Methods in differential equations – mixed exercise 7

1 The integrating factor is $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(y \sec x) = 2\sec^2 x$$

$$\Rightarrow y \sec x = \int 2\sec^2 x \, dx = 2\tan x + c$$

$$\Rightarrow y = 2\sin x + c\cos x$$

2 Rewrite the equation as $\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{5x}{1-x^2}$

The integrating factor is $e^{\int \frac{x}{1-x^2} dx} = e^{-\frac{1}{2}\ln(1-x^2)} = e^{\ln(1-x^2)^{-\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}}$

Multiplying the equation by this factor gives:

$$\frac{1}{(1-x^2)^{\frac{1}{2}}} \frac{dy}{dx} + \frac{x}{(1-x^2)^{\frac{3}{2}}} y = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \int \frac{5x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5}{(1-x^2)^{\frac{1}{2}}} + c$$

So $y = 5 + c(1-x^2)^{\frac{1}{2}}$

3 Rewrite the equation as $x\frac{dy}{dx} + y = -x$

The left-hand side is the derivative of the product xy, so the equation is equivalent to

$$\frac{d}{dx}(xy) = -x$$

$$\Rightarrow xy = -\int x dx = -\frac{x^2}{2} + c$$

So $y = -\frac{x}{2} + \frac{c}{x}$

4 The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = x\sqrt{x}$$

$$\Rightarrow \frac{d}{dx}(xy) = x\sqrt{x}$$

$$\Rightarrow xy = \int x\sqrt{x}\,dx = \frac{2}{5}x^{\frac{5}{2}} + c$$

So $y = \frac{2}{5}x^{\frac{3}{2}} + \frac{c}{x}$

5 The integrating factor is $e^{\int 2x\,dx} = e^{x^2}$

Multiplying the equation by this factor gives:

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

$$\Rightarrow e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$$

So $y = \frac{1}{2} + ce^{-x^2}$

6 Rewrite the equation in the form $\dfrac{dy}{dx} + \dfrac{2x^2 - 1}{x(1-x^2)} y = \dfrac{2x^2}{(1-x^2)}$

The integrating factor is $e^{\int \frac{2x^2-1}{x(1-x^2)} dx}$ and this simplifies as follows:

$$\int \frac{2x^2-1}{x(1-x^2)} dx = \int \frac{3x^2 - 1 - x^2}{x - x^3} dx = \int \frac{3x^2-1}{x-x^3} - \frac{x}{1-x^2} dx$$

$$= -\ln(x - x^3) + \frac{1}{2}\ln(1-x^2) = \ln(1-x^2)^{\frac{1}{2}} - \ln(x - x^3)$$

$$= \ln\left(\frac{(1-x^2)^{\frac{1}{2}}}{(x-x^3)}\right) = \ln\left(\frac{1}{x(1-x^2)^{\frac{1}{2}}}\right)$$

So the integrating factor simplifies to $\dfrac{1}{x(1-x^2)^{\frac{1}{2}}}$

Multiplying the equation by this factor gives:

$$\frac{1}{x(1-x^2)^{\frac{1}{2}}} \frac{dy}{dx} + \frac{2x^2-1}{x^2(1-x^2)^{\frac{3}{2}}} y = \frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x(1-x^2)^{\frac{1}{2}}}\right) = \frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{y}{x(1-x^2)^{\frac{1}{2}}} = \int \frac{2x}{(1-x^2)^{\frac{3}{2}}} dx = \frac{2}{(1-x^2)^{\frac{1}{2}}} + c$$

So $y = 2x + cx(1-x^2)^{\frac{1}{2}} = 2x + cx\sqrt{1-x^2}$

7 a Given that $\dfrac{dy}{dx} - ay = ke^{\lambda x}$, the integrating factor is $e^{\int -a\,dx} = e^{-ax}$

