=-\frac{\pi}{4}$$

So $\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}=\dfrac{1}{\sqrt{2}}\left[\cos\left(-\dfrac{\pi}{4}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right]$

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{-16}&=\left[\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-16}\\&=\left(2^{-\frac{1}{2}}\right)^{-16}\left(\cos\left(\frac{16\pi}{4}\right)+\mathrm{i}\sin\left(\frac{16\pi}{4}\right)\right)\\&=2^8(\cos 4\pi+\mathrm{i}\sin 4\pi)\\&=256(1+\mathrm{i}(0))\\&=256\end{aligned}$$

Therefore, $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=256$

5 a $z = \cos\theta + \mathrm{i}\sin\theta$

$z^n = (\cos\theta + \mathrm{i}\sin\theta)^{-n}$

$= \cos n\theta + \mathrm{i}\sin n\theta$

de Moivre's Theorem.

$\frac{1}{z^n} = z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^n$

de Moivre's Theorem.

$= \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$

$= \cos n\theta - \mathrm{i}\sin n\theta$

$\cos(-n\theta) = \cos n\theta$
$\sin(-n\theta) = -\sin n\theta$

Therefore $z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$

i.e. $z^n + \frac{1}{z^n} = 2\cos n\theta$ (as required)

b $\left(z^2 + \frac{1}{z^2}\right)^3 = (z^2)^3 + {}^3C_1(z^2)^2\left(\frac{1}{z^2}\right) + {}^3C_2(z^2)\left(\frac{1}{z^2}\right)^2 + \left(\frac{1}{z^2}\right)^3$

$= z^6 + 3z^4\left(\frac{1}{z^2}\right) + 3z^2\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^6}\right)$

$= z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) = 3\left(z^2 + \frac{1}{z^2}\right)$

$= 2\cos 6\theta + 3(2)\cos 2\theta$

$= 2\cos 6\theta + 6\cos 2\theta$

Hence, $\left(z^2 + \frac{1}{z^2}\right)^3 = 2\cos 6\theta + 6\cos 2\theta$

c $\left(z^2 + \frac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$

$\therefore \cos^3 2\theta = \frac{2}{8}\cos 6\theta + \frac{6}{8}\cos 2\theta$

Hence, $\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$

So $a = \frac{1}{4}, b = \frac{3}{4}$

5 d $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta \, d\theta$

$$= \left[\frac{1}{24}\sin 6\theta + \frac{3}{8}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{24}\sin \pi + \frac{3}{8}\sin\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{24}\sin 0 + \frac{3}{8}\sin 0\right)$$

$$= \left(\frac{1}{24}(0) + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right)\right) - (0)$$

$$= \frac{3}{16}\sqrt{3}$$

So, $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{3}{16}\sqrt{3}$ and $k = \frac{3}{16}$

6 a We wish to show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

Starting with $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ we have

$$\cos^5\theta = \frac{1}{32}(e^{i\theta} + e^{-i\theta})^5 = \frac{1}{32}(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta})$$

$$= \frac{1}{32}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta) = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

b The area of the region is given by the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x dx = 2\int_0^{\frac{\pi}{2}} \cos^5 x dx$$

Using the first part of the question, this is equivalent to

$$\int_0^{\frac{\pi}{2}} \frac{1}{8}(\cos 5x + 5\cos 3x + 10\cos x)dx$$

$$= \left[\frac{1}{8}\left(\frac{1}{5}\sin 5x + \frac{5}{3}\sin 3x + 10\sin x\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2}\right) = \frac{1}{8}\left(\frac{1}{5} - \frac{5}{3} + 10\right) = \frac{16}{15}$$

7 a We have $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ so that

$$\sin^6\theta = -\frac{1}{64}(e^{i\theta} - e^{-i\theta})^6$$

$$= -\frac{1}{64}(e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{2i\theta} - 6e^{4i\theta} + e^{6i\theta})$$

$$= -\frac{1}{64}(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20) = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

