Methods in calculus 3E

1 a $\frac{1}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3}$

$1 \equiv (Ax+B)(x+3) + C(x^2+1)$

If $x = -3$, $C = \frac{1}{10}$

$A + C = 0 \Rightarrow A = -\frac{1}{10}$

$3B + C = 1 \Rightarrow B = \frac{3}{10}$

Therefore,

$\frac{1}{(x^2+1)(x+3)} = \frac{-x+3}{10(x^2+1)} + \frac{1}{10(x+3)}$

b $\frac{1}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$

$1 \equiv (Ax+B)(x-1) + C(x^2+2)$

If $x = 1$, $C = \frac{1}{3}$

$A + C = 0 \Rightarrow A = -\frac{1}{3}$

$-B + 2C = 1 \Rightarrow B = -\frac{1}{3}$

Therefore,

$\frac{1}{(x^2+2)(x-1)} = -\frac{x+1}{3(x^2+2)} + \frac{1}{3(x-1)}$

c $\frac{x-4}{x(x^2+7)} = \frac{Ax+B}{x^2+7} + \frac{C}{x}$

$x - 4 \equiv (Ax+B)(x) + C(x^2+7)$

$B = 1$

Set $x = 0$, so $C = -\frac{4}{7}$

$A + C = 0 \Rightarrow A = \frac{4}{7}$

Therefore,

$\frac{x-4}{x(x^2+7)} = \frac{4x+7}{7(x^2+7)} - \frac{4}{7x}$

2 $\int \frac{x^2-3x}{(x^2+6)(x+2)}\,dx = \int \frac{Ax+B}{x^2+6}+\frac{C}{x+2}\,dx$

$x^2-3x \equiv (Ax+B)(x+2)+C(x^2+6)$

Set $x=-2$, so $C=1$

$A+C=1 \Rightarrow A=0$

$2B+6C=0 \Rightarrow B=-3$

Therefore,

$\int \frac{x^2-3x}{(x^2+6)(x+2)}\,dx$

$= -3\int \frac{1}{x^2+6}\,dx + \int \frac{1}{x+2}\,dx$

$= -\frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{6}}\right)+\ln|x+2|+c$

Therefore, $A=1$ and $B=-\frac{3}{\sqrt{2}}$

3 **a** Writing $x^4-x^3-4x^2-2x-12=(x+2)(x^3+ax^2+bx-6)$ and solving we obtain

$\mathrm{f}(x)=(x-3)(x+2)(x^2+2)$

b $\int \frac{x^3-20x^2+4x-24}{x^4-x^3-4x^2-2x-12}\,dx$

$= \int \frac{x^3-20x^2+4x-24}{(x+2)(x-3)(x^2+2)}\,dx$

$= \int \frac{A}{x+2}+\frac{B}{x-3}+\frac{Cx+D}{x^2+2}\,dx$

$$x^3-20x^2+4x-24 \equiv \begin{pmatrix} A(x-3)(x^2+2) \\ +B(x+2)(x^2+2)+ \\ Cx(x+2)(x-3)+ \\ D(x+2)(x-3) \end{pmatrix}$$

Set $x=3$, so $B=-3$

Set $x=-2$, so $A=4$

$A+B+C=1 \Rightarrow C=0$

$-6A+4B-6D=-24 \Rightarrow D=-2$

3 b $\int \frac{x^3-20x^2+4x-24}{(x+2)(x-3)(x^2+2)}\mathrm{d}x$

$=4\int \frac{1}{x+2}\mathrm{d}x-3\int \frac{1}{x-3}\mathrm{d}x-2\int \frac{1}{x^2+2}\mathrm{d}x$

$=\begin{pmatrix}4\ln|x+2|-3\ln|x-3| \\ -\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)+c\end{pmatrix}$

$=\left(\ln\left|\frac{(x+2)^4}{(x-3)^3}\right|-\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)+c\right)$

