Series – mixed exercise 2

1 a $\dfrac{2}{(r+2)(r+4)} = \dfrac{A}{r+2} + \dfrac{B}{r+4}$

$2 = A(r+4) + B(r+2)$

$2 = 2A = -2B$

$\dfrac{2}{(r+2)(r+4)} = \dfrac{1}{r+2} - \dfrac{1}{r+4}$

b Let $\mathrm{f}(r) = \dfrac{1}{r+2}$

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \sum_{r=1}^{n}\left(\frac{1}{r+2} - \frac{1}{r+4}\right)$$

$$= \sum_{r=1}^{n}\left(\mathrm{f}(r) - \mathrm{f}(r+2)\right)$$

$$= \mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+3) - 12(n+4)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{12(n+3)(n+4)}$$

2 a $\dfrac{4}{(4r-1)(4r+3)} = \dfrac{A}{4r-1} + \dfrac{B}{4r+3}$

$4 = A(4r+3) + B(4r-1)$

$4 = 4A = -4B$

$\dfrac{4}{(4r-1)(4r+3)} = \dfrac{1}{4r-1} - \dfrac{1}{4r+3}$

b Let $\mathrm{f}(r) = \dfrac{1}{4r-1}$

$$\sum_{r=1}^{n} \frac{4}{(4r-1)(4r+3)} = \sum_{r=1}^{n}\left(\frac{1}{4r-1} - \frac{1}{4r+3}\right)$$

$$= \sum_{r=1}^{n}\left(\mathrm{f}(r) - \mathrm{f}(r+1)\right) = \mathrm{f}(1) - \mathrm{f}(n+1)$$

$$= \frac{1}{3} - \frac{1}{4n+3}$$

$$= \frac{4n}{3(4n+3)}$$

2 c $\sum_{r=100}^{200} \frac{4}{(4r-1)(4r+3)}$

$= \sum_{r=1}^{200} \frac{4}{(4r-1)(4r+3)} - \sum_{r=1}^{99} \frac{4}{(4r-1)(4r+3)}$

$= \frac{800}{3(803)} - \frac{396}{3(399)} = \frac{404}{320397} = 0.00126 \text{ (3 s.f.)}$

3 a $(r+1)^3 - (r-1)^3$

$= r^3 + 3r^2 + 3r + 1 - (r^3 - 3r^2 + 3r - 1)$

$= 6r^2 + 2$

b $r^2 = \frac{1}{6}\left((r+1)^3 - (r-1)^3 - 2\right)$

$\sum_{r=1}^{n} (r+1)^3 - (r-1)^3 = (n+1)^3 + n^3 - 1$

$= 2n^3 + 3n^2 + 3n$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2\sum_{r=1}^{n} 1\right)$

$= \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right) = \frac{1}{6}n(n+1)(2n+1)$

4 $\frac{4}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$

$4 = A(r+3) + B(r+1)$

$4 = 2A = -2B$

Let $\mathrm{f}(r) = \frac{1}{r+1}$

$\sum_{r=1}^{n} \frac{4}{(r+1)(r+3)} = 2\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$= 2(\mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2))$

$= 2\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = 2\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$

$= \frac{1}{3}\left(\frac{5(n+2)(n+3) - 6(n+2) - 6(n+3)}{(n+2)(n+3)}\right)$

$= \frac{5n^2 + 13n}{3(n+2)(n+3)}$

5 $\sum_{r=1}^{n}(r+1)^3-(r-1)^3=(n+1)^3+n^3-1$

$\sum_{r=n}^{2n}(r+1)^3-(r-1)^3$

$=(2n+1)^3+(2n)^3-1-(n^3+(n-1)^3-1)$

$=16n^3+12n^2+6n-(2n^3-3n^2+3n-2)$

$=14n^3+15n^2+3n+2$

So $a=14$, $b=15$, $c=3$ and $\mathrm{d}=2$

6 a $y=e^{1-2x}$

$y'=-2e^{1-2x}=-2y$

$y''=-2y'=(-2)^2y$

$\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=(-2)^ny$

b $\frac{\mathrm{d}^8y}{\mathrm{d}x^8}=(-2)^8y$

When $x=\log 32$, $\frac{\mathrm{d}^8y}{\mathrm{d}x^8}=2^8e^{1-2\log 32}$

$=2^8e(32)^{-2}=2^{8-2\times5}e=\frac{e}{4}$

7 a $\mathrm{f}(x)=\ln(1+\mathrm{e}^x)$ so $\mathrm{f}(0)=\ln 2$

$\mathrm{f}'(x)=\frac{\mathrm{e}^x}{1+\mathrm{e}^x}$ $\boxed{=1-\frac{1}{1+\mathrm{e}^x}=1-(1+\mathrm{e}^x)^{-1}}$ $\mathrm{f}'(0)=\frac{1}{2}$