7 b Let $\alpha = \frac{\pi}{2} - \theta$ then recall that we have $\cos\alpha = \sin\theta$ hence we have

$$\cos^6\alpha = \sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

$$= -\frac{1}{32}\left(\cos 6\left(\frac{\pi}{2} - \alpha\right) - 6\cos 4\left(\frac{\pi}{2} - \alpha\right) + 15\cos 2\left(\frac{\pi}{2} - \alpha\right) - 10\right)$$

$$= -\frac{1}{32}(\cos(3\pi - 6\alpha) - 6\cos(2\pi - 4\alpha) + 15\cos(\pi - \alpha) - 10)$$

$$= -\frac{1}{32}(-\cos 6\alpha - 6\cos 4\alpha - 15\cos\alpha - 10) = \frac{1}{32}(\cos 6\alpha + 6\cos 4\alpha + 15\cos\alpha + 10)$$

c We have $\int_0^a \cos^6\theta + \sin^6\theta \, d\theta = \frac{5\pi}{32}$ using the previous two parts we can evaluate the integral as

$$\int_0^a \cos^6\theta + \sin^6\theta \, d\theta = \int_0^a \frac{1}{32}(12\cos 4\theta + 20)\, d\theta = \frac{1}{32}[3\sin 4\theta + 20\theta]_0^a$$

$$= \frac{1}{32}(3\sin 4a + 20a)$$

Hence we require $3\sin 4a + 20a = 5\pi$ and observe that $a = \frac{\pi}{4}$ solves this and this is the only Solution since the function $3\sin 4a + 20a$ is increasing.

8 Let $z = e^{i\theta}$ then we have

$$\sin 6\theta = \frac{1}{2i}(z^6 - z^{-6}) = \frac{1}{2i}\left((\cos\theta + i\sin\theta)^6 - (\cos\theta - i\sin\theta)^6\right)$$

When we expand the right hand side the only terms that survive are odd powers of sin hence we get

$$= \frac{1}{2i}(12i\cos^5\theta\sin\theta - 40i\cos^3\theta\sin^3 + 12i\cos\theta\sin^5\theta)$$

$$= 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3 + 6\cos\theta\sin^5\theta$$

$$= 2\sin\theta\cos\theta(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta)$$

$$= \sin 2\theta(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + 3(1 - \cos^2\theta)^2\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + 3(1 - 2\cos^2\theta + \cos^4\theta)\right)$$

$$= \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3)$$

9 **a** Let $z = e^{i\theta}$ then we have

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

Now note the only terms that survive are even powers of sin so this becomes

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

$$= \frac{1}{2i}\left(2i\cos^5\theta - 20i\cos^3\theta\sin^2\theta + 10i\cos\theta\sin^4\theta\right)$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta\right)$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\left(1-\cos^2\theta\right) + 5\left(1-\cos^2\theta\right)^2\right)$$

$$\cos\theta\left(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5\left(1 - 2\cos^2\theta + \cos^4\theta\right)\right)$$

$$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right)$$

b Now consider the equation

$16x^5 - 20x^3 + 5x + 1 = 0$

We make the substitution $x = \cos\theta$ so the equation becomes

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

i.e. $\cos 5\theta = -1$ for which the general solution is $5\theta = \pi + 2k\pi$ for $k \in \mathbb{Z}$ we take 5 different values of θ

$$\theta_1 = \frac{\pi}{5}$$

$$\theta_2 = \frac{3\pi}{5}$$

$$\theta_3 = \pi$$

$$\theta_4 = \frac{7\pi}{5}$$

$$\theta_5 = \frac{9\pi}{5}$$

Let $x_i = \cos\theta_i$ so

$$x_1 = \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} = 0.809$$

$$x_2 = \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4} = -0.309$$

$$x_3 = \cos\pi = -1$$

$$x_4 = \cos\frac{7\pi}{5} = \frac{1-\sqrt{5}}{4} = -0.309$$

$$x_5 = \cos\frac{9\pi}{5} = \frac{1+\sqrt{5}}{4} = 0.809$$

Now we only have 3 distinct solutions so we still have to check this is all solutions, factorising out $x+1$ gives

$16x^5 - 20x^3 + 5x + 1 = \left(x+1\right)\left(4x^2 - 2x - 1\right)^2 = 0$

9 b (Continued)

And observe that

$$\left(x-\left(\frac{1-\sqrt{5}}{4}\right)\right)\left(x-\left(\frac{1+\sqrt{5}}{4}\right)\right)=x^2-\frac{1}{2}x-\frac{1}{4}=\frac{1}{4}\left(4x^2-2x-1\right)$$

In other words the 3 distinct roots that we have obtained completely factorise the equation and so these three solutions are the only solutions.