Therefore, $A=4$, $B=-3$, and $D=-\sqrt{2}$

4 $\int \frac{2-x}{x^3+4x}\mathrm{d}x=\int \frac{2-x}{x(x^2+4)}\mathrm{d}x$

$=\frac{Ax+B}{x^2+4}+\frac{C}{x}$

$2-x\equiv(Ax+B)(x)+C(x^2+4)$

$B=-1$

Set $x=0$, so $C=\frac{1}{2}$

$A+C=0\Rightarrow A=-\frac{1}{2}$

Therefore,

$\int \frac{2-x}{x^3+4x}\mathrm{d}x$

$=-\frac{1}{2}\int \frac{x}{x^2+4}\mathrm{d}x-\int \frac{1}{x^2+4}\mathrm{d}x+\frac{1}{2}\int \frac{1}{x}\mathrm{d}x$

$=-\frac{1}{4}\ln|x^2+4|-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+\frac{1}{2}\ln|x|+c$

$=\frac{1}{4}\ln\left|\frac{x^2}{x^2+4}\right|-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+c$

Therefore, $A=\frac{1}{4}$, $B=-\frac{1}{2}$

5 $\int \frac{x^2+1}{4x^4+9x^2}dx=\int \frac{x^2+1}{x^2(4x^2+9)}dx$

$=\frac{Ax+B}{x^2}+\frac{Cx+D}{4x^2+9}$

$x^2+1\equiv(Ax+B)(4x^2+9)+(Cx+D)(x^2)$

$B=\frac{1}{9}$

$4B+D=1\Rightarrow D=\frac{5}{9}$

$A=0$

5 $C=0$

Therefore,

$$\int \frac{x^2+1}{4x^4+9x^2}\,\mathrm{d}x$$
$$=\frac{1}{9}\int \frac{1}{x^2}\,\mathrm{d}x+\frac{5}{9}\int \frac{1}{4x^2+9}\,\mathrm{d}x$$
$$=\frac{1}{9}\int \frac{1}{x^2}\,\mathrm{d}x+\frac{5}{36}\int \frac{1}{x^2+\frac{9}{4}}\,\mathrm{d}x$$
$$=-\frac{1}{9x}+\frac{5}{36}\left(\frac{1}{\frac{3}{2}}\arctan\left(\frac{x}{\frac{3}{2}}\right)\right)+c$$
$$=-\frac{1}{9x}+\frac{5}{54}\arctan\left(\frac{2x}{3}\right)+c$$

Therefore, $A=-\frac{1}{9}$, $B=\frac{5}{54}$

6 $\int \frac{x^3+9x^2+x+1}{x^4-1}\,\mathrm{d}x$

$$=\int \frac{x^3+9x^2+x+1}{(x-1)(x+1)(x^2+1)}\,\mathrm{d}x$$
$$=\int \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\,\mathrm{d}x$$
$$x^3+9x^2+x+1\equiv\begin{pmatrix}A(x+1)(x^2+1)+\\ B(x-1)(x^2+1)+\\ Cx(x-1)(x+1)+\\ D(x-1)(x+1)\end{pmatrix}$$

Set $x=-1$, so $B=-2$

Set $x=1$, so $A=3$

$A+B+C=1\Rightarrow C=0$

$A-B-D=1\Rightarrow D=4$

Therefore,

$$\int \frac{x^3+9x^2+x+1}{(x-1)(x+1)(x^2+1)}\,\mathrm{d}x$$
$$=3\int \frac{1}{x-1}\,\mathrm{d}x-2\int \frac{1}{x+1}\,\mathrm{d}x+4\int \frac{1}{x^2+1}\,\mathrm{d}x$$
$$=3\ln|x-1|-2\ln|x+1|+4\arctan x+c$$
$$=\ln\left|\frac{(x-1)^3}{(x+1)^2}\right|+4\arctan x+c$$