So $\mathrm{f}''(x)=\frac{\mathrm{e}^x}{(1+\mathrm{e}^x)^2}$ or use the quotient rule $\mathrm{f}''(0)=\frac{1}{4}$

b $\mathrm{f}'''(x)=\frac{(1+\mathrm{e}^x)^2\mathrm{e}^x-\mathrm{e}^x2(1+\mathrm{e}^x)\mathrm{e}^x}{(1+\mathrm{e}^x)^4}$

Use the quotient rule and chain rule.

$=\frac{(1+\mathrm{e}^x)\mathrm{e}^x\{(1+\mathrm{e}^x)-2\mathrm{e}^x\}}{(1+\mathrm{e}^x)^4}=\frac{\mathrm{e}^x(1-\mathrm{e}^x)}{(1+\mathrm{e}^x)^3}$ $\mathrm{f}'''(0)=0$

c Using Maclaurin's expansion

$\ln(1+\mathrm{e}^x)=\ln 2+\frac{x}{2}+\frac{x^2}{8}+\ldots$

The expansion is valid for $-1<\mathrm{e}^x\le 1\Rightarrow 0,\ \mathrm{e}^x\le 1$ so for $x\le 0$.

8 a $\cos 4x=1-\frac{(4x)^2}{2!}+\frac{(4x)^4}{4!}-\frac{(4x)^6}{6!}+\ldots$

$=1-8x^2+\frac{32}{3}x^4-\frac{256}{45}x^6+\ldots$

8 b $\cos 4x = 1 - 2\sin^2 2x,$

so $2\sin^2 2x = 1 - 2\cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \ldots$

$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \ldots$

9 Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$

$$e^{\cos x} = e^{\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}}$$

$$= e\left\{1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\} \quad \text{no other terms required}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \ldots\right\} = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + \ldots\right\}$$

10 $\ln[(1+x)^2(1-2x)] = 2\ln(1+x) + \ln(1-2x)$

$$= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} - \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots\right\}$$

$$= 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 - 2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \ldots$$

$$= -3x^2 - 2x^3 - \ldots$$

11 $f(x) = \ln(\sec x + \tan x)$ $\qquad f(0) = \ln 1 = 0$

$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$ $\qquad f'(0) = 1$

$f''(x) = \sec x \tan x$ $\qquad f''(0) = 0$

$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$ $\qquad f'''(0) = 1$

Substituting into Maclaurin's expansion gives $y = x + \frac{x^3}{6} + \ldots$

12 a $\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \ldots\right)$

$$= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots + \frac{(r+1)x^r}{(r+1)!} + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$$

$$= e^x$$

12 b $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + (-1)^r \dfrac{x^{2r+1}}{(2r+1)!} + \ldots\right)$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \ldots + (-1)^r \frac{(2r+1)x^{2r}}{(2r+1)!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots = \cos x$$

c $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots + (-1)^r \dfrac{x^{2r}}{(2r)!} + (-1)^{r+1} \dfrac{x^{2r+2}}{(2r+2)!} + \ldots\right)$

$$= \left(-\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \ldots + (-1)^r \frac{2rx^{2r-1}}{(2r)!} + (-1)^{r+1} \frac{(2r+2)x^{2r+1}}{(2r+2)!} + \ldots\right)$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots + (-1)^{r+1} \frac{x^{2r+1}}{(2r+1)!} + \ldots$$

$$= -\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \ldots\right) = -\sin x$$

13 a You can write $\cos x = 1 - \left(\dfrac{x^2}{2} - \dfrac{x^4}{24} + \ldots\right)$; it is not necessary to have higher powers

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)\right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\sec x = 1 + (-1)\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\} + \frac{(-1)(-2)}{2!}\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\}^2 + \ldots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{x^4}{4} + \text{higher powers of } x$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$$

13 b $\tan x = \dfrac{\sin x}{\cos x} = \sin x \times \sec x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \ldots$$

$$= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

14 Using $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ and $\cos 3x = 1 - \dfrac{(3x)^2}{2!} + \ldots$

$$e^x \cos 3x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{9x^2}{2} + \ldots\right)$$

$$= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \ldots\right\}$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

15 $f(x) = (1 + x)^2 \ln(1 + x)$.

