Hyperbolic Functions 6E

1 a $\int (\sinh x + 3\cosh x)\,dx = \cosh x + 3\sinh x + C$

b $\int \left(\cosh x - \frac{1}{\cosh^2 x}\right) dx = \int (\cosh x - \text{sech}^2 x)\,dx = \sinh x - \tanh x + C$

c $\int \frac{\sinh x}{\cosh^2 x}\,dx = \int \frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x}\,dx = \int \text{sech}x \tanh x \; dx = -\text{sech}x + C$

2 a $\int \sinh 2x\,dx = \frac{1}{2}\cosh 2x + C$

b $\int \cosh\left(\frac{x}{3}\right) dx = \frac{1}{\left(\frac{1}{3}\right)}\sinh\left(\frac{x}{3}\right) + C = 3\sinh\left(\frac{x}{3}\right) + C$

3 a
$$\int \frac{1+x}{\sqrt{(x^2-1)}}\,dx = \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int \frac{x}{\sqrt{(x^2-1)}}\,dx$$
$$= \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int x(x^2-1)^{-\frac{1}{2}}\,dx$$
$$= \text{arcosh}x + \sqrt{(x^2-1)} + C$$

b
$$\int \frac{x-3}{\sqrt{(1+x^2)}}\,dx = \int \frac{x}{\sqrt{(1+x^2)}}\,dx - \int \frac{3}{\sqrt{(1+x^2)}}\,dx$$
$$= \int x(1+x^2)^{-\frac{1}{2}}\,dx - \int \frac{3}{\sqrt{(1+x^2)}}\,dx$$
$$= \sqrt{(1+x^2)} - 3\text{arsinh}x + C$$

4 a $\int \sinh^3 x \cosh x\,dx = \int (\sinh x)^3 \cosh x\,dx = \frac{1}{4}\sinh^4 x + C$

b $\int \tanh 4x\,dx = \int \frac{\sinh 4x}{\cosh 4x}\,dx = \frac{1}{4}\ln\cosh 4x + C$

c
$$\int \sqrt{\cosh 2x}\,\sinh 2x\,dx = \frac{1}{2}\int (\cosh 2x)^{\frac{1}{2}}(2\sinh 2x)\,dx$$
$$= \frac{1}{2}\left\{\frac{(\cosh 2x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right\} + C$$
$$= \frac{1}{3}(\cosh 2x)^{\frac{3}{2}} + C$$

5 a $\int \frac{\sinh x}{2+3\cosh x}\,dx = \frac{1}{3}\int \frac{3\sinh x}{2+3\cosh x}\,dx$

$= \frac{1}{3}\ln(2+3\cosh x)+C$

b $\int \frac{1+\tanh x}{\cosh^2 x}\,dx = \int (1+\tanh x)\,\mathrm{sech}^2 x\,dx$

$= \int (\mathrm{sech}^2 x + \tanh x\,\mathrm{sech}^2 x)\,dx$

$= \tanh x + \frac{1}{2}\tanh^2 x + C$ or $\tanh x - \frac{1}{2}\mathrm{sech}^2 x + C$

c $\int \frac{5\cosh x + 2\sinh x}{\cosh x}\,dx = \int (5+2\tanh x)\,dx$

$= 5x + 2\ln\cosh x + C$

6 $\int x\sinh 3x\,dx = \frac{1}{3}x\cosh 3x - \int \frac{1}{3}\cosh 3x\,dx$

$= \frac{1}{3}x\cosh 3x - \frac{1}{9}\sinh 3x + C$

Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with $u = x$ and $\frac{dv}{dx} = \sinh 3x$

7 a $\int e^x\cosh x\,dx = \int e^x\left(\frac{e^x+e^{-x}}{2}\right)dx$

$= \frac{1}{2}\int (e^{2x}+1)\,dx$

$= \frac{1}{4}e^{2x} + \frac{1}{2}x + C$

You cannot use integration by parts.

b $\int e^{-2x}\sinh 3x\,dx = \int e^{-2x}\left(\frac{e^{3x}-e^{-3x}}{2}\right)dx$

