Modelling with differential equations 8D

1 The system of equations is

$$\frac{dx}{dt} = x + y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = x - y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{dx}{dt} - x \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} - \frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} = x - \frac{dx}{dt} + x$$

$$\frac{d^2x}{dt^2} - 2x = 0$$

Solving the auxiliary equation

$$m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$$

So $x = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t}$

Then differentiating with respect to t and substituting in equation (**3**) gives:

$$y = \frac{dx}{dt} - x = \sqrt{2}Ae^{\sqrt{2}t} - \sqrt{2}Be^{-\sqrt{2}t} - Ae^{\sqrt{2}t} - Be^{-\sqrt{2}t}$$

$$y = A(\sqrt{2}-1)e^{\sqrt{2}t} - B(\sqrt{2}+1)e^{-\sqrt{2}t}$$

Using the initial conditions at $t = 0$, $x = 1$ and $y = 2$ gives

$$A + B = 1 \Rightarrow A = 1 - B \qquad (\mathbf{4})$$

$$A(\sqrt{2}-1) - B(\sqrt{2}+1) = 2 \qquad (\mathbf{5})$$

Substituting equation (**4**) into equation (**5**)

$$(1-B)(\sqrt{2}-1) - B(\sqrt{2}+1) = 2$$

$$\Rightarrow \sqrt{2} - 1 - \sqrt{2}B + B - \sqrt{2}B - B = 2$$

$$\Rightarrow B = \frac{\sqrt{2}-3}{2\sqrt{2}} = \frac{2-3\sqrt{2}}{4}$$

So $A = 1 - \dfrac{2-3\sqrt{2}}{4} = \dfrac{2+3\sqrt{2}}{4}$

Substituting for A and B gives the particular solutions

$$x = \frac{2+3\sqrt{2}}{4}e^{\sqrt{2}t} + \frac{2-3\sqrt{2}}{4}e^{-\sqrt{2}t} = \frac{1}{4}(2+3\sqrt{2})e^{\sqrt{2}t} + \frac{1}{4}(2-3\sqrt{2})e^{-\sqrt{2}t}$$

$$y = \frac{2+3\sqrt{2}}{4}(\sqrt{2}-1)e^{\sqrt{2}t} - \frac{2-3\sqrt{2}}{4}(\sqrt{2}+1)e^{-\sqrt{2}t}$$

$$= \frac{4-\sqrt{2}}{4}e^{\sqrt{2}t} + \frac{4+\sqrt{2}}{4}e^{-\sqrt{2}t} = \frac{1}{4}(4-\sqrt{2})e^{\sqrt{2}t} + \frac{1}{4}(4+\sqrt{2})e^{-\sqrt{2}t}$$

2 a The system of equations is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 5y \qquad (\mathbf{1})$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3y - x \qquad (\mathbf{2})$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = -3y - \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