Multiplying the equation by this factor gives:

$$e^{-ax}\frac{dy}{dx} - ae^{-ax} y = ke^{\lambda x} e^{-ax}$$

$$\Rightarrow \frac{d}{dx}(ye^{-ax}) = ke^{\lambda x}e^{-ax}$$

$$\Rightarrow ye^{-ax} = \int ke^{(\lambda - a)x}\, dx = \frac{ke^{(\lambda - a)x}}{\lambda - a} e^{(\lambda - a)x} + c$$

So $y = \dfrac{ke^{(\lambda - a)x} e^{ax}}{\lambda - a} + ce^{ax} = \dfrac{ke^{\lambda x}}{\lambda - a} + ce^{ax}$ for $\lambda \neq a$

b When $Q(x) = kx^n e^{ax}$

$$\frac{d}{dx}(ye^{-ax}) = ke^{\lambda x}e^{ax}e^{-ax} = ke^{\lambda x}$$

$$\Rightarrow ye^{-ax} = \int kx^n dx = \frac{kx^{n+1}}{n+1} + c$$

So $y = \dfrac{kx^{n+1}}{n+1} e^{ax} + ce^{ax}$

8 Rewrite the equation in the form $\frac{dy}{dx}+\frac{y}{\tan x}=2\cos x$

The integrating factor is $e^{\int \frac{1}{\tan x}dx}=e^{\int \cot x\,dx}=e^{\ln \sin x}=\sin x$

Multiplying both sides by this factor gives:

$\sin x\frac{dy}{dx}+y\cos x=2\cos x\sin x=\sin 2x$ using a trigonometric identity

$\Rightarrow \frac{d}{dx}(y\sin x)=\sin 2x$

$\Rightarrow y\sin x=\int \sin 2x\,dx=-\frac{1}{2}\cos 2x+c=-\frac{1}{2}+\sin^2 x+c$ using $\cos 2x=1-\sin^2 x$

So $y=\sin x+A\operatorname{cosec} x$ defining $A=c-\frac{1}{2}$

Note that without making the simplification using $\cos 2x=1-\sin^2 x$, an alternative but correct answer is obtained.

$y\sin x=\int \sin 2x\,dx=-\frac{1}{2}\cos 2x+c$

$\Rightarrow y=-\frac{1}{2}\cos 2x\operatorname{cosec} x+c\operatorname{cosec} x$

9 **a** The integrating factor is $e^{\int \tan x\,dx}=e^{\ln\sec x}=\frac{1}{\cos x}$

Multiplying both sides by this factor gives:

$\left(\frac{1}{\cos x}\right)\frac{dy}{dx}+y\frac{\sin x}{\cos^2 x}=e^x$

$\Rightarrow \frac{d}{dx}\left(\frac{y}{\cos x}\right)=\int e^x\,dx$

$\Rightarrow \frac{y}{\cos x}=e^x+c$

So $y=e^x\cos x+c\cos x$

b When $x=\pi, y=1$ so $-e^{\pi}-c=1\Rightarrow c=-e^{\pi}-1$

So $y=e^x\cos x-(1+e^{\pi})\cos x$

10 The integrating factor is $e^{\int -3 dx} = e^{-3x}$

Multiplying the equation by this factor gives:

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\sin x$$

$$\Rightarrow \frac{d}{dx}(e^{-3x}y) = e^{-3x}\sin x$$

$$\Rightarrow e^{-3x}y = \int e^{-3x}\sin x\,dx \qquad \textbf{(1)}$$

The expression $e^{-3x}\sin x$ can be integrated by using integration by parts twice

$$\int e^{-3x}\sin x\,dx = -\frac{1}{3}e^{-3x}\sin x + \int \frac{1}{3}e^{-3x}\cos x\,dx$$ using integration by parts with $u = \sin x, v = -\frac{1}{3}e^{x}$

$$= -\frac{1}{3}e^{-3x}\sin x - \frac{1}{9}e^{-3x}\cos x - \int \frac{1}{9}e^{-3x}\sin x\,dx$$ this time with $u = \cos x, v = -\frac{1}{9}e^{x}$