$= \frac{1}{2}\int (e^x - e^{-5x})\,dx$

$= \frac{1}{2}e^x + \frac{1}{10}e^{-5x} + C$

You could use integration by parts twice.

c $\int \cosh x\cosh 3x\,dx = \int \left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{3x}+e^{-3x}}{2}\right)dx$

or write as $\frac{1}{2}(\cosh 4x + \cosh 2x)$

$= \frac{1}{4}\int (e^{4x}+e^{-4x}+e^{2x}+e^{-2x})\,dx$

$= \frac{1}{16}e^{4x} - \frac{1}{16}e^{-4x} + \frac{1}{8}e^{2x} - \frac{1}{8}e^{-2x} + C$ or $\frac{1}{8}\sinh 4x + \frac{1}{4}\sinh 2x + C$

8 $\sinh x + \cosh x = \frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x}) = e^x$

So $\int_0^1 \left(\frac{1}{\sinh x + \cosh x}\right) dx = \int_0^1 e^{-x} dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

9 a $\int \sinh^2 x \, dx = \frac{1}{2}\int (\cosh 2x - 1)\, dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C$

b $\int \sinh^2 x \cosh^2 x \, dx = \int \left(\frac{1}{2}\sinh 2x\right)^2 dx$

$= \frac{1}{4}\int \sinh^2 2x \, dx$

$= \frac{1}{4}\int \left(\frac{\cosh 4x - 1}{2}\right) dx$ ← Using $\cosh 2u = 1 + 2\sinh^2 u$

$= -\frac{1}{8}x + \frac{1}{32}\sinh 4x + C$

c $\int \cosh^5 x \, dx = \int \cosh^4 x \cosh x \, dx$

$= \int (1 + \sinh^2 x)^2 \cosh x \, dx$

$= \int (1 + 2\sinh^2 x + \sinh^4 x)\cosh x \, dx$

$= \int (\cosh x + 2\sinh^2 x \cosh x + \sinh^4 x \cosh x)\, dx$

$= \sinh x + \frac{2}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x + C$

10 $\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx = \int_0^{\ln 2} \left(\frac{1 + \cosh x}{2}\right) dx$

$= \frac{1}{2}\left[x + \sinh x\right]_0^{\ln 2}$

$= \frac{1}{2}\left[\ln 2 + \left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)\right]$ ← $e^{\ln 2} = 2,\ e^{-\ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

$= \frac{1}{2}\left[\ln 2 + \frac{3}{4}\right]$

$= \frac{1}{8}\left[3 + 4\ln 2\right]$

$= \frac{1}{8}(3 + \ln 16)$

11 a Let $x = 3\cosh u$, so $\mathrm{d}x = 3\sinh u\,\mathrm{d}u$

$$\int \frac{1}{\sqrt{x^2-9}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9\cosh^2 u - 9}}\,3\sinh u\ \mathrm{d}u$$
$$= \int \frac{1}{3\sqrt{\cosh^2 u - 1}}\,3\sinh u\ \mathrm{d}u$$
$$= \int \frac{3\sinh u}{3\sinh u}\,\mathrm{d}u$$
$$= \int 1\,\mathrm{d}u$$
$$= u + C$$
$$= \operatorname{arcosh}\left(\frac{x}{3}\right) + C$$

b You need $4x^2 = 25\sinh^2 u$, or $2x = 5\sinh u$, then $\mathrm{d}x = \frac{5}{2}\cosh u\,\mathrm{d}u$

$$\int \frac{1}{\sqrt{4x^2+25}}\,\mathrm{d}x = \int \frac{1}{\sqrt{25\sinh^2 u + 25}}\left(\frac{5}{2}\cosh u\right)\mathrm{d}u$$
$$= \frac{5}{2}\int \frac{\cosh u}{5\sqrt{\sinh^2 u + 1}}\,\mathrm{d}u$$
$$= \frac{1}{2}\int \frac{\cosh u}{\cosh u}\,\mathrm{d}u$$
$$= \frac{1}{2}\int 1\,\mathrm{d}u$$
$$= \frac{1}{2}u + C$$
$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{5}\right) + C$$

12 a $\int \frac{3}{\sqrt{x^2+9}}\,\mathrm{d}x = 3\operatorname{arsinh}\left(\frac{x}{3}\right) + C$