Therefore, $A=3$, $B=2$, and $D=4$

7 a $\mathrm{f}(x)=x^3-4x^2+6x-24$

Since $\mathrm{f}(4)=0$, $(x-4)$ is a factor. Using long division, we find that

$\mathrm{f}(x)=(x-4)(x^2+6)$

b $\dfrac{2x^2-3x+24}{x^3-4x^2+6x-24}=\dfrac{2x^2-3x+24}{(x-4)(x^2+6)}$

$=\dfrac{A}{x-4}+\dfrac{Bx+C}{x^2+6}$

$2x^2-3x+24\equiv\begin{pmatrix}A(x^2+6)+\\(Bx+C)(x-4)\end{pmatrix}$

Set $x=-2$, so $A=2$

$A+B=2\Rightarrow B=0$

$6A-4C=24\Rightarrow C=-3$

So,

$\dfrac{2x^2-3x+24}{x^3-4x^2+6x-24}=\dfrac{2}{x-4}-\dfrac{3}{x^2+6}$

c $\displaystyle\int\frac{2x^2-3x+24}{x^3-4x^2+6x-24}\,\mathrm{d}x$

$\displaystyle=\int\frac{2}{x-4}-\frac{3}{x^2+6}\,\mathrm{d}x$

$\displaystyle=2\int\frac{1}{x-4}\,\mathrm{d}x-3\int\frac{1}{x^2+6}\,\mathrm{d}x$

$=2\ln|x-4|-\dfrac{3}{\sqrt{6}}\arctan\left(\dfrac{x}{\sqrt{6}}\right)+c$

$=2\ln|x-4|-\sqrt{\dfrac{3}{2}}\arctan\left(\dfrac{x}{\sqrt{6}}\right)+c$

8 a $\displaystyle\int\frac{1}{(x-2)(2x-1)}\,\mathrm{d}x=\int\frac{A}{x-2}+\frac{B}{2x-1}\,\mathrm{d}x$

$1\equiv A(2x-1)+B(x-2)$

Set $x=2$, so $A=\dfrac{1}{3}$

Set $x=\dfrac{1}{2}$, so $B=-\dfrac{2}{3}$

$\displaystyle\int\frac{1}{(x-2)(2x-1)}\,\mathrm{d}x$

$\displaystyle=\frac{1}{3}\int\frac{1}{x-2}\,\mathrm{d}x-\frac{2}{3}\int\frac{1}{2x-1}\,\mathrm{d}x$

$=\dfrac{1}{3}\ln|x-2|-\dfrac{1}{3}\ln|2x-1|+c$

$=\dfrac{1}{3}\ln\left|\dfrac{x-2}{2x-1}\right|+c$

8 b $\int_{\frac{1}{2}}^{2} \frac{1}{(x-2)(2x-1)} \mathrm{d}x$

Split the integral as

$\int_{\frac{1}{2}}^{1} \frac{1}{(x-2)(2x-1)} \mathrm{d}x + \int_{1}^{2} \frac{1}{(x-2)(2x-1)} \mathrm{d}x$

Consider $\int_{1}^{2} \frac{1}{(x-2)(2x-1)} \mathrm{d}x$

$= \lim_{t \to 2} \int_{1}^{t} \frac{1}{(x-2)(2x-1)} \mathrm{d}x$

$= \lim_{t \to 2} \left[\frac{1}{3} \ln \left| \frac{x-2}{2x-1} \right| \right]_{1}^{t}$

$= \lim_{t \to 2} \left(\frac{1}{3} \ln \left| \frac{t-2}{2t-1} \right| \right)$

$\ln \left| \frac{t-2}{2t-1} \right| \to \infty$ as $t \to 2$, so the integral diverges.