Simplifying this equation gives:

$$\frac{10}{9}\int e^{-3x}\sin x\,dx = -\frac{1}{3}e^{-3x}\sin x - \frac{1}{9}e^{-3x}\cos x$$

$$\Rightarrow \int e^{-3x}\sin x\,dx = -\frac{3}{10}e^{-3x}\sin x - \frac{1}{10}e^{-3x}\cos x$$

Using this expression in equation (**1**) and adding in a constant gives:

$$e^{-3x}y = \int e^{-3x}\sin x\,dx = -\frac{3}{10}e^{-3x}\sin x - \frac{1}{10}e^{-3x}\cos x + c$$

So $y = -\frac{3}{10}\sin x - \frac{1}{10}\cos x + ce^{3x}$

Applying the condition that when $x = 0, y = 0$ so $-\frac{1}{10} + c = 0 \Rightarrow c = \frac{1}{10}$

So the solution is $y = -\frac{3}{10}\sin x - \frac{1}{10}\cos x + \frac{1}{10}e^{3x}$

11 a Separating the variables and integrating:

$$\frac{dy}{dx} = y\sinh x$$

$$\Rightarrow \int \frac{1}{y}dy = \int \sinh x\,dx$$

$\Rightarrow \ln y = \cosh x + c$ where c is a constant

$$\Rightarrow y = e^{\cosh x + c} = e^{c} \times e^{\cosh x}$$

$\Rightarrow y = Ae^{\cosh x}$ where A is a constant ($A = e^{c}$)

b When $x = 0, y = 1$ so $Ae = 1 \Rightarrow A = e^{-1}$

So the solution is $y = e^{-1}e^{\cosh x} = e^{\cosh x - 1}$

12 Separating the variables and integrating:

$$\frac{dy}{dx} = x(4 - y^2)$$

$$\Rightarrow \int \frac{1}{4 - y^2}\,dy = \int x\,dx$$

$$\Rightarrow \int \frac{1}{(2 - y)(2 + y)}\,dy = \int x\,dx$$ rearranging the left-hand side so that it is easier to integrate

$$\Rightarrow \frac{1}{4}\int \frac{1}{(2 - y)} + \frac{1}{(2 + y)}\,dy = \int x\,dx$$

$$\Rightarrow -\frac{1}{4}\ln(2 - y) + \frac{1}{4}\ln(2 + y) = \frac{1}{2}x^2 + c$$

$$\Rightarrow \ln\left(\frac{2 + y}{2 - y}\right)^{\frac{1}{4}} = \frac{1}{2}x^2 + c$$ simplifying using the rules of logarithms

$$\Rightarrow \left(\frac{2 + y}{2 - y}\right)^{\frac{1}{4}} = e^{\frac{1}{2}x^2 + c} = Ae^{\frac{x^2}{2}}$$

$$\Rightarrow \frac{2 + y}{2 - y} = Ce^{2x^2}$$ taking the power of 4 of each side

$$\Rightarrow 2 + y = 2Ce^{2x^2} - yCe^{2x^2}$$ rearranging to find an expression in y

$$\Rightarrow y = \frac{2Ce^{2x^2} - 2}{Ce^{2x^2} + 1}$$

When $x = 0, y = 1$ so $\frac{2C - 2}{C + 1} = 1 \Rightarrow 2C - 2 = C + 1 \Rightarrow C = 3$

So the solution is $y = \frac{6e^{2x^2} - 2}{3e^{2x^2} + 1} = \frac{2(3e^{2x^2} - 1)}{3e^{2x^2} + 1}$

13 The auxiliary equation is

$m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

So the general solution is $y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

> If the auxiliary equation has complex roots, the solution is of the form $y = e^{px}(A\cos qx + B\sin qx)$

14 The auxiliary equation is

$m^2 - 12m + 36 = 0$

$(m - 6)(m - 6) = 0$

$m = 6$

So the general solution is $y = (A + Bx)e^{6x}$

> If the auxiliary equation has a repeated solution, the solution is of the form $y = (A + Bx)e^{\alpha x}$

