Volumes of revolution – mixed exercise 4

1 a $I = \int x\cos 2x\,\mathrm{d}x$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \frac{1}{2}\sin 2x$

Integrating by parts

$$I = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,\mathrm{d}x$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

b $y = 2x^{\frac{1}{2}}\sin x$

$y^2 = 4x\sin^2 x = 2x(1-\cos 2x)$

$$\text{Volume } = 2\pi\int_0^{\frac{\pi}{2}} x(1-\cos 2x)\,\mathrm{d}x$$
$$= 2\pi\left[\frac{x^2}{2} - \frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= 2\pi\left(\left(\frac{\pi^2}{8} + \frac{1}{4}\right) + \frac{1}{4}\right)$$
$$= \pi\left(\frac{\pi^2}{4} + 1\right)$$

2 a $I = \int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \tan x$

$$I = [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x$$
$$= \left(\frac{\pi}{4} - 0\right) + \left[\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$
$$= \frac{\pi}{4} - \ln\sqrt{2}$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2 b
$$\text{Volume } = \pi\int_0^{\frac{\pi}{4}} y^2\,\mathrm{d}x$$
$$= \pi\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$$
$$= \pi\left(\frac{1}{4}\pi - \frac{1}{2}\ln 2\right)$$

3 Volume generated by area under C

$$= \pi\int_1^{\frac{5}{2}} \left(\frac{x+2}{x}\right)^2 \mathrm{d}x$$
$$= \pi\int_1^{\frac{5}{2}} \frac{x^2+4x+4}{x^2}\,\mathrm{d}x$$
$$= \pi\int_1^{\frac{5}{2}} \left(1 + \frac{4}{x} + \frac{4}{x^2}\right)\mathrm{d}x$$
$$= \pi\left[x + 4\ln x - \frac{4}{x}\right]_1^{\frac{5}{2}}$$
$$= \pi\left(\left(\frac{5}{2} + 4\ln\frac{5}{2} - \frac{8}{5}\right) - (1-4)\right)$$
$$= \pi\left(\frac{39}{10} + 4\ln\frac{5}{2}\right)$$

Volume generated by T is a cone of radius 3 and height 1.5.

Volume of cone $= \frac{1}{3}\pi\times 9\times 1.5 = \frac{9}{2}\pi$

So volume generated by R

$$= \pi\left(\frac{39}{10} + 4\ln\frac{5}{2}\right) - \frac{9}{2}\pi$$
$$= \pi\left(4\ln\frac{5}{2} - \frac{3}{5}\right)$$

4 $x=\frac{\pi}{4}\Rightarrow \sec x-\cos x=\sqrt{2}-\frac{1}{\sqrt{2}}$

Also $\operatorname{cosec} x-\sin x=\sqrt{2}-\frac{1}{\sqrt{2}}$

So the curves meet at $x=\frac{\pi}{4}$

Volume generated by $y=\sec x-\cos x$ between $x=0$ and $x=\frac{\pi}{4}$ is given by

$$\pi\int_0^{\frac{\pi}{4}}(\sec x-\cos x)^2\,dx$$
$$=\pi\int_0^{\frac{\pi}{4}}\left(\sec^2 x-2+\cos^2 x\right)dx$$
$$=\pi\int_0^{\frac{\pi}{4}}\left(\sec^2 x-2+\frac{1}{2}(1+\cos 2x)\right)dx$$
$$=\pi\left[\tan x-\frac{3}{2}x+\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$=\pi\left(1-\frac{3\pi}{8}+\frac{1}{4}\right)$$
$$=\frac{\pi}{8}(10-3\pi)$$

Due to the symmetry of the curves, the total volume generated by R will be

$$2\times\frac{\pi}{8}(10-3\pi)$$
$$=\frac{\pi}{4}(10-3\pi)$$

5 $x=e^{\frac{1}{2}y}-2\Rightarrow x^2=e^y-4e^{\frac{1}{2}y}+4$

Volume $=\pi\int_2^4\left(e^y-4e^{\frac{1}{2}y}+4\right)dy$

$$=\pi\left[e^y-8e^{\frac{1}{2}y}+4y\right]_2^4$$
$$=\pi\left(\left(e^4-8e^2+16\right)-\left(e^2-8e+8\right)\right)$$
$$=\pi\left(e^4-9e^2+8e+8\right)$$