9 a $\frac{x^4 + 5x^2 + 2x}{x^4 + 10x^2 + 24} = \frac{x^4 + 5x^2 + 2x}{(x^2+4)(x^2+6)}$

$= A + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{x^2 + 6}$

$x^4 + 5x^2 + 2x \equiv \begin{pmatrix} A(x^2+4)(x^2+6) + \\ (Bx+C)(x^2+6) + \\ (Dx+E)(x^2+4) \end{pmatrix}$

$A = 1$

$B + D = 0$ and $6B + 4D = 2$

So $B = 1$ and $D = -1$

$24A + 6C + 4E = 0$

$10A + C + E = 5$

So $C = -2$ and $E = -3$

Therefore,

$\frac{x^4 + 5x^2 + 2x}{x^4 + 10x^2 + 24} = 1 + \frac{x-2}{x^2+4} - \frac{x+3}{x^2+6}$

b $\int \frac{x^4 + 5x^2 + 2x}{x^4 + 10x^2 + 24} \mathrm{d}x$

$= \int 1 + \frac{x-2}{x^2+4} - \frac{x+3}{x^2+6} \mathrm{d}x$

$= \int \mathrm{d}x + \int \frac{x}{x^2+4} \mathrm{d}x - 2\int \frac{1}{x^2+4} \mathrm{d}x$

9 b $-\int \frac{x}{x^2+6}\,\mathrm{d}x - 3\int \frac{1}{x^2+6}\,\mathrm{d}x$

$= x + \frac{1}{2}\ln|x^2+4| - \arctan\left(\frac{x}{2}\right)$

$-\frac{1}{2}\ln|x^2+6| - \frac{3}{\sqrt{6}}\arctan\left(\frac{x}{\sqrt{6}}\right) + c$

$= x + \frac{1}{2}\ln\left|\frac{x^2+4}{x^2+6}\right| - \arctan\left(\frac{x}{2}\right)$

$-\sqrt{\frac{3}{2}}\arctan\left(\frac{x}{\sqrt{6}}\right) + c$

10 Setting up the model $\frac{x^2+4x+10}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$

$\Rightarrow x^2+4x+10 \equiv A(x^2+5) + (Bx+C)x$

$x = 0 \quad \Rightarrow 10 = 5A \quad \Rightarrow \quad A = 2$

Coefficient of $x \quad \Rightarrow \quad 4 = C$

Coefficient of $x^2 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad B = -1$

So $\int \frac{x^2+4x+10}{x^3+5x}\,\mathrm{d}x = \int\left(\frac{2}{x} + \frac{-x+4}{x^2+5}\right)\mathrm{d}x$

$= \int\left(\frac{2}{x} + \frac{4}{x^2+5} - \frac{1}{2}\frac{2x}{x^2+5}\right)\mathrm{d}x$

$= 2\ln x + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) - \frac{1}{2}\ln(x^2+5) + C$

$= \frac{1}{2}\ln\left(\frac{x^4}{x^2+5}\right) + \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$

11 Setting up the model $\dfrac{2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$$\Rightarrow 2 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$$x=-1 \Rightarrow 2 = 2A \Rightarrow A = -1$$

Coefficient of $x^2 \Rightarrow 0 = A + B \Rightarrow B = -1$

Coefficient of $x \Rightarrow 0 = B + C \Rightarrow C = 1$

So $\displaystyle\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{1}{(x+1)}\,dx + \int_0^1 \frac{1-x}{(x^2+1)}\,dx$

$$= \int_0^1 \frac{1}{(x+1)}\,dx + \int_0^1 \frac{1}{(x^2+1)}\,dx - \int_0^1 \frac{x}{(x^2+1)}\,dx$$

$$= [\ln(1+x)]_0^1 + [\arctan x]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \ln 2 + \arctan 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

$$= \frac{1}{4}(\pi + 2\ln 2)$$

12 a $\dfrac{x^4+1}{x(x^2+2)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2}$

$$x^4 + 1 \equiv \begin{pmatrix} A(x^2+2)^2 + \\ (Bx+C)x(x^2+2) + \\ (Dx+E)x \end{pmatrix}$$

Set $x = 0$, so $A = \dfrac{1}{4}$

$A + B = 1 \Rightarrow B = \dfrac{3}{4}$

$4A + 2B + D = 0 \Rightarrow D = -\dfrac{5}{2}$

$C = 0$

$2C + E = 0 \Rightarrow E = 0$

Therefore,

$$\frac{x^4+1}{x(x^2+2)^2} = \frac{1}{4x} + \frac{3x}{4(x^2+2)} - \frac{5x}{2(x^2+2)^2}$$