$$f'(x) = (1 + x)^2 \frac{1}{1 + x} + 2(1 + x)\ln(1 + x) = (1 + x)\{1 + 2\ln(1 + x)\}$$

$$f''(x) = (1 + x)\left(\frac{2}{1 + x}\right) + \{1 + 2\ln(1 + x)\} = 3 + 2\ln(1 + x)$$

$$f'''(x) = \left(\frac{2}{1 + x}\right)$$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 3,\ f'''(0) = 2$

Using Maclaurin's expansion

$$(1 + x)^2 \ln(1 + x) = 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$

$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

16 a $\ln(1+\sin x) = \ln\left\{1+\left(x-\frac{x^3}{3!}+\ldots\right)\right\}$

$$= \left(x-\frac{x^3}{3!}+\ldots\right)-\frac{1}{2}\left(x-\frac{x^3}{3!}+\ldots\right)^2+\frac{1}{3}\left(x-\frac{x^3}{3!}+\ldots\right)^3-\frac{1}{4}\left(x-\frac{x^3}{3!}+\ldots\right)^4+\ldots$$

$$= \left(x-\frac{x^3}{6}+\ldots\right)-\frac{1}{2}\left(x^2-\frac{x^4}{3}+\ldots\right)+\frac{1}{3}(x^3+\ldots)-\frac{1}{4}(x^4+\ldots) \text{ no other terms necessary}$$

$$= x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots$$

b $\int_0^{\frac{\pi}{6}} \ln(1+\sin x)\,dx \approx \int_0^{\frac{\pi}{6}}\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}\right)dx$

$$\approx \left[\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{60}\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}-\frac{\pi^3}{1296}+\frac{\pi^4}{31104}-\frac{\pi^5}{466560} = 0.116\,(3\text{d.p.})$$

17 a $f(x) = e^{\tan x} = e^{x+\frac{x^3}{3}+\ldots} = e^x \times e^{\frac{x^3}{3}}$ (As only terms up to x^3 are required, only first two terms of $\tan x$ are needed.)

$$= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(1+\frac{x^3}{3}+\ldots\right) \text{ no other terms required.}$$

$$= \left(1+\frac{x^3}{3}+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{2}+\ldots$$

b $e^{-\tan x} = e^{\tan(-x)}$, so replacing x by $-x$ in **a** gives

$$e^{-\tan x} = 1-x+\frac{x^2}{2}-\frac{x^3}{2}+\ldots$$

18 a

$f(x) = \ln\cos x$ $\quad f(0)=0$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ $\quad f'(0)=0$

$f''(x) = -\sec^2 x$ $\quad f''(0) = -1$

$f'''(x) = -2\sec^2 x\tan x$ $\quad f'''(0)=0$

$f''''(x) = -2\sec^4 x - 4\sec^2 x\tan^2 x$ $\quad f''''(0) = -2$

Substituting into Maclaurin:

$$\ln\cos x = (-1)\frac{x^2}{2!}+(-2)\frac{x^4}{4!}+\ldots = -\frac{x^2}{2}-\frac{x^4}{12}-\ldots$$

18 b Using $1+\cos x = 2\cos^2\left(\frac{x}{2}\right)$, $\ln(1+\cos x) = \ln 2\cos^2\left(\frac{x}{2}\right) = \ln 2 + 2\ln\cos\left(\frac{x}{2}\right)$

so $\ln(1+\cos x) = \ln 2 + 2\left\{-\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \ldots\right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \ldots$

19 a

$$y = e^{3x} - e^{-3x}$$
$$y' = 3e^{3x} + 3e^{-3x}$$
$$y'' = 9e^{3x} - 9e^{-3x} = 9y$$
$$y''' = 9y', y'''' = 9y'' = 81y$$

b When $x = 0$,

$$y = 0$$
$$y' = 6$$
$$y'' = 9y = 0$$
$$y''' = 9y' = 54$$
$$y'''' = 81y = 0$$
$$y''''' = 81y' = 486$$

so $y = 6x + \frac{54}{3!}x^3 + \frac{486}{5!}x^5 + \ldots$

$$= 6x + 9x^3 + \frac{81}{20}x^5 + \ldots$$

c

$$e^{3x} = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!}, e^{-3x} = \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}$$
$$y = e^{3x} - e^{-3x}$$
$$= \sum_{n=0}^{\infty}\frac{(3x)^n}{n!} - \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}(1-(-1)^n)\frac{(3x)^n}{n!}$$

n^{th} non-zero term is $(1-(-1)^{2n-1})\frac{3^{2n-1}x^{2n-1}}{(2n-1)!} = 2\frac{3^{2n-1}x^{2n-1}}{(2n-1)!}$

Challenge

$$e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots$$
$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \cos x + i\sin x$$