Hyperbolic Functions Mixed Exercise 6

1 a $\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

b $\cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2}$

$= \frac{5 + \frac{1}{5}}{2} = \frac{13}{5}$

$e^{\ln 5} = 5$, and $e^{-\ln 5} = e^{\ln 5^{-1}} = \frac{1}{5}$

c $\tanh\left(\ln \frac{1}{4}\right) = \frac{e^{2\ln\frac{1}{4}} - 1}{e^{2\ln\frac{1}{4}} + 1}$

$= \frac{\left(\frac{1}{16} - 1\right)}{\left(\frac{1}{16} + 1\right)}$

$= -\frac{15}{17}$

$e^{2\ln\frac{1}{4}} = e^{\ln\left(\frac{1}{4}\right)^2} = \frac{1}{16}$

2 $\text{artanh}x - \text{artanh}y$

$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$

Use $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$= \frac{1}{2}\ln\left(\frac{1+x}{1-x} \times \frac{1-y}{1+y}\right)$

$= \frac{1}{2}\ln\left(\frac{1+x-y-xy}{1-x+y-xy}\right)$

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$

$= \ln\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)}$

So $\sqrt{\left(\frac{1+x-y-xy}{1-x+y-xy}\right)} = 5$

$\frac{1+x-y-xy}{1-x+y-xy} = 25$

$1+x-y-xy = 25 - 25x + 25y - 25xy$

$24xy - 26y = 24 - 26x$

$y(12x - 13) = 12 - 13x$

$y = \frac{12 - 13x}{12x - 13}$

3 RHS $= \sinh A \cosh B - \cosh A \sinh B$

$$= \left(\frac{e^{A} - e^{-A}}{2}\right)\left(\frac{e^{B} + e^{-B}}{2}\right) - \left(\frac{e^{A} + e^{-A}}{2}\right)\left(\frac{e^{B} - e^{-B}}{2}\right)$$

$$= \frac{e^{A+B} - e^{-A+B} + e^{A-B} - e^{-A-B}}{4} - \frac{e^{A+B} + e^{-A+B} - e^{A-B} - e^{-A-B}}{4}$$

$$= \frac{2(e^{A-B} - e^{-A+B})}{4}$$

$$= \frac{e^{A-B} - e^{-(A-B)}}{2}$$

$$= \sinh(A - B) = \text{LHS}$$

4 RHS $= \dfrac{2\tanh\frac{1}{2}x}{1 - \tanh^2\frac{1}{2}x}$

$$2\tanh\frac{1}{2}x = \frac{2(e^{x} - 1)}{e^{x} + 1}$$

$$1 - \tanh^2\frac{1}{2}x = 1 - \left(\frac{e^{x} - 1}{e^{x} + 1}\right)^2$$

$$= \frac{(e^{x} + 1)^2 - (e^{x} - 1)^2}{(e^{x} + 1)^2}$$

$$= \frac{4e^{x}}{(e^{x} + 1)^2}$$

So RHS $= \dfrac{2(e^{x} - 1)}{e^{x} + 1} \times \dfrac{(e^{x} + 1)^2}{4e^{x}}$

$$= \frac{(e^{x} - 1)(e^{x} + 1)}{2e^{x}}$$

$$= \frac{e^{2x} - 1}{2e^{x}}$$

$$= \frac{e^{x} - e^{-x}}{2}$$

$$= \sinh x = \text{LHS}$$

5 $9\cosh x - 5\sinh x = 15$

$$9\frac{\left(e^{x}+e^{-x}\right)}{2} - 5\frac{\left(e^{x}-e^{-x}\right)}{2} = 15$$

$$9e^{x} + 9e^{-x} - 5e^{x} + 5e^{-x} = 30$$

$$4e^{x} - 30 + 14e^{-x} = 0$$

$$2e^{x} - 15 + 7e^{-x} = 0$$

Multiply throughout by e^x.

$$2e^{2x} - 15e^{x} + 7 = 0$$

$$(2e^{x}-1)(e^{x}-7) = 0$$

Solve as a quadratic in e^x.