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$.

b $\int \frac{1}{\sqrt{x^2-2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{\sqrt{2}}\right) + C$

Using $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + C$

13 a
$$\int \frac{1}{\sqrt{4x^2-12}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4(x^2-3)}}\,\mathrm{d}x$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2-3}}\,\mathrm{d}x$$
$$= \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\sqrt{3}}\right) + C$$

13 b $\int \frac{1}{\sqrt{9x^2+16}}\,dx = \int \frac{1}{\sqrt{9\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,dx$$

$$= \frac{1}{3}\text{arsinh}\left(\frac{x}{\left(\frac{4}{3}\right)}\right)+C$$

$$= \frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)+C$$

14 $\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx = 3\int_1^2 \frac{1}{2\sqrt{\frac{1}{4}+x^2}}\,dx$

$$= \frac{3}{2}\left[\text{arsinh}\frac{x}{\left(\frac{1}{2}\right)}\right]_1^2$$

$$= \frac{3}{2}\left[\text{arsinh}\,(2x)\right]_1^2$$

$$= \frac{3}{2}\left[\text{arsinh}4-\text{arsinh}2\right]$$

$$= 0.977 \quad (3 \text{ s.f.})$$

15 Reminder: The logarithmic form of an inverse hyperbolic function is in the Edexcel formulae booklet.

a $\int_0^4 \frac{1}{\sqrt{x^2+16}}\,dx = \left[\text{arsinh}\left(\frac{x}{4}\right)\right]_0^4$

$$= \text{arsinh}1-\text{arsinh}0$$

$$= \ln\left(1+\sqrt{2}\right)$$

Using $\text{arsinh}x = \ln\left\{x+\sqrt{x^2+1}\right\}$

b $\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,dx = \left[\text{arcosh}\left(\frac{x}{12}\right)\right]_{13}^{15}$

$$= \text{arcosh}\left(\frac{5}{4}\right)-\text{arcosh}\left(\frac{13}{12}\right)$$

Using $\text{arcosh}x = \ln\left\{x+\sqrt{x^2-1}\right\}$

$$= \ln\left\{\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right\}-\ln\left\{\frac{13}{12}+\sqrt{\frac{169}{144}-1}\right\}$$

$$= \ln\left\{\frac{5}{4}+\sqrt{\frac{9}{16}}\right\}-\ln\left\{\frac{13}{12}+\sqrt{\frac{25}{144}}\right\}$$

$$= \ln 2-\ln\left(\frac{3}{2}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

Using $\ln a-\ln b = \ln\left(\frac{a}{b}\right)$

16 With $x = \sinh^2 u$ and $\mathrm{d}x = 2\sin u \cosh u \,\mathrm{d}u$,

and $\dfrac{x}{x+1} = \dfrac{\sinh^2 u}{\sinh^2 u + 1} = \dfrac{\sinh^2 u}{\cosh^2 u}$

$$\int \sqrt{\frac{x}{x+1}}\,\mathrm{d}x = \int \frac{\sinh u}{\cosh u} 2\sinh u \cosh u \,\mathrm{d}u$$
$$= \int 2\sinh^2 u \,\mathrm{d}u$$
$$= \int (\cosh 2u - 1)\,\mathrm{d}u$$
$$= \frac{\sinh 2u}{2} - u + C$$
$$= \sinh u \cosh u - \operatorname{arsinh}(\sqrt{x}) + C$$
$$= \sqrt{x}\sqrt{1+x} - \operatorname{arsinh}(\sqrt{x}) + C$$

$\sinh u = \sqrt{x}$ and
$\cosh u = \sqrt{1+\sinh^2 u}$

17 With $u = x^2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$,

$$\int_2^3 \frac{2x}{\sqrt{x^4 - 1}}\,\mathrm{d}x = \int_4^9 \frac{\mathrm{d}u}{\sqrt{u^2 - 1}}$$
$$= [\operatorname{arcosh} u]_4^9$$
$$= \operatorname{arcosh} 9 - \operatorname{arcosh} 4$$
$$= 0.824 \quad (3 \text{ s.f.})$$