12 b $\int \frac{x^4+1}{x\left(x^2+2\right)^2}\mathrm{d}x$

$=\int \frac{1}{4x}+\frac{3x}{4\left(x^2+2\right)}-\frac{5x}{2\left(x^2+2\right)^2}\mathrm{d}x$

$=\frac{1}{4}\int \frac{1}{x}\mathrm{d}x+\frac{3}{4}\int \frac{x}{x^2+2}\mathrm{d}x-\frac{5}{2}\int \frac{x}{\left(x^2+2\right)^2}\mathrm{d}x$

Consider $\int \frac{x}{\left(x^2+2\right)^2}\mathrm{d}x$

Let $u=x^2+2$ and $du=2x\mathrm{d}x$

$\int \frac{x}{\left(x^2+2\right)^2}\mathrm{d}x=\frac{1}{2}\int \frac{1}{u^2}du$

$=-\frac{1}{2u}+c=-\frac{1}{2\left(x^2+2\right)}+c_1$

So,

$\frac{1}{4}\int \frac{1}{x}\mathrm{d}x+\frac{3}{4}\int \frac{x}{x^2+2}\mathrm{d}x-\frac{5}{2}\int \frac{x}{\left(x^2+2\right)^2}\mathrm{d}x$

$=\frac{1}{4}\ln|x|+\frac{3}{8}\ln\left|x^2+2\right|+\frac{5}{4\left(x^2+2\right)}+c$

Challenge

a $\int \frac{1}{x^2-8x+8}\,\mathrm{d}x$

$= \int \frac{1}{(x-4)^2-8}\,\mathrm{d}x$

$= \int \frac{1}{(x-4)^2-\left(2\sqrt{2}\right)^2}\,\mathrm{d}x$

$= \int \frac{1}{\left(x-4+2\sqrt{2}\right)\left(x-4-2\sqrt{2}\right)}\,\mathrm{d}x$

$= \int \frac{A}{x-4+2\sqrt{2}} + \frac{B}{x-4-2\sqrt{2}}\,\mathrm{d}x$

$1 \equiv A\left(x-4-2\sqrt{2}\right) + B\left(x-4+2\sqrt{2}\right)$

Set $x = 4-2\sqrt{2}$, so $A = -\frac{1}{4\sqrt{2}}$

$A + B = 0 \Rightarrow B = \frac{1}{4\sqrt{2}}$

So,

$\int \frac{1}{x^2-8x+8}\,\mathrm{d}x$

$= \frac{1}{4\sqrt{2}} \begin{pmatrix} -\int \frac{1}{x-4+2\sqrt{2}}\,\mathrm{d}x + \\ \int \frac{1}{x-4-2\sqrt{2}}\,\mathrm{d}x \end{pmatrix}$

$= \frac{1}{4\sqrt{2}} \begin{pmatrix} -\ln\left|x-4+2\sqrt{2}\right| + \\ \ln\left|x-4+2\sqrt{2}\right| \end{pmatrix} + c$

$= \frac{1}{4\sqrt{2}} \ln\left|\frac{x-4-2\sqrt{2}}{x-4+2\sqrt{2}}\right| + c$

Challenge

b $\int \frac{1}{2x^2+4x+11}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{(x+1)^2+\frac{9}{2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{1}{(x+1)^2+\left(\frac{3}{\sqrt{2}}\right)^2}\,\mathrm{d}x$$

(Let $u = x+1$, so $du = \mathrm{d}x$)

$$= \frac{1}{2}\int \frac{1}{u^2+\left(\frac{3}{\sqrt{2}}\right)^2}\,du$$

$$= \frac{1}{3\sqrt{2}}\arctan\left(\frac{u}{\frac{3}{\sqrt{2}}}\right)+c$$

$$= \frac{1}{3\sqrt{2}}\arctan\left(\frac{\sqrt{2}(x+1)}{3}\right)+c$$