Hence the three solutions are given by $-1, \frac{1+\sqrt{5}}{4}=0.809, \frac{1-\sqrt{5}}{4}=-0.309$

10 a Let $z=e^{i\theta}=\cos\theta+i\sin\theta$ then $\sin\theta=\frac{1}{2i}\left(z-z^{-1}\right)$ so we have

$$\sin^5\theta=\frac{1}{32i}\left(z-z^{-1}\right)^5=\frac{1}{32i}\left(z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5}\right)$$

$$=\frac{1}{32i}\left(2i\sin 5\theta-10i\sin 3\theta+20i\sin\theta\right)=\frac{1}{16}\left(\sin 5\theta-5\sin 3\theta+10\sin\theta\right)$$

b We wish to solve

$\sin 5\theta-5\sin 3\theta+9\sin\theta=0$

Which we can rearrange to give

$\sin 5\theta-5\sin 3\theta+10\sin\theta=\sin\theta$

So $16\sin^5\theta=\sin\theta$

So we either have $\sin\theta=0$ in which case $\theta=0$ or $16\sin^4\theta=1$ hence $\sin\theta=\pm\frac{1}{2}$ and on $0\le\theta<\pi$ the solutions correspond to $\theta=\frac{\pi}{6},\frac{5\pi}{6}$

11 a $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos 5\theta+\mathrm{i}\sin 5\theta$ — de Moivre's Theorem.

$$=\cos^5\theta+{}^5C_1\cos^4\theta(\mathrm{i}\sin\theta)+{}^5C_2\cos^3\theta(\mathrm{i}\sin\theta)^2$$
$$+{}^5C_3\cos^2\theta(\mathrm{i}\sin\theta)^3+{}^5C_4\cos^3\theta(\mathrm{i}\sin\theta)^4+\left(\mathrm{i}\sin\theta\right)^5$$

Binomial expansion.

$$=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta+10\mathrm{i}^2\cos^3\theta\sin^2\theta$$
$$+10\mathrm{i}^3\cos^2\theta\sin^3\theta+5\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$

Hence,

$$\cos 5\theta+\mathrm{i}\sin 5\theta=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin\theta$$
$$-10\mathrm{i}\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+\mathrm{i}\sin^5\theta$$

Equating the real parts gives,

$$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta\sin^2\theta+5\sin^4\theta)$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta(1-\cos^2\theta)+5(1-\cos^2\theta)^2)$$

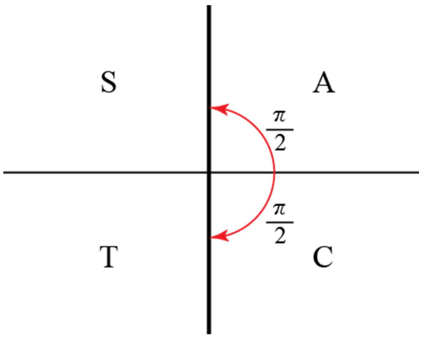
Applying $\sin^2\theta=1-\cos^2\theta$.

$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5(1-2\cos^2\theta+\cos^4\theta))$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5-10\cos^2\theta+5\cos^4\theta)$$
$$=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$$

Hence, $\cos 5\theta=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$ (as required)

11 b $\cos 5\theta = 0$

$\alpha = \dfrac{\pi}{2}$



So $5\theta = \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}\right\}$

$\theta = \left\{\dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{5\pi}{10}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}\right\}$

$\theta = \left\{\dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{5\pi}{10}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}\right\}$ for $0 < \theta \le \pi$

$\cos 5\theta = 0 \Rightarrow \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Five solutions must come from: $\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Solution **(1)** $\cos\theta = 0$