15 The auxiliary equation is

$$m^2 - 4m = 0$$
$$m(m-4) = 0$$
$$m = 0 \text{ or } 4$$

So the general solution is $y = Ae^{0x} + Be^{4x} = A + Be^{4x}$

16 The auxiliary equation is $m^2 + k^2 = 0 \Rightarrow m = \pm \mathrm{i}k$

The general solution is $y = A\cos kx + B\sin kx$

When $x = 0,\ y = 1$ so $A = 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -kA\sin kx + kB\cos kx$$

When $x = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 1$ so $kB = 1 \Rightarrow B = \frac{1}{k}$

Substituting values for A and B into the general solution gives the particular solution

$$y = \cos kx + \frac{1}{k}\sin kx$$

If the auxiliary equation has imaginary solutions, and the general solution has the form $y = A\cos \omega x + B\sin \omega x$.

17 The auxiliary equation is

$$m^2 - 2m + 10 = 0$$
$$m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3\mathrm{i}$$

The general solution is $y = \mathrm{e}^x(A\cos 3x + B\sin 3x)$

When $x = 0,\ y = 0$ so $A = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x B\sin 3x + 3\mathrm{e}^x B\cos 3x$$

When $x = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 3$ so $3B = 3 \Rightarrow B = 1$

Substituting values for A and B into the general solution gives the particular solution

$$y = \mathrm{e}^x \sin 3x$$

18 a $y = kx\mathrm{e}^{2x}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = k\mathrm{e}^{2x} + 2kx\mathrm{e}^{2x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4k\mathrm{e}^{2x} + 4kx\mathrm{e}^{2x}$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = \mathrm{e}^{2x}$ gives:

$$4k\mathrm{e}^{2x} + 4kx\mathrm{e}^{2x} - 4(k\mathrm{e}^{2x} + 2kx\mathrm{e}^{2x}) + 13kx\mathrm{e}^{2x} = \mathrm{e}^{2x}$$
$$\Rightarrow 9kx\mathrm{e}^{2x} = \mathrm{e}^{2x} \Rightarrow k = \frac{1}{9}$$

So a particular integral is $\frac{1}{9}x\mathrm{e}^{2x}$

18 b Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$

The auxiliary equation is

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3\text{i}$

So the complementary function is $y = e^{2x}(A\cos 3x + B\sin 3x)$

Using the particular integral from part **a**, the general solution is $y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$

19 Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - y = 0$

The auxiliary equation is

$m^2 - 1 = 0$

$(m+1)(m-1) = 0$

$m = 1$ or -1

So the complementary function is $y = Ae^x + Be^{-x}$

As the complementary function has an e^x term, try a particular integral in the form λxe^x

So $\frac{dy}{dx} = \lambda e^x + \lambda xe^x$ and $\frac{d^2y}{dx^2} = 2\lambda e^x + \lambda xe^x$

Substituting into $\frac{d^2y}{dx^2} - y = 4e^x$ gives:

$2\lambda e^x + \lambda xe^x - \lambda xe^x = 4e^x$

$\Rightarrow 2\lambda e^x = 4e^x \Rightarrow \lambda = 2$

So a particular integral is $2xe^x$

The general solution is $y = Ae^x + Be^{-x} + 2xe^x$

20 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

So the complementary function is $y = (A + Bx)e^{2x}$

b As the complementary function has e^{2x} and xe^{2x} terms, these cannot be terms in the particular integral.

20 c $y = kx^2e^{2x}$, so $\frac{dy}{dx} = 2kx^2e^{2x} + 2kxe^{2x}$

$\frac{d^2y}{dx^2} = 4kx^2e^{2x} + 4kxe^{2x} + 4kxe^{2x} + 2ke^{2x} = 4kx^2e^{2x} + 8kxe^{2x} + 2ke^{2x}$

Substituting into $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ gives:

$(4kx^2 + 8kx + 2k - 8kx^2 - 8kx + 4kx^2)e^{2x} = 4e^{2x}$

$\Rightarrow 2ke^{2x} = 4e^{2x} \Rightarrow k = 2$

So a particular interval is $2x^2e^{2x}$

Using the complementary function from part **a**, the general solution is $y = (A + Bx + 2x^2)e^{2x}$