18 Using $x = 2\cosh u$, $\mathrm{d}x = 2\sinh u \,\mathrm{d}u$

$$\int \sqrt{x^2 - 4}\,\mathrm{d}x = \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u \,\mathrm{d}u$$
$$= 4\int \sinh^2 u \,\mathrm{d}u$$
$$= 2\int (\cosh 2u - 1)\,\mathrm{d}u$$
$$= 2\left\{\frac{\sinh 2u}{2} - u\right\} + C$$
$$= 2\sinh u \cosh u - 2u + C$$
$$= 2\left(\sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$
$$= 2\left(\frac{\sqrt{x^2 - 4}}{2}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$
$$= \frac{1}{2}x\sqrt{x^2 - 4} - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$

$\cosh u = \dfrac{x}{2}$ and
$\sinh u = \sqrt{\cosh^2 u - 1}$

19 a $2\cosh x - \sinh x = 2\left(\dfrac{e^x + e^{-x}}{2}\right) - \left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{e^x + 3e^{-x}}{2}$

So $$\int \frac{1}{2\cosh x - \sinh x}\,\mathrm{d}x = \int \frac{2}{e^x + 3e^{-x}}\,\mathrm{d}x$$
$$= \int \frac{2e^x}{e^{2x} + 3}\,\mathrm{d}x$$

Multiplying numerator and denominator by e^x.

19 b Using the substitution $u = e^x$, $du = e^x\ dx$ and

$$\int \frac{2e^x}{e^{2x}+3}dx = 2\int \frac{du}{u^2+3}$$
$$= \frac{2}{\sqrt{3}}\text{artan}\left(\frac{u}{\sqrt{3}}\right)+C$$
$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)+C$$

20 With $u = \frac{2}{3}\sinh x$, $du = \frac{2}{3}\cosh x dx$ or $\cosh x dx = \frac{3}{2}du$

$$4\sinh^2 x + 9 = 4\left(\frac{3u}{2}\right)^2 + 9 = 9u^2 + 9 = 9(u^2+1)$$

$$\text{So} \int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x+9}}dx = \int_0^{\frac{2}{3}\sinh 1} \frac{1}{3\sqrt{u^2+1}} \times \frac{3}{2}du$$
$$= \left[\frac{1}{2}\text{arsinh}(u)\right]_0^{\frac{2}{3}\sinh 1}$$
$$= \frac{1}{2}\text{arsinh}\left(\frac{2}{3}\sinh 1\right)$$
$$= 0.360 \quad (3 \text{ s.f.})$$

21 a $x^2 - 4x - 12 = \{(x-2)^2 - 16\}$

$$\text{So} \int \frac{1}{\sqrt{x^2-4x-12}}dx = \int \frac{1}{\sqrt{(x-2)^2-16}}dx$$

Let $u = (x-2)$, so $du = dx$.

$$\text{Then} \int \frac{1}{\sqrt{x^2-4x-12}}dx = \int \frac{1}{\sqrt{u^2-16}}dx$$
$$= \text{arcosh}\left(\frac{u}{4}\right)+C$$
$$= \text{arcosh}\left(\frac{x-2}{4}\right)+C$$

b $x^2 + 6x + 10 = \{(x+3)^2 + 1\}$

$$\text{So} \int \frac{1}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{\sqrt{(x+3)^2+1}}dx$$

Let $u = (x+3)$, so $du = dx$.

$$\text{Then} \int \frac{1}{\sqrt{x^2+6x+10}}dx = \int \frac{1}{\sqrt{u^2+1}}du$$
$$= \text{arsinh}(u) + C$$
$$= \text{arsinh}(x+3) + C$$

21 c $2x^2+4x+7=2\left(x^2+2x+\frac{7}{2}\right)=2\left\{(x+1)^2+\frac{5}{2}\right\}$

Let $u=(x+1)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int \frac{1}{2x^2+4x+7}\,\mathrm{d}x=\frac{1}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2}\,\mathrm{d}u$

$$=\frac{1}{2}\left\{\frac{\sqrt{2}}{\sqrt{5}}\arctan\left(\frac{\sqrt{2}u}{\sqrt{5}}\right)\right\}+C$$

$$=\frac{\sqrt{10}}{10}\arctan\left(\frac{\sqrt{2}(x+1)}{\sqrt{5}}\right)+C$$

d $9x^2-8x+1=9\left(x^2-\frac{8}{9}x+\frac{1}{9}\right)=9\left\{\left(x-\frac{4}{9}\right)^2-\frac{7}{81}\right\}$