6 a $x=2\sin^2 t\Rightarrow\frac{dx}{dt}=4\sin t\cos t$

$y=2\cos t\Rightarrow\frac{dy}{dt}=-2\sin t$

$\frac{dy}{dx}=\frac{-2\sin t}{4\sin t\cos t}=-\frac{1}{2\cos t}$

At point $P\left(\frac{3}{2},1\right)$ $t=\frac{\pi}{3}$ and $\frac{dy}{dx}=-1$

Equation of l is

$y-1=-1\left(x-\frac{3}{2}\right)$

$y=-x+\frac{5}{2}$

6 b $x = 2\sin^2 t \Rightarrow \sin^2 t = \frac{x}{2}$

$y = 2\cos t \Rightarrow \cos^2 t = \frac{y^2}{4}$

$\frac{y^2}{4} + \frac{x}{2} = 1$

$x = 2 - \frac{y^2}{2}$

Volume generated by the curve C

$= \pi\int_0^2 x^2\,dy$

$= \pi\int_0^2 \left(2 - \frac{y^2}{2}\right)^2 dy$

$= \pi\int_0^2 \left(4 - 2y^2 + \frac{y^4}{4}\right) dy$

$= \pi\left[4y - \frac{2y^3}{3} + \frac{y^5}{20}\right]_0^2$

$= \pi\left(8 - \frac{16}{3} + \frac{32}{20}\right)$

$= \frac{\pi}{15}(120 - 80 + 24)$

$= \frac{64\pi}{15}$

Volume generated by line l

$= \frac{1}{3}\pi \times \left(\frac{5}{2}\right)^2 \times \frac{5}{2}$

$= \frac{125}{24}\pi$

Volume generated by region R

$= \frac{125}{24}\pi - \frac{64}{15}\pi$

$= \frac{\pi}{120}(625 - 512)$

$= \frac{113}{120}\pi$

7 a $x = (t+1)^2, \quad y = \frac{1}{2}t^3 + 3$

$\frac{dx}{dt} = 2(t+1)$

Area of $R = \int_0^2 y\frac{dx}{dt}\,dt$

$= 2\int_0^2 \left(\frac{1}{2}t^3 + 3\right)(t+1)\,dt$

$= 2\int_0^2 \left(\frac{1}{2}t^4 + \frac{1}{2}t^3 + 3t + 3\right) dt$

$= 2\left[\frac{1}{10}t^5 + \frac{1}{8}t^4 + \frac{3}{2}t^2 + 3t\right]_0^2$

$= 2\left(\left(\frac{32}{10} + 2 + 6 + 6\right) - 0\right)$

$= \frac{172}{5}$

b Volume generated by R by rotating about the y axis

$= \pi\int_0^2 x^2 \frac{dy}{dt}\,dt$

$= \frac{3}{2}\pi\int_0^2 (t+1)^4 t^2\,dt$

$= \frac{3}{2}\pi\int_0^2 \left(t^4 + 4t^3 + 6t^2 + 4t + 1\right)t^2\,dt$

$= \frac{3}{2}\pi\int_0^2 \left(t^6 + 4t^5 + 6t^4 + 4t^3 + t^2\right)dt$

$= \frac{3}{2}\pi\left[\frac{1}{7}t^7 + \frac{2}{3}t^6 + \frac{6}{5}t^5 + t^4 + \frac{1}{3}t^3\right]_0^2$

$= \frac{3}{2}\pi\left(\frac{128}{7} + \frac{128}{3} + \frac{192}{5} + 16 + \frac{8}{3}\right)$

$= \frac{3}{2}\pi\left(\frac{1920 + 4480 + 4032 + 1680 + 280}{105}\right)$

$= \frac{6196}{35}\pi$

8 Parameters are $x = \cos t,\ \ y = \sin t$

A unit sphere can be generated by rotating the semi-circle around the y axis between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

i.e. Volume $= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \frac{dy}{dt}\, dt$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \cos t\, dt$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sin^2 t\right) \cos t\, dt$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos t - \cos t \sin^2 t\right) dt$

$= \pi \left[\sin t - \frac{1}{3}\sin^3 t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \pi \left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right)$

$= \frac{4\pi}{3}$

9 a $\sin 3\theta = \sin\left(2\theta + \theta\right)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta \cos^2\theta + \left(1 - 2\sin^2\theta\right)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

B $x = 15\cos\theta,\ \ y = 10\sin\theta$

When $\theta = 0,\ x = 15,$ when $\theta = \pi,\ x = -15$

So lower limit is $\theta = \pi$ and upper limit is $\theta = 0$

Volume $= \pi \int_{\pi}^{0} \left(10\sin\theta\right)^2 \left(-15\sin\theta\right) d\theta$

$= -1500\pi \int_{\pi}^{0} \sin^3\theta\, d\theta$

$= -1500\pi \int_{\pi}^{0} \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \right) d\theta$

$= -1500\pi \left[-\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta \right]_{\pi}^{0}$