21 Solving the corresponding homogeneous equation $\frac{d^2y}{dt^2} + 4y = 0$

The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

So the complementary function is $y = A\cos 2t + B\sin 2t$

The form of the particular integral is $y = \lambda\cos 3t + \mu\sin 3t$

$\frac{dy}{dt} = -3\lambda\sin 3t + 3\mu\cos 3t$ and $\frac{d^2y}{dt^2} = -9\lambda\cos 3t - 9\mu\sin 3t$

Substituting into $\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ gives:

$-9\lambda\cos 3t - 9\mu\sin 3t + 4\lambda\cos 3t + 4\mu\sin t = 5\cos 3t$

$\Rightarrow -5\lambda\cos 3t - 5\mu\sin 3t = 5\cos 3t$

$\Rightarrow \lambda = -1$ and $\mu = 0$

So a particular integral is $-\cos 3t$

The general solution is $y = A\cos 2t + B\sin 2t - \cos 3t$

When $t = 0,\ y = 1$ so $A - 1 = 1 \Rightarrow A = 2$

$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t + 3\sin 3t$

When $t = 0,\ \frac{dy}{dt} = 2$ so $2B = 2 \Rightarrow B = 1$

Substituting values for A and B into the general solution gives the particular solution

$y = 2\cos 2t + \sin 2t - \cos 3t$

22 a Let $y = \lambda + \mu x + kxe^{2x}$

$$\frac{dy}{dx} = \mu + 2kxe^{2x} + ke^{2x}$$

$$\frac{d^2y}{dx^2} = 4kxe^{2x} + 2ke^{2x} + 2ke^{2x} = 4kxe^{2x} + 4ke^{2x}$$

Substituting into $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$ gives:

$$4kxe^{2x} + 4ke^{2x} - 3\mu - 6kxe^{2x} - 3ke^{2x} + 2\lambda + 2\mu x + 2kxe^{2x} = 4x + e^{2x}$$

$$\Rightarrow ke^{2x} + 2\mu x + 2\lambda - 3\mu = 4x + e^{2x}$$

Equating the coefficients of e^{2x}: $k = 1$
Equating the coefficients of x: $2\mu = 4 \Rightarrow \mu = 2$
Equating constants: $2\lambda - 3\mu = 0 \Rightarrow \lambda = 3$

So a particular integral is $3 + 2x + xe^{2x}$

b Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 1 \text{ or } 2$$

So the complementary function is $y = Ae^x + Be^{2x}$
Using the particular integral from part **a**, the general solution is

$$y = Ae^x + Be^{2x} + 3 + 2x + xe^{2x} = Ae^x + (B + x)e^{2x} + 2x + 3$$

23 a Solving the corresponding homogeneous equation $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 0$

The auxiliary equation is
$$16m^2 + 8m + 5 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 320}}{32} = -\frac{1}{4} \pm \frac{\sqrt{-256}}{32} = -\frac{1}{4} \pm \frac{1}{2}\text{i}$$

So the complementary function is $y = e^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x\right)$

The particular integral has the form $y = \lambda x + \mu$, so $\frac{dy}{dx} = \lambda$ and $\frac{d^2y}{dx^2} = 0$

Substituting into $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23$ gives:

$$8\lambda + 5\lambda x + 5\mu = 5x + 23$$

Equating the coefficients of x: $5\lambda = 5 \Rightarrow \lambda = 1$
Equating constant terms: $8\lambda + 5\mu = 23 \Rightarrow \mu = 3$

23 a So a particular integral is $x+3$

The general solution is $y=\mathrm{e}^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x+B\sin\frac{1}{2}x\right)+x+3$

When $x = 0$, $y = 3$ so $A + 3 = 3 \Rightarrow A = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{4}\mathrm{e}^{-\frac{1}{4}x}\left(A\cos\frac{1}{2}x+B\sin\frac{1}{2}x\right)-\frac{1}{2}\mathrm{e}^{-\frac{1}{4}x}\left(A\sin\frac{1}{2}x+B\cos\frac{1}{2}x\right)+1$