So $\int \frac{1}{\sqrt{9x^2-8x+1}}\,\mathrm{d}x=\frac{1}{3}\int \frac{1}{\sqrt{\left(x-\frac{4}{9}\right)^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,\mathrm{d}x$

Let $u=\left(x-\frac{4}{9}\right)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int \frac{1}{\sqrt{9x^2-8x+1}}\,\mathrm{d}x=\frac{1}{3}\int \frac{1}{\sqrt{u^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,\mathrm{d}u$

$$=\frac{1}{3}\operatorname{arcosh}\left(\frac{9u}{\sqrt{7}}\right)+C$$

$$=\frac{1}{3}\operatorname{arcosh}\left(\frac{9x-4}{\sqrt{7}}\right)+C$$

22 a $4x^2-12x+10=4\left(x^2-3x+\frac{5}{2}\right)=4\left\{\left(x-\frac{3}{2}\right)^2+\frac{1}{4}\right\}$

So $\int \frac{1}{\sqrt{4x^2-12x+10}}\,\mathrm{d}x=\frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\,\mathrm{d}x$

Let $u=\left(x-\frac{3}{2}\right)$, so $\mathrm{d}u=\mathrm{d}x$.

Then $\int \frac{1}{\sqrt{4x^2-12x+10}}\,\mathrm{d}x=\frac{1}{2}\int \frac{1}{\sqrt{u^2+\left(\frac{1}{2}\right)^2}}\,\mathrm{d}u$

$$=\frac{1}{2}\operatorname{arsinh}(2u)+C$$

$$=\frac{1}{2}\operatorname{arsinh}(2x-3)+C$$

22 b $4x^2-12x+4=4(x^2-3x+1)=4\left\{\left(x-\frac{3}{2}\right)^2-\frac{5}{4}\right\}$

So $\int\frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{2}\right)$, so $du=dx$.

Then $\int\frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{u^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,du$

$$=\frac{1}{2}\text{arcosh}\left(\frac{2u}{\sqrt{5}}\right)+C$$

$$=\frac{1}{2}\text{arcosh}\left(\frac{2x-3}{\sqrt{5}}\right)+C$$

23 $x^2+2x+5=(x+1)^2+4$

So $\int_0^1\frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_0^1\frac{1}{\sqrt{(x+1)^2+4}}\,dx$

Let $u=(x+1)$, so $du=dx$.

Then $\int_0^1\frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_1^2\frac{1}{\sqrt{u^2+2^2}}\,du$

$$=\left[\text{arsinh}\left(\frac{u}{2}\right)\right]_1^2$$

$$=\left[\text{arsinh}1-\text{arsinh}\left(\frac{1}{2}\right)\right]$$

$$=0.400\quad(3\text{ s.f.})$$

24 $x^2-2x+2=(x-1)^2+1$

So $\int_1^3\frac{1}{\sqrt{x^2-2x+2}}\,dx=\int_1^3\frac{1}{\sqrt{(x-1)^2+1}}\,dx$

$$=\left[\text{arsinh}(x-1)\right]_1^3$$

$$=\text{arsinh}2$$

$$=\ln\{2+\sqrt{5}\}$$

$ar\sinh x=\ln\{x+\sqrt{x^2+1}\}$

25 $3x^2-6x+7=3\left(x^2-2x+\frac{7}{3}\right)=3\left\{(x-1)^2+\frac{4}{3}\right\}$

So $\int\frac{1}{\sqrt{3x^2-6x+7}}\,dx=\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{(x-1)^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$

Let $u=(x-1)$, so $du=dx$.