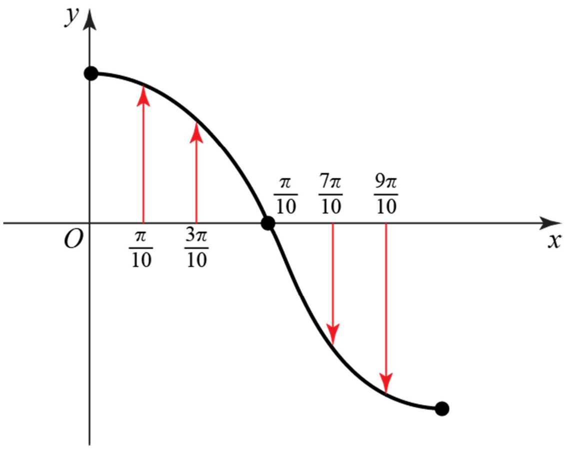
$\alpha = \dfrac{\pi}{2}$

For $0 < \theta \le \pi$, $\theta = \dfrac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$
$$= \frac{20 \pm \sqrt{400 - 320}}{32}$$
$$= \frac{20 \pm \sqrt{80}}{32}$$
$$= \frac{20 \pm \sqrt{16}\sqrt{5}}{32}$$
$$= \frac{20 \pm 4\sqrt{5}}{32}$$
$$\therefore \cos^2\theta = \frac{5 + \sqrt{5}}{8}$$

11 b (Continued)



Due to symmetry and as $\cos\left(\frac{\pi}{10}\right) > \cos\left(\frac{3\pi}{10}\right)$

$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$

$\therefore \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$

c $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$

$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$

$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$

Therefore, $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$, $\cos^2\left(\frac{7\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$, $\cos^2\left(\frac{9\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$

12 a Let $z = e^{i\theta} = \cos\theta + i\sin\theta$ then we have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\frac{1}{2i}\left(z^3 - z^{-3}\right)}{\frac{1}{2}\left(z^3 + z^{-3}\right)} = \frac{z^3 - z^{-3}}{i\left(z^3 + z^{-3}\right)} = \frac{(\cos\theta + i\sin\theta)^3 - (\cos\theta - i\sin\theta)^3}{i\left((\cos\theta + i\sin\theta)^3 + (\cos\theta - i\sin\theta)^3\right)}$$

$$= \frac{6i\cos^2\theta\sin\theta - 2i\sin^3\theta}{i\left(2\cos^3\theta - 6\cos\theta\sin^2\theta\right)} = \frac{6\cos^2\theta\sin\theta - 2\sin^3\theta}{2\cos^3\theta - 6\cos\theta\sin^2\theta}$$

Now divide top and bottom by $\cos^3\theta$ to give

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

12 b We have

$$\cot 3\theta = \frac{1}{\tan 3\theta} = \frac{1-3\tan^2\theta}{3\tan\theta - \tan^3\theta}$$

Now divide top and bottom by $\tan^3\theta$ to give

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$$

13 We have

$C = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \ldots$

$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + \ldots$

So we have

$C + iS = 1 + ke^{i\theta} + k^2e^{2i\theta} + k^3e^{3i\theta} + \ldots$

Since $|k| < 1$ this is a geometric series which we can sum giving

$$C + iS = \frac{1}{1-ke^{i\theta}}$$

$$= \frac{1-ke^{-i\theta}}{(1-ke^{i\theta})(1-ke^{-i\theta})} = \frac{1-k\cos\theta + ki\sin\theta}{(1-k\cos\theta)^2 + k^2\sin^2\theta}$$

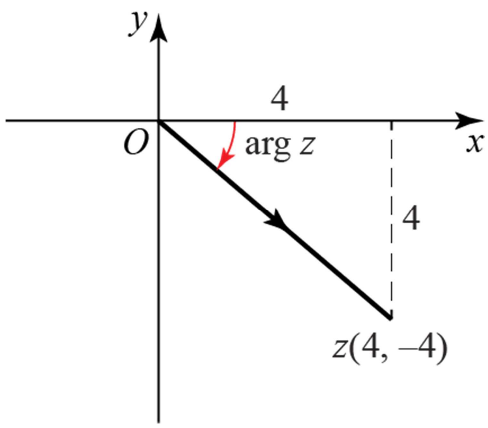
$$= \frac{1-k\cos\theta + ki\sin\theta}{1+k^2-2k\cos\theta}$$