When x = 0, $\frac{\mathrm{d}y}{\mathrm{d}x}=3$ so $-\frac{1}{4}A+\frac{1}{2}B+1=3$

As $A = 0 \Rightarrow B = 4$

Substituting values for A and B into the general solution gives the particular solution

$y=4\mathrm{e}^{-\frac{1}{4}x}B\sin\frac{1}{2}x+x+3$

b As $x\to\infty$, $\mathrm{e}^{-\frac{1}{4}x}\to 0$ so $y\to x+3$

Therefore $y\approx x+3$ for large values of x.

24 Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-6y=0$

The auxiliary equation is

$$m^2-m-6=0$$
$$(m-3)(m+2)=0$$
$$m=3 \text{ or } -2$$

So the complementary function is $y=A\mathrm{e}^{3x}+B\mathrm{e}^{-2x}$

The particular integral is of the form $y=\lambda\sin 3x+\mu\cos 3x$

$\frac{\mathrm{d}y}{\mathrm{d}x}=3\lambda\cos 3x-3\mu\sin 3x$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-9\lambda\sin 3x-9\mu\cos 3x$

Substituting into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-6y=3\sin 3x-2\cos 3x$ gives:

$-9\lambda\sin 3x-9\mu\cos 3x-3\lambda\cos 3x+3\mu\sin 3x-6\lambda\sin 3x-6\mu\cos 3x=3\sin 3x-2\cos 3x$

$\Rightarrow(3\mu-15\lambda)\sin 3x-(15\mu+3\lambda)\cos 3x=3\sin 3x-2\cos 3x$

Equating the coefficients of sin $3x$: $3\mu-15\lambda=3$ **(1)**

Equating the coefficients of sin $3x$: $15\mu+3\lambda=2$ **(2)**

Adding 5 × equation (**2**) to equation (**1**) gives: $78\mu=13\Rightarrow\mu=\frac{1}{6}$

Substituting in equation (**1**) gives: $\frac{1}{2}-15\lambda=3\Rightarrow 15\lambda=-\frac{5}{2}=-\frac{1}{6}$

24 So a particular integral is $\frac{1}{6}(\cos 3x - \sin 3x)$

The general solution is $y = Ae^{3x} + Be^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

When $x = 0, y = 1$ so $A + B + \frac{1}{6} = 1 \Rightarrow A + B = \frac{5}{6}$

However as y remains finite for large values of x, $A = 0 \Rightarrow B = \frac{5}{6}$

Substituting values for A and B into the general solution gives the particular solution

$y = \frac{5}{6}e^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

25 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

The auxiliary equation is

$$m^2 + 8m + 16 = 0$$
$$(m+4)(m+4) = 0$$
$$m = -4$$

So the complementary function is $y = (A + Bt)e^{-4t}$

The particular integral is of the form $x = \lambda \sin 4t + \mu \cos 4t$

$$\frac{dx}{dt} = 4\lambda \cos 4t - 4\mu \sin 4t$$
$$\frac{d^2x}{dt^2} = -16\lambda \sin 4t - 16\mu \cos 4t$$

Substituting into $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = \cos 4t$ gives:

$$-16\lambda \sin 4t - 16\mu \cos 4t + 32\lambda \cos 4t - 32\mu \sin 4t + 16\lambda \sin 4t + 16\mu \cos 4t = \cos 4t$$
$$\Rightarrow -32\mu \sin 4t + 32\lambda \cos 4t = \cos 4t$$
$$\Rightarrow \lambda = \frac{1}{32} \text{ and } \mu = 0$$