$$e^{x} = \frac{1}{2},\ e^{x} = 7$$

$$x = \ln\left(\frac{1}{2}\right),\ x = \ln 7$$

6 $23\sinh x - 17\cosh x + 7 = 0$

$$23\frac{(e^{x}-e^{-x})}{2} - 17\frac{(e^{x}+e^{-x})}{2} + 7 = 0$$

$$23e^{x} - 23e^{-x} - 17e^{x} - 17e^{-x} + 14 = 0$$

$$6e^{x} + 14 - 40e^{-x} = 0$$

$$3e^{x} + 7 - 20e^{-x} = 0$$

$$3e^{2x} + 7e^{x} - 20 = 0$$

Multiply throughout by e^x.

$$(3e^{x}-5)(e^{x}+4) = 0$$

$$e^{x} = \frac{5}{3}$$

$e^{x} = -4$ is not possible for real x.

$$x = \ln\left(\frac{5}{3}\right)$$

7 $3\cosh^2 x + 11\sinh x = 17$

Using $\cosh^2 x - \sinh^2 x = 1$

$$3(1+\sinh^2 x) + 11\sinh x = 17$$

$$3\sinh^2 x + 11\sinh x - 14 = 0$$

$$(3\sinh x + 14)(\sinh x - 1) = 0$$

$$\sinh x = -\frac{14}{3},\ \sinh x = 1$$

$$x = \operatorname{arsinh}\left(-\frac{14}{3}\right),\ x = \operatorname{arsinh}1$$

Use $\operatorname{arsinh}x = \ln(x+\sqrt{x^2+1})$.

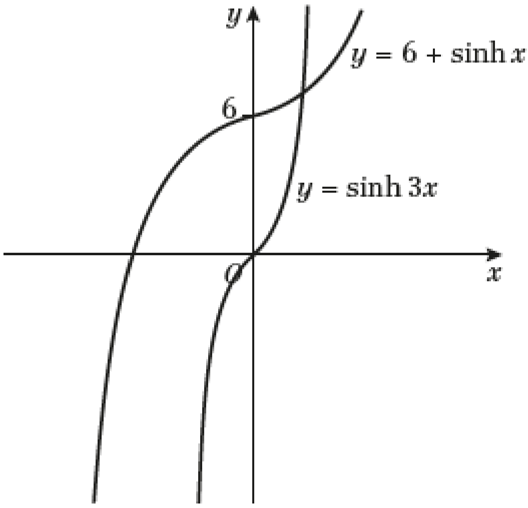
$$x = \ln\left(-\frac{14}{3} + \sqrt{\frac{196}{9}+1}\right)$$

$$= \ln\left(\frac{-14+\sqrt{205}}{3}\right)$$

$$x = \ln(1+\sqrt{1+1})$$

$$= \ln(1+\sqrt{2})$$

8 a



b At the interaction,

$$6+\sinh x = \sinh 3x$$
$$6+\sinh x = 3\sinh x + 4\sinh^3 x$$
$$4\sinh^3 x + 2\sinh x - 6 = 0$$
$$2\sinh^3 x + \sinh x - 3 = 0$$
$$(\sinh x - 1)(2\sinh^2 x + 2\sinh x + 3) = 0$$

You can see, by inspection that $\sinh x = 1$ satisfies this equation.

The equation $2\sinh^2 x + 2\sinh x + 3 = 0$ has no real roots, because $b^2 - 4ac = 4 - 24 < 0$.

The only intersection is where $\sinh x = 1$

For $\sinh x = 1$,

$$x = \text{arsinh}1$$
$$= \ln(1+\sqrt{1^2+1})$$
$$= \ln(1+\sqrt{2})$$

Using $y = 6+\sinh x$

$y = 7$

Coordinates of the point of intersection are $(\ln(1+\sqrt{2}),\ 7)$

9 a $13\cosh x + 5\sinh x = R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

So $R\cosh\alpha = 13$

$R\sinh\alpha = 5$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 13^2 - 5^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 144$

$R^2 = 144$

$R = 12$

$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{5}{13}$

$\tanh\alpha = \frac{5}{13}$

$\alpha = 0.405$ ← Direct from calculator.

b $13\cosh x + 5\sinh x = 12\cosh(x + 0.405)$ ← For any value A, $\cosh A \geqslant 1$.