Hence taking real and imaginary parts gives

$$C = \frac{1-k\cos\theta}{1+k^2-2k\cos\theta}$$

$$S = \frac{k\sin\theta}{1+k^2-2k\cos\theta}$$

14 a $4-4i$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

argument $= \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$

$\therefore 4-4i = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

14 b $z^5 = 4 - 4\text{i}$

for $4 - 4\text{i}$, $r = 4\sqrt{2}$, $\theta = -\dfrac{\pi}{4}$

So, $z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}\right)}$

$z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}+2k\pi\right)},\ k \in \mathbb{Z}$

Hence, $z = \left[4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}+2k\pi\right)}\right]^{\frac{1}{5}}$

$= \left(4\sqrt{2}\right)^{\frac{1}{5}}\text{e}^{\text{i}\left(\frac{-\frac{\pi}{4}+2k\pi}{5}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20}+\frac{2k\pi}{5}\right)}$

$4\sqrt{2} = 2^{\frac{5}{2}}$
So, $(4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$
$= 2^{\frac{1}{2}} = \sqrt{2}$

$k = 0,\ z_1 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20}\right)}$

$k = 1,\ z_2 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{7\pi}{20}\right)}$

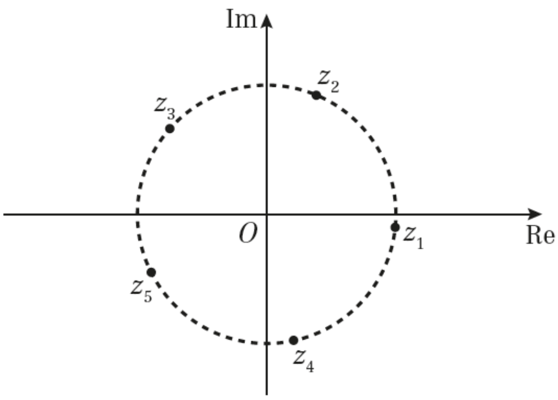
$k = 2,\ z_3 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{3\pi}{4}\right)}$

$k = -1,\ z_4 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{9\pi}{20}\right)}$

$k = -2,\ z_5 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{17\pi}{20}\right)}$

Therefore, $z = \sqrt{2}\,\text{e}^{-\frac{\pi i}{20}},\ \sqrt{2}\,\text{e}^{\frac{7\pi i}{20}},\ \sqrt{2}\,\text{e}^{\frac{3\pi i}{4}},\ \sqrt{2}\,\text{e}^{-\frac{9\pi i}{20}},\ \sqrt{2}\,\text{e}^{-\frac{17\pi i}{20}}$

c



15 a We have $z = 2 - 2i = 2\sqrt{2}e^{-\frac{i\pi}{4}}$, now suppose $\omega^3 = z$ then let $\omega = re^{i\theta}$ then we have $r^3 = \sqrt{8}$ so

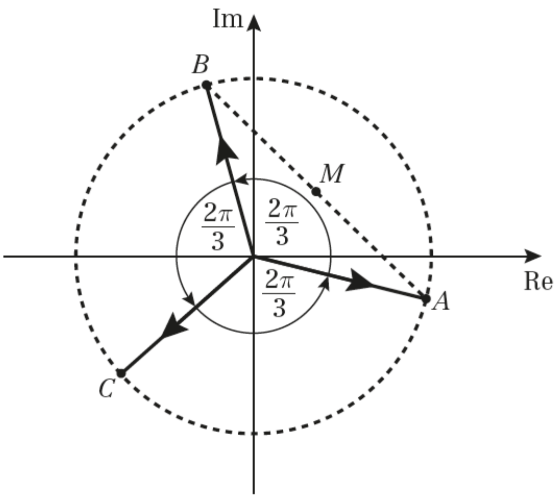
$r = \left(\sqrt{8}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = \sqrt{2}$ and $3\theta = -\frac{\pi}{4} + 2k\pi$ hence three distinct roots are given by