Then $\int_1^3\frac{1}{\sqrt{3x^2-6x+7}}=\frac{1}{\sqrt{3}}\int_0^2\frac{1}{\sqrt{u^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,du$

$=\frac{1}{\sqrt{3}}\left[\text{arsinh}\left(\frac{\sqrt{3}u}{2}\right)\right]_0^2$

$=\frac{1}{\sqrt{3}}\text{arsinh}\sqrt{3}$

$=\frac{1}{\sqrt{3}}\ln\{\sqrt{3}+\sqrt{3+1}\}$ $\quad$ $\text{arsinh}x=\ln\{x+\sqrt{x^2+1}\}$

$=\frac{1}{\sqrt{3}}\ln\{2+\sqrt{3}\}$

26 a In order to find the intersection point of the two curves, we solve

$5\cosh x=7-\sinh x$

$\frac{5}{2}\left(e^x+e^{-x}\right)=7-\frac{1}{2}\left(e^x-e^{-x}\right)$

$\frac{5}{2}\left(e^{2x}+1\right)=7e^x-\frac{1}{2}\left(e^{2x}-1\right)$

$3e^{2x}-7e^x+2=0.$

Set $u=e^x$ in order to clearly see the quadratic equation $3u^2-7u+2=0$ with solutions

$u=\frac{7\pm\sqrt{49-24}}{6}$

$u=\frac{7\pm5}{6}$

i.e.

$u_1=\frac{1}{3}$

$u_2=2.$

This gives us the x values for the intersection points as

$x_1=\ln\frac{1}{3}=-\ln 3$

$x_2=\ln 2$

26 b We find the area of the region R by calculating

$$Area_R = \int_{x_1}^{x_2} (7 - \sinh x)\,dx - \int_{x_1}^{x_2} (5\cosh x)\,dx$$

$$= \int_{x_1}^{x_2} (7 - \sinh x - 5\cosh x)\,dx$$

$$= [7x - \cosh x - 5\sinh x]_{-\ln 3}^{\ln 2}$$

$$= (7\ln 2 - 5) - (-7\ln 3 + 5)$$

$$= 7\ln 6 - 10$$

$$= \ln 279936 - 10.$$

27 Volume $= \pi\int_0^1 \sinh^2 x\,dx = \frac{\pi}{2}\int_0^1 (\cosh 2x - 1)dx$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2 - 1\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{4}(e^2 - e^{-2}) - 1\right]$$

$$= \frac{\pi}{8}\left[e^2 - 4 - e^{-2}\right]$$

$$= \frac{\pi}{8e^2}(e^4 - 4e^2 - 1).$$

Challenge

1 Using the substitution $x = 1 + \sinh\theta$, $dx = \cosh\theta\ d\theta$

$x^2 - 2x + 2 = (\sinh^2\theta + 2\sinh\theta + 1) - 2(\sinh\theta + 1) + 2 = \sinh^2\theta + 1 = \cosh^2\theta$

So $\displaystyle\int \frac{1}{(x^2 - 2x + 2)^{\frac{3}{2}}}dx = \int \frac{1}{\cosh^3\theta}\cdot\cosh\theta\,d\theta$

$$= \int \text{sech}^2\theta\,d\theta$$

$$= \tanh\theta + C$$

$$= \frac{x-1}{\sqrt{x^2 - 2x + 2}} + C$$

$\sinh\theta = x - 1$
$\cosh\theta = \sqrt{1 + \sinh^2\theta} = \sqrt{2 - 2x + x^2}$

Challenge

2 Using the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$,

a So $\int x\cosh^2(x^2)\,\mathrm{d}x = \frac{1}{2}\int \cosh^2 \mathrm{u}\,\mathrm{d}u$

$$= \frac{1}{4}\int (\cosh 2u + 1)\,\mathrm{d}u$$

$$= \frac{1}{8}\sinh 2u + \frac{u}{4} + C$$

$$= \frac{1}{8}\sinh(2x^2) + \frac{x^2}{4} + C$$

b So $\int \frac{x}{\cosh^2(x^2)}\mathrm{d}x = \int x\,\mathrm{sech}^2(x^2)\,\mathrm{d}x$

$$= \frac{1}{2}\int \mathrm{sech}^2 u\ \mathrm{d}u$$

$$= \frac{1}{2}\tanh u + C$$

$$= \frac{1}{2}\tanh(x^2) + C$$