The minimum value of $13\cosh x + 5\sinh x$ is 12.

10 a $3\cosh x + 5\sinh x = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$

So $R\cosh\alpha = 5$

$R\sinh\alpha = 3$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 5^2 - 3^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 16$

$R^2 = 16$

$R = 4$

$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{3}{5}$

$\tanh\alpha = \frac{3}{5}$

$\alpha = 0.693$ ← Direct from calculator.

$3\cosh x + 5\sinh x = 4\sinh(x + 0.693)$

b $4\sinh(x + 0.693) = 8$

$\sinh(x + 0.693) = 2$

$x + 0.693 = \text{arsinh}2$

$= 1.44 \quad (3\text{ s.f.})$ ← Direct from calculator.

$x = 0.75 \quad (2\text{ d.p.})$

10 c $3\cosh x + 5\sinh = 8$

$$3\frac{(e^x + e^{-x})}{2} + 5\frac{(e^x - e^{-x})}{2} = 8$$

$$3e^x + 3e^{-x} + 5e^x - 5e^{-x} = 16$$

$$8e^x - 16 - 2e^{-x} = 0$$

$$4e^x - 8 - e^{-x} = 0$$

Multiply throughout by e^x.

$$4e^{2x} - 8e^x - 1 = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{8 \pm \sqrt{64+16}}{8}$$

$$e^x = 1 \pm \frac{\sqrt{80}}{8} = 1 \pm \frac{\sqrt{5}}{2}$$

$$e^x = 1 + \frac{\sqrt{5}}{2}$$

$e^x = 1 - \frac{\sqrt{5}}{2}$ is negative, so not possible for real x.

$$x = \ln\left(1 + \frac{\sqrt{5}}{2}\right)$$

$$= 0.75 \quad (2 \text{ d.p.})$$

11 $y = \cosh 2x$

$$\frac{dy}{dx} = 2\sinh 2x$$

12 a $y = \text{arsinh}3x$

Let $t = 3x$ $\quad y = \text{arsinh}t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$

$$= \frac{3}{\sqrt{9x^2+1}}$$

b $y = \text{arsinh}x^2$

Let $t = x^2$ $\quad y = \text{arsinh}t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}} \times 2x$$

$$= \frac{2x}{\sqrt{x^4+1}}$$

12 c $y = \text{arcosh}\dfrac{x}{2}$

Let $t = \dfrac{x}{2}$ $\quad y = \text{arcosh}t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2 - 1}} \times \frac{1}{2}$$

$$= \frac{1}{2\sqrt{\frac{x^2}{4} - 1}} = \frac{1}{\sqrt{x^2 - 4}}$$

d $y = x^2\text{arcosh}2x$

$$\frac{dy}{dx} = 2x\text{arcosh}2x + x^2 \times \frac{2}{\sqrt{4x^2 - 1}}$$

$$= 2x\left(\text{arcosh}2x + \frac{x}{\sqrt{4x^2 - 1}}\right)$$

13 $y = (\text{arsinh}x)^2$

$$\frac{dy}{dx} = \frac{2(\text{arsinh}x)^1}{\sqrt{x^2 + 1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2}{\sqrt{x^2+1}} \times \sqrt{x^2 + 1} - \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \times 2x \times 2\text{arsinh}x}{\left(\sqrt{x^2 + 1}\right)^2}$$

$$(x^2 + 1)\frac{d^2y}{dx^2} = 2 - 2x(x^2 + 1)^{-\frac{1}{2}}\text{arsinh}x$$

$$= 2 - x\frac{dy}{dx}$$

$$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$$

14 a Differentiating with respect to x gives us $f'(x) = 5\sinh x - 3\cosh x$.

b $y = f(x)$ has a turning point when $f'(x) = 0$.