$\omega_1 = \sqrt{2}e^{-\frac{i\pi}{12}}$

$\omega_2 = \sqrt{2}e^{\frac{7i\pi}{12}}$

$\omega_1 = \sqrt{2}e^{-\frac{9i\pi}{12}}$

b



c We have $w = \frac{1}{2}\left(\sqrt{2}e^{\frac{7i\pi}{12}} + \sqrt{2}e^{-\frac{i\pi}{12}}\right)$ but by geometrical considerations the argument of w is just $\frac{1}{2}\left(\frac{-\pi}{12} + \frac{7\pi}{12}\right) = \frac{3\pi}{12} = \frac{\pi}{4}$ and by considering the triangle with vertices at the origin, A, B we have that the modulus is given by $\sqrt{2}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}$ hence we have $w = \frac{\sqrt{2}}{2}e^{\frac{i\pi}{4}}$

d We have

$$w^6 = \left(\frac{\sqrt{2}}{2}e^{\frac{i\pi}{4}}\right)^6 = \frac{8}{64}e^{\frac{6i\pi}{4}} = \frac{1}{8}e^{\frac{3i\pi}{2}} = \frac{1}{8}e^{\frac{-i\pi}{2}} = -\frac{i}{8}$$

16 a Let $z = 2 + i$ now let $\omega = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be a primitive 3rd root of unity then the coordinates of the other two vertices are given by

$$z\omega = (2+i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{-2-\sqrt{3}}{2} + i\frac{-1+2\sqrt{3}}{2}$$

$$z\omega^2 = (2+i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{-2+\sqrt{3}}{2} - i\frac{1+2\sqrt{3}}{2}$$

16 b The length of one side is given by $|z - z\omega|$ we have

$$|z - z\omega| = |z||1-\omega| = \sqrt{5}\left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{5}}{2}\sqrt{9+3} = \sqrt{5}\times\sqrt{3} = \sqrt{15}$$

Challenge

We consider the solutions of $\left(\frac{z+1}{z}\right)^6 = 1$, let $\omega = \frac{z+1}{z}$ then we have $\omega^6 = 1$ so $\omega = e^{\frac{2ki\pi}{6}}$ for $k \in \mathbb{Z}$ hence we have

$$\frac{z+1}{z} = e^{\frac{2ki\pi}{6}}$$

So

$$z + 1 = ze^{\frac{2ki\pi}{6}}$$

So

$$z\left(e^{\frac{2ki\pi}{6}} - 1\right) = 1$$

Therefore

$$z = \frac{1}{e^{\frac{ki\pi}{3}} - 1} = \frac{e^{-\frac{ki\pi}{3}} - 1}{\left(e^{\frac{ki\pi}{3}} - 1\right)\left(e^{-\frac{ki\pi}{3}} - 1\right)} = \frac{\cos\frac{k\pi}{3} - 1 - i\sin\frac{k\pi}{3}}{\left(\cos\frac{k\pi}{3} - 1\right)^2 + \sin\frac{k\pi}{3}} = \frac{\cos\frac{k\pi}{3} - 1 - i\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}}$$

Assuming $e^{\frac{2ki\pi}{6}} - 1 \neq 0$ i.e. $1 \le k \le 5$, when $k = 0$ the equation becomes $z = z + 1$ which has no solutions, however, for $1 \le k \le 5$ we get a different solution. So our six values of z are $z_0 = -\frac{1}{2}$

$$z_1 = \frac{\cos\frac{\pi}{3} - 1 - i\sin\frac{\pi}{3}}{2 - 2\cos\frac{\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$z_2 = \frac{\cos\frac{2\pi}{3} - 1 - i\sin\frac{2\pi}{3}}{2 - 2\cos\frac{2\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{6}$$

$$z_3 = \frac{\cos\pi - 1 - i\sin\pi}{2 - 2\cos\pi} = -\frac{1}{2}$$

$$z_4 = \frac{\cos\frac{4\pi}{3} - 1 - i\sin\frac{4\pi}{3}}{1 - 2\cos\frac{4\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{6}$$

$$z_5 = \frac{\cos\frac{5\pi}{3} - 1 - i\sin\frac{5\pi}{3}}{1 - 2\cos\frac{5\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Which all lie on the straight line $\text{Re}(z) = -\frac{1}{2}$