Modelling with differential equations 8A

1 Let s be the displacement along the x-axis from O.

So $\frac{ds}{dt} = v$, and therefore $s = \int v \, dt = \int t \sin t \, dt$

Using integration by parts with $u = t$ and $v = -\cos t$

$s = -t\cos t + \int \cos t \, dt = -t\cos t + \sin t + c$

When $t = 0, s = 0,$ so $0 + 0 + c = 0 \Rightarrow c = 0$

So the equation for displacement is $s = -t\cos t + \sin t$

When $t = \frac{\pi}{2}, s = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

Hence P is 1 metre from O, as required.

2 $a = \frac{dv}{dt} = \frac{6t}{(2+t^2)^2}$ so $v = \int a \, dt = \int \frac{6t}{(2+t^2)^2} dt$

Using integration by substitution, let $u = 2 + t^2$ and so $\frac{du}{dt} = 2t$

$v = \int \frac{6t}{(2+t^2)^2} dt = \int \frac{3}{(2+t^2)^2} \times 2t \, dt$

$= \int \frac{3}{u^2} du = \int 3u^{-2} \, du = \frac{3u^{-1}}{-1} + c = c - \frac{3}{u}$

$= c - \frac{3}{2+t^2}$

When $t = 0, v = 0$ so $c - \frac{3}{2} = 0 \Rightarrow c = \frac{3}{2}$

So the solution is $v = \frac{3}{2} - \frac{3}{2+t^2}$

3 **a** $v = \int a \, dt = \int -4e^{0.2t} \, dt = -20e^{0.2t} + c$

When $t = 0, v = 20$ so $-20 + c = 20 \Rightarrow c = 40$

The solution is $v = 40 - 20e^{0.2t}$

b $x = \int v \, dt = \int (40 - 20e^{0.2t}) \, dt = 40t - 100e^{0.2t} + c$

When $t = 0, x = 0$ so $0 - 100 + c = 0 \Rightarrow c = 100$

$x = 40t - 100e^{0.2t} + 100$

The maximum value of x occurs when

$\frac{dx}{dt} = v = 40 - 20e^{0.2t} = 0 \Rightarrow e^{0.2t} = 2$

Taking logarithms of both sides gives $0.2t = \ln 2 \Rightarrow t = 5\ln 2$

The maximum value of x is given by

$x = 40 \times 5\ln 2 - 100e^{0.2\times 5\ln 2} + 100 = 40 \times 5\ln 2 - 100 \times 2 + 100 = 200\ln 2 - 100$

4 Velocity is the derivative of the displacement with respect to time, hence $\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{-\frac{x}{2}}$

So $\frac{\mathrm{d}t}{\mathrm{d}x} = \mathrm{e}^{\frac{x}{2}}$

$t = \int \mathrm{e}^{\frac{x}{2}}\mathrm{d}x = 2\mathrm{e}^{\frac{x}{2}} + c$

When $t = 0$ the particle is at the origin, so $x = 0$: $2\mathrm{e}^{\frac{0}{2}} + c = 2 + c = 0 \Rightarrow c = -2$

So an equation for t in terms of x is $t = 2\mathrm{e}^{\frac{x}{2}} - 2$
Solving for x:

$\mathrm{e}^{\frac{x}{2}} = \frac{t+2}{2}$

$\frac{x}{2} = \ln\left(\frac{t+2}{2}\right)$

$x = 2\ln\left(\frac{t}{2} + 1\right)$

5 a To find an expression for v integrate $\frac{\mathrm{d}v}{\mathrm{d}t} - 2vt = t$. The integrating factor is $\mathrm{e}^{-\int 2t\,\mathrm{d}t} = \mathrm{e}^{-t^2}$

Multiplying by this factor, the original equations becomes

$\mathrm{e}^{-t^2}\frac{\mathrm{d}v}{\mathrm{d}t} - \mathrm{e}^{-t^2}2vt = \mathrm{e}^{-t^2}t$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t^2}v) = \mathrm{e}^{-t^2}t$

$\Rightarrow \mathrm{e}^{-t^2}v = \int \mathrm{e}^{-t^2}t\,\mathrm{d}t$

Using integration by substitution, let $u = -t^2$ and so $\frac{\mathrm{d}u}{\mathrm{d}t} = -2t$

$\mathrm{e}^{-t^2}v = \int \mathrm{e}^{-t^2}t\,\mathrm{d}t = -\frac{1}{2}\int \mathrm{e}^{u}\mathrm{d}u = -\frac{1}{2}\mathrm{e}^{u} + c = -\frac{1}{2}\mathrm{e}^{-t^2} + c$

At $t = 0$, $v = 1\,\mathrm{m\,s^{-1}}$, so $\mathrm{e}^0 \times 1 = -\frac{1}{2}\mathrm{e}^0 + c \Rightarrow c = \frac{3}{2}$