$f'(x) = 5\sinh x - 3\cosh x = 0$

$5\sinh x = 3\cosh x$

$5\tanh x = 3$

$\tanh x = \dfrac{3}{5}$

$x = \text{ar}\tanh\left(\dfrac{3}{5}\right)$

$x = \ln 2$

Thus, $x = \ln 2$, and $y = 5\cosh(\ln 2) - 3\sinh(\ln 2)$

Therefore $(\ln 2, 4)$ are the coordinates of the turning point.

15 $y = \text{arcosh}(\sinh 2x)$

Let $t = \sinh 2x \quad y = \text{arcosh}t$

$\frac{dt}{dx} = 2\cosh 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{t^2 - 1}} \times 2\cosh 2x$

$= \frac{2\cosh 2x}{\sqrt{\sinh^2 2x - 1}}$

16 a Differentiating y with respect to x four times and applying the product rule where necessary gives

$\frac{dy}{dx} = \cos x \cosh x + \sin x \sinh x,$

$\frac{d^2y}{dx^2} = 2\cos x \sinh x,$

$\frac{d^3y}{dx^3} = -2\sin x \sinh x + 2\cos x \cosh x,$

$\frac{d^4y}{dx^4} = -4\sin x \cosh x.$

So, $\frac{d^4y}{dx^4} = -4y$

b Evaluating the expressions we found in part a at $x = 0$ gives

$y(0) = 0,$

$\frac{dy}{dx}(0) = 1,$

$\frac{d^2y}{dx^2}(0) = 0,$

$\frac{d^3y}{dx^3}(0) = 2,$

$\frac{d^4y}{dx^4}(0) = 0.$

Since we know that $\frac{d^4y}{dx^4} = -4y$, we conclude $\frac{d^5y}{dx^5} = -4$.

This gives us the first three terms in the Maclaurin expansion as

$y \approx 0 + 1x + \frac{0x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{-4x^5}{5!}$

$y \approx x + \frac{x^3}{3} - \frac{x^5}{30}$

17 Differentiating y with respect to x five times and applying the product rule where necessary give

$$\frac{dy}{dx} = 2\cosh 2x\cosh x + \sinh 2x\sinh x,$$

$$\frac{d^2y}{dx^2} = 5\sinh 2x\cosh x + 4\cosh 2x\sinh x,$$

$$\frac{d^3y}{dx^3} = 14\cosh 2x\cosh x + 13\sinh 2x\sinh x,$$

$$\frac{d^4y}{dx^4} = 41\sinh 2x\cosh x + 40\cosh 2x\sinh x,$$

$$\frac{d^5y}{dx^5} = 122\cosh 2x\cosh x + 121\sinh 2x\sinh x.$$

Evaluating these expressions at 0 gives

$$y(0) = 0$$

$$\frac{dy}{dx}(0) = 2$$

$$\frac{d^2y}{dx^2}(0) = 0$$

$$\frac{d^3y}{dx^3}(0) = 14$$

$$\frac{d^4y}{dx^4}(0) = 0$$

$$\frac{d^5y}{dx^5}(0) = 122$$

So $y \approx 2x + \frac{7x^3}{3} + \frac{61x^5}{60}$.

18 $4x^2 + 4x + 17 \equiv (ax+b)^2 + c, \quad a > 0$

a $4x^2 + 4x + 17 \equiv (2x+b)^2 + c \quad a = 2$

$\equiv 4x^2 + 4bx + b^2 + c$

Comparing coefficient of x: $b = 1$

Comparing constant term: $17 = 1 + c \Rightarrow c = 16$

18 b Using **a**, $\int \frac{1}{4x^2+4x+17}\,dx = \int \frac{1}{(2x+1)^2+16}\,dx$