So a particular integral is $\frac{1}{32}\sin 4t$

The general solution is $x = (A + Bt)e^{-4t} + \frac{1}{32}\sin 4t$

b When $t = 0, x = \frac{1}{2}$ so $A = \frac{1}{2}$

$$\frac{dx}{dt} = -4(A + Bt)e^{-4t} + Be^{-4t} + \frac{1}{8}\cos 4t$$

When $t = 0, \frac{dx}{dt} = 0$ so $-4A + B + \frac{1}{8} = 0$

$$\Rightarrow B = 4A - \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8}$$

25 b Substituting values for A and B into the general solution from part **a** gives the particular solution

$$x = \frac{1}{2}e^{-4t} + \frac{15}{8}te^{-4t} + \frac{1}{32}\sin 4t$$

c As $t \to \infty$, $e^{-4t} \to 0$ and $te^{-4t} \to 0$, as e^{-4t} decreases faster than the linear factor.

So for large values of t, the function behaves like $x = \frac{1}{32}\sin 4t$.

It will oscillate like a sine wave with amplitude $\frac{1}{32}$ and period $\frac{\pi}{2}$.

Challenge

1 a Given that $z = y^2$, then $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The equation $2(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$ becomes

$$2(1+x^2)\frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx} + 2xz^{\frac{1}{2}} = z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1+x^2}$ gives

$$\frac{dz}{dx} + \frac{2x}{1+x^2}z = \frac{1}{1+x^2}$$

The integrating factor is $e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$

$$(1+x^2)\frac{dz}{dx} + 2xz = 1$$

$$\Rightarrow \frac{d}{dx}\left((1+x^2)z\right) = 1$$

$$\Rightarrow (1+x^2)z = \int 1\,dx = x + c$$

$$\Rightarrow z = \frac{x+c}{1+x^2}$$

As $y = z^{\frac{1}{2}}$, $y = \sqrt{\frac{x+c}{1+x^2}}$

b When $x = 0$, $y = 2$ so $\sqrt{c} = 2 \Rightarrow c = 4$

So the particular solution is $y = \sqrt{\frac{x+4}{1+x^2}}$

Challenge

2 a Let $x = e^u$, then $\frac{dx}{du} = e^u$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x\frac{dy}{dx}$$

$$\frac{d^2y}{du^2} = \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

Find $\frac{dy}{du}$ in terms of x and $\frac{dy}{dx}$, and show that $\frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ then substitute into the differential equation.

So $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x$ transforms to $\frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \ln x = u$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1 \text{ or } -2$$

So the complementary function is $y = Ae^{-u} + Be^{-2u}$

Let the particular integral be of the form $y = \lambda u + \mu$, so $\frac{dy}{du} = \lambda$ and $\frac{d^2y}{du^2} = 0$

Substituting into $\frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = u$ gives:

$$3\lambda + 2\lambda u + 2\mu = u$$

Equating the coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Equating constants: $3\lambda + 2\mu = 0 \Rightarrow 2\mu = -\frac{3}{2} \Rightarrow \mu = -\frac{3}{4}$

So a particular integral is $\frac{1}{2}u - \frac{3}{4}$

The general solution is $y = Ae^{-u} + Be^{-2u} + \frac{1}{2}u - \frac{3}{4}$

But $x = e^u \Rightarrow u = \ln x$ and $e^{-u} = x^{-1} = \frac{1}{x}$, $e^{-2u} = x^{-2} = \frac{1}{x^2}$

So the general solution of the original equation is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{2}\ln x - \frac{3}{4}$

Challenge

2 b When $x = 1$, $y = 1$ so $A + B - \frac{3}{4} = 1 \Rightarrow A + B = \frac{7}{4}$ **(1)**

$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{1}{2x}$

When $x = 1$, $\frac{dy}{dx} = 1$ so $-A - 2B + \frac{1}{2} = 1 \Rightarrow A + 2B = -\frac{1}{2}$ **(2)**

Subtracting equation (**1**) from equation (**2**) gives $B = -\frac{9}{4}$ and so from equation (**1**) $A = \frac{9}{4} + \frac{7}{4} = 4$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y = \frac{4}{x} - \frac{9}{4x^2} + \frac{1}{2}\ln x - \frac{3}{4}$