This gives

$\mathrm{e}^{-t^2}v = -\frac{1}{2}\mathrm{e}^{-t^2} + \frac{3}{2}$

$\Rightarrow v = \frac{3}{2}\mathrm{e}^{t^2} - \frac{1}{2} = \frac{1}{2}(3\mathrm{e}^{t^2} - 1)$ as required

b After 2 seconds, the velocity of the car will be equal to

$v = \frac{1}{2}(3\mathrm{e}^{2^2} - 1) = \frac{1}{2}(3\mathrm{e}^{4} - 1) = 81.4\,\mathrm{m\,s^{-1}}$ (3 s.f.)

c For $t = 4$, $\frac{1}{2}(3\mathrm{e}^{4^2} - 1) = \frac{1}{2}(3\mathrm{e}^{16} - 1) = 1.34 \times 10^7\,\mathrm{m\,s^{-1}}$

No car can travel at 13 million $\mathrm{ms^{-1}}$ (nearly 30 million mph), so the model is not appropriate.

6 a $(t+4)\dfrac{dv}{dt}+4v=9.8(t+4)$

As t is positive, rewrite as $\dfrac{dv}{dt}+4v(t+4)^{-1}=9.8$

The integrating factor is $e^{\int\frac{4}{t+4}dt}=e^{4\ln(t+4)}=(t+4)^4$

So the equation becomes

$$(t+4)^4\frac{dv}{dt}+4v(t+4)^3=9.8(t+4)^4$$

$$\Rightarrow \frac{d}{dt}\left(v(t+4)^4\right)=9.8(t+4)^4$$

$$\Rightarrow v(t+4)^4=9.8\int(t+4)^4 dt$$

$$\Rightarrow v(t+4)^4=\frac{9.8}{5}(t+4)^5+C$$

$$\Rightarrow v=\frac{49(t+4)^5}{25(t+4)^4}+\frac{C}{(t+4)^4}=\frac{49(t+4)^5-c}{25(t+4)^4} \qquad \left(\text{rearranging by } C=\frac{-c}{25}\right)$$

At time $t=0$, the drop is at rest so $v=0$, hence

$$\frac{49\times4^5-c}{25\times4^4}=0\Rightarrow c=49\times4^5=50176$$

b So after 5 seconds, $t=5$ and

$$v=\frac{49\times9^5-50176}{25\times9^4}=17.3\,\text{m s}^{-1}\ (3\text{ s.f.})$$

c The function is increasing in t. This means that as $t\to\infty, v\to\infty$, i.e. the velocity will keep increasing without any limit. This is an unrealistic scenario.

7 a The gas mixture added to the tank in an hour contains $0.05\times50=2.5\text{cm}^3$ of oxygen.

The total volume of the gas mixture in the tank is

$$V=500+50t-20t=500+30t$$

There is x cm^3 of oxygen in the tank at time t and so the proportion of oxygen in the tank is

$$\frac{x}{500+30t}$$

Since the oxygen is distributed uniformly throughout the tank, the proportion of oxygen in the 20 cm^3 gas leaking every hour is

$$20\times\frac{x}{500+30t}$$

Thus the rate of change of the amount of oxygen in the tank can be expressed as

$$\frac{dx}{dt}=2.5-20\times\frac{x}{500+30t}=2.5-\frac{2x}{50+3t}\quad\text{as required.}$$

7 b To find x solve $\dfrac{dx}{dt}+\dfrac{2x}{50+3t}=2.5$

The integrating factor is $e^{\int \frac{2}{50+3t}dt}=e^{\frac{2}{3}\ln(3t+50)}=(3t+50)^{\frac{2}{3}}$

Multiplying by this factor gives:

$$\left(3t+50\right)(3t+50)^{\frac{2}{3}}\frac{dx}{dt}+\frac{2x}{(3t+50)^{\frac{1}{3}}}=2.5(3t+50)^{\frac{2}{3}}$$

$$\Rightarrow \frac{d}{dt}\left(x(3t+50)^{\frac{2}{3}}\right)=2.5(3t+50)^{\frac{2}{3}}$$

$$\Rightarrow x(3t+50)^{\frac{2}{3}}=\int 2.5(3t+50)^{\frac{2}{3}}dt=\frac{1}{2}(3t+50)^{\frac{5}{3}}+c$$

$$\Rightarrow x=\frac{1}{2}(3t+50)+\frac{c}{(3t+50)^{\frac{2}{3}}}$$

We know that at time t = 0 there is no oxygen in the tank, so

$$\frac{50}{2}+\frac{c}{50^{\frac{2}{3}}}=0\Rightarrow c=-25\times 50^{\frac{2}{3}}=339.3 \text{ (4 s.f.)}$$

Thus, after 4 hours

$$x=\frac{1}{2}(3\times 4+50)-\frac{339.3}{(3\times 4+50)^{\frac{2}{3}}}=31-\frac{339.3}{62^{\frac{2}{3}}}=9.34\,\text{cm}^3 \text{ (3 s.f.)}$$

c The model as it is currently formulated assumes that the oxygen mixes in to the tank immediately and uniformly. In reality, this is unlikely and the model should account for that.