Let $2x+1 = 4\tan\theta$, then $2dx = 4\sec^2\theta\, d\theta$

and $\int \frac{1}{(2x+1)^2+16}\,dx = \int \frac{2\sec^2\theta}{16\tan^2\theta+16}\,d\theta$

$$= \int \frac{2\sec^2\theta}{16\sec^2\theta}\,d\theta$$

$$= \frac{1}{8}\theta$$

$$= \frac{1}{8}\arctan\left(\frac{2x+1}{4}\right)(+C)$$

So $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,dx = \frac{1}{8}[\arctan 1 - \arctan 0]$

$$= \frac{\pi}{32}$$

19 Using the definitions of $\sinh 4x$ and $\cosh 6x$

$$\int \sinh 4x \cosh 6x\,dx = \int \left(\frac{e^{4x}-e^{-4x}}{2}\right)\left(\frac{e^{6x}+e^{-6x}}{2}\right)dx$$

$$= \frac{1}{4}\int \left(e^{10x}+e^{-2x}-e^{2x}-e^{-10x}\right)dx$$

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10}+\frac{e^{-2x}}{-2}-\frac{e^{2x}}{2}-\frac{e^{-10x}}{-10}\right\}+C$$

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10}+\frac{e^{-10x}}{10}-\frac{e^{2x}}{2}-\frac{e^{-2x}}{2}\right\}+C$$

$$= \frac{1}{20}\cosh 10x - \frac{1}{4}\cosh 2x + C$$

> You could use hyperbolic identities to split up into a difference of two $\sinh s$.

> as $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

b You cannot use by parts for $\int e^x \sinh x\,dx$

Using the definition of $\sinh x$

$$\int e^x \sinh x\,dx = \int e^x\left(\frac{e^x-e^{-x}}{2}\right)dx$$

$$= \frac{1}{2}\int \left(e^{2x}-1\right)dx$$

$$= \frac{1}{2}\left(\frac{1}{2}e^{2x}-x\right)+C$$

$$= \frac{1}{4}e^{2x}-\frac{1}{2}x+C$$

20 Area under curve $= \int_0^5 y\,\mathrm{d}x = \int_0^5 \frac{10}{\sqrt{4x^2+9}}\,\mathrm{d}x$

$$= 5\int_0^5 \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,\mathrm{d}x$$

$$= 5\left[\text{arsinh}\left(\frac{2x}{3}\right)\right]_0^5$$

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \text{arsinh}\left(\frac{x}{a}\right)$

$$= 5\,\text{arsinh}\left(\frac{10}{3}\right)$$

'Real' area $= 5\,\text{arsinh}\left(\frac{10}{3}\right) \times 100 = 960\ \text{m}^2$.

21 $x^2 - 2x + 10 = (x-1)^2 + 9$

So $\int \frac{\mathrm{d}x}{\sqrt{x^2-2x+10}} = \int \frac{\mathrm{d}x}{\sqrt{(x-1)^2+9}}$

Let $x-1 = 3\sinh u$, then $\mathrm{d}x = 3\cosh u\,\mathrm{d}u$

so $\int \frac{\mathrm{d}x}{\sqrt{x^2-2x+10}} = \int \frac{3\cosh u}{3\cosh u}\,\mathrm{d}u$

$$= u + C$$

$$= \text{arsinh}\left(\frac{x-1}{3}\right) + C$$

22 a Using the exponential forms

$$\int \frac{1}{\sinh x + 2\cosh x}\,\mathrm{d}x = \int \frac{1}{\left(\frac{e^x-e^{-x}}{2}\right) + 2\left(\frac{e^x+e^{-x}}{2}\right)}\,\mathrm{d}x$$

$$= \int \frac{2}{3e^x + e^{-x}}\,\mathrm{d}x$$

$$= \int \frac{2e^x}{3e^{2x}+1}\,\mathrm{d}x$$

Using the substitution $u = e^x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = e^x = u$

So $\int \frac{2e^x}{3e^{2x}+1}\,\mathrm{d}x = \int \frac{2u}{3u^2+1}\left(\frac{\mathrm{d}u}{u}\right) = \int \frac{2}{3u^2+1}\,\mathrm{d}u$

$$= \frac{2}{3}\int \frac{1}{u^2+\frac{1}{3}}\,\mathrm{d}u$$

$$= \frac{2}{3}\left(\sqrt{3}\right)\arctan\left(\sqrt{3}u\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}e^x\right) + C$$