$= -1500\pi \left(\left(-\frac{3}{4} + \frac{1}{12} \right) - \left(\frac{3}{4} - \frac{1}{12} \right) \right)$

$= 2000\pi \text{ cm}^3$

10 $x = 2\sin 2t,\ \ y = 4\cos t$

Note that R is defined by $0 \le t \le \frac{\pi}{2}$

When $t = 0,\ y = 4,$ when $t = \frac{\pi}{2},\ y = 0$

So lower limit is $t = \frac{\pi}{2}$ and upper limit is $t = 0$

So volume $= \pi \int_{\frac{\pi}{2}}^{0} x^2 \frac{dy}{dt}\, dt$

$= -16\pi \int_{\frac{\pi}{2}}^{0} \sin^2 2t \sin t\, dt$

$= -16\pi \int_{\frac{\pi}{2}}^{0} 4\sin^2 t \cos^2 t \sin t\, dt$

$= -64\pi \int_{\frac{\pi}{2}}^{0} \sin^3 t \cos^2 t\, dt$

$= -64\pi \int_{\frac{\pi}{2}}^{0} \left(1 - \cos^2 t\right) \sin t \cos^2 t\, dt$

$= -64\pi \int_{\frac{\pi}{2}}^{0} \left(\sin t \cos^2 t - \sin t \cos^4 t \right) dt$

$= -64\pi \left[-\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t \right]_{\frac{\pi}{2}}^{0}$

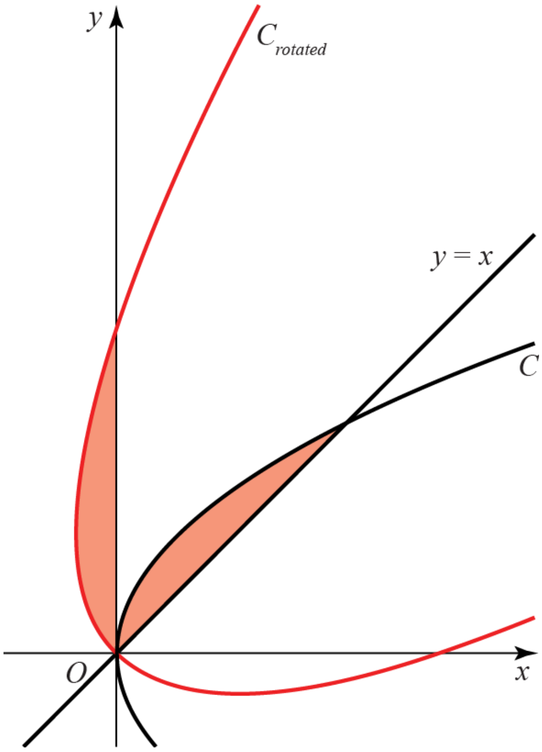
$= -64\pi \left(\left(-\frac{1}{3} + \frac{1}{5} \right) - 0 \right)$

$= -64\pi \times -\frac{2}{15}$

$= \frac{128\pi}{15}$

Challenge

If the curve C and he line $y = x$ are rotated $\pi/4$ radians anticlockwise about the origin, then the required volume will be given by the rotated curve about the y axis between the points at which it crosses the y axis.



The parametric equations of the rotated curve are found using

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ where } \theta = \frac{\pi}{4}$$

$$\text{i.e.} \begin{pmatrix} x_{rotated} \\ y_{rotated} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(t^2 - 2t) \\ \frac{1}{\sqrt{2}}(t^2 + 2t) \end{pmatrix}$$

$$\frac{dy_{rotated}}{dt} = \frac{1}{\sqrt{2}}(2t + 2)$$

$$x_{rotated} = 0 \Rightarrow t^2 - 2t = 0$$

So $t = 0$ or 2

$$\text{Volume } = \pi\int_0^2 \left(\frac{1}{2}(t^2 - 2t)^2 \times \frac{1}{\sqrt{2}}(2t + 2)\right) dt$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^2 (t^4 - 4t^3 + 4t^2)(2t + 2)\, dt$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^2 (2t^5 - 8t^4 + 8t^3 + 2t^4 - 8t^3 + 8t^2)\, dt$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^2 (2t^5 - 6t^4 + 8t^2)\, dt$$

$$= \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3}t^6 - \frac{6}{5}t^5 + \frac{8}{3}t^3\right]_0^2$$

$$= \frac{\pi}{2\sqrt{2}} \left(\frac{64}{3} - \frac{192}{5} + \frac{64}{3}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \left(\frac{320 - 576 + 320}{15}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \times \frac{64}{15}$$

$$= \frac{64\pi}{30\sqrt{2}}$$

$$\left(= \frac{32\pi}{15\sqrt{2}}\right)$$