22 b $x^2-2x+10=(x-1)^2+9$

So let $x-1=3\sinh u$, then $\mathrm{d}x=3\cosh u\ \mathrm{d}u$

$$\text{and}\int\frac{3x-1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x=\int\frac{9\sinh u+2}{\sqrt{9\sinh^2 u+9}}3\cosh u\,\mathrm{d}u$$

$$=\int\frac{9\sinh u+2}{3\cosh u}3\cosh u\,\mathrm{d}u$$

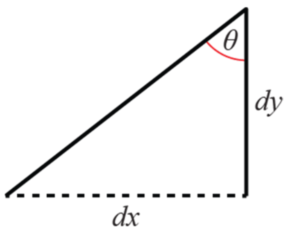
$$=9\cosh u+2u+C$$

$$=9\sqrt{1+\left(\frac{x-1}{3}\right)^2}+2\operatorname{arsinh}\left(\frac{x-1}{3}\right)+C$$

$$\text{So}\int_1^4\frac{3x-1}{\sqrt{x^2-2x+10}}=\left[9\sqrt{2}+2\operatorname{arsinh}1\right]-[9]$$

$$=9\left(\sqrt{2}-1\right)+2\operatorname{arsinh}1$$

23 a An infinitesimal region close to the point of intersection can be seen in this diagram.



So $\tan\theta=\dfrac{\mathrm{d}x}{\mathrm{d}y}$.

We differentiate with respect to x and obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}\sinh\left(\frac{x}{3}\right)$$

That is, $\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{3}{\sinh\left(\frac{x}{3}\right)}$.

So at $x=4$,

$$\theta=\arctan\left(\frac{3}{\sinh\left(\frac{4}{3}\right)}\right)\approx 59.5°\ \ (3\text{s.f.})$$

b $\displaystyle\int_{-3}^{4}\cosh\left(\frac{x}{3}\right)\mathrm{d}x=\left[3\sinh\left(\frac{x}{3}\right)\right]_{-3}^{4}=8.82\,\mathrm{m}^2.$

24 a $3\cosh 2x = \frac{3}{2}(e^{2x} + e^{-2x}),\ 8 + \sinh 2x = 8 + \frac{1}{2}(e^{2x} - e^{-2x}),$

we find the intersection points by setting these expressions equal to each other and solving for x.

$\frac{3}{2}(e^{2x} + e^{-2x}) = 8 + \frac{1}{2}(e^{2x} - e^{-2x})$

$e^{2x} - 8 + 2e^{-2x} = 0.$

Multiplying through by e^{2x} and using the substitution $z = e^{2x}$, we obtain the quadratic equation $z^2 - 8z + 2 = 0$, with solutions $z_1 = 4 - \sqrt{14},\ z_2 = 4 + \sqrt{14}.$

Therefore, the solutions are $x_1 = \frac{1}{2}\ln\left(4 - \sqrt{14}\right)$ and $x_2 = \frac{1}{2}\ln\left(4 + \sqrt{14}\right)$.

b $\int_{\frac{1}{2}\ln\left(4-\sqrt{14}\right)}^{\frac{1}{2}\ln\left(4+\sqrt{14}\right)} (8 + \sinh 2x - 3\cosh 2x)\ \mathrm{d}x$

$= \left[8x + \frac{1}{2}\cosh 2x - \frac{3}{2}\sinh 2x\right]_{\frac{1}{2}\ln\left(4-\sqrt{14}\right)}^{\frac{1}{2}\ln\left(4+\sqrt{14}\right)}$

≈ 6.12 (3s.f.)

25 First we find the cross-sectional area of the loaf. Note that since a unit is 5cm, we will have a number which we will multiply by $(5\text{cm})^2 = 25\text{cm}^2$ as opposed to the usual cm^2. Let A denote cross-sectional area.

$$A = \int_{-2}^{2} \frac{5}{\sqrt{x^2 + 4}}\ \mathrm{d}x$$
$$= \frac{5}{2}\int_{-2}^{2} \frac{1}{\sqrt{(\frac{x}{2})^2 + 1}}\ \mathrm{d}x$$

Let $\frac{x}{2} = \sinh u$, so that the integral becomes

$$A = \frac{5}{2}\int_{\operatorname{arsinh}(-1)}^{\operatorname{arsinh}(1)} \frac{1}{\sqrt{(\sinh u)^2 + 1}} \times 2\cosh u\ \mathrm{d}u$$
$$= 5\int_{\operatorname{arsinh}(-1)}^{\operatorname{arsinh}(1)} 1\,\mathrm{d}u$$
$$= 5\left[u\right]_{\operatorname{arsinh}(-1)}^{\operatorname{arsinh}(1)}$$
$$= 10\operatorname{arsinh}(1)\ \text{units}^2$$
$$= 10\operatorname{arsinh}(1)\ (5\text{cm})^2$$
$$= 250\operatorname{arsinh}(1)\ \text{cm}^2.$$

Then the volume, V, is calculated by multiplying the cross-sectional area in centimetres by the length in centimetres.

$$V = 250\operatorname{arsinh}(1) \times 30$$
$$= 7500\operatorname{arsinh}(1)$$
$$\approx 6610\,\text{cm}^3$$

26 a $y = 3 - \sinh x$ crosses the $x-$axis
when $x = \text{arsinh}(3)$

$$= \ln\left(3 + \sqrt{3^2 + 1}\right)$$

$$= \ln\left(3 + \sqrt{10}\right).$$

So the coordinates are $\left(\ln\left(3 + \sqrt{10}\right), 0\right)$.

b The volume of revolution, V, is found by

$$V = \int_0^{\ln\left(3+\sqrt{10}\right)} \pi(3 - \sinh x)^2 \, dx$$

$$V = \int_0^{\ln\left(3+\sqrt{10}\right)} \pi\left(3 - \frac{1}{2}(e^x - e^{-x})\right)^2 \, dx$$

$$= \int_0^{\ln\left(3+\sqrt{10}\right)} \pi\left(9 - 3(e^x - e^{-x}) + \frac{1}{4}(e^{2x} - 2 + e^{-2x})\right) dx$$

$$= \pi\left[9x - 3(e^x + e^{-x}) + \frac{1}{4}\left(\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right)\right]_0^{\ln\left(3+\sqrt{10}\right)}$$

$$\approx 22.7 \quad \text{(3s.f.)}$$

Challenge

First we find the area between two arbitrarily large values equidistant from the origin; we will call these values $-R$ and R.

So area $= \int_{-R}^{R} \operatorname{sech} x \, dx$

We convert to exponential form and use the substitution $u = e^x$, noting that $\frac{du}{dx} = e^x = u$.

$$\int_{-R}^{R} \operatorname{sech} x \, dx = \int_{-R}^{R} \frac{2}{e^x + e^{-x}} \, dx$$
$$= \int_{-R}^{R} \frac{2e^x}{e^{2x} + 1} \, dx$$
$$= \int_{e^{-R}}^{e^R} \frac{2u}{u^2 + 1} \, \frac{du}{u}$$
$$= 2\int_{e^{-R}}^{e^R} \frac{1}{u^2 + 1} \, du$$
$$= 2\Big[\arctan(u)\Big]_{e^{-R}}^{e^R}$$
$$= 2\Big[\arctan(e^x)\Big]_{-R}^{R}$$

Now that we have this expression for arbitrarily large R, we take the limit as $R \to \infty$.

$$A = 2\lim_{R\to\infty}\Big[\arctan(e^x)\Big]_{-R}^{R}$$
$$= 2\lim_{R\to\infty}\left(\arctan(e^R) - \arctan(e^{-R})\right)$$
$$= 2\left(\frac{\pi}{2} - 0\right)$$
$$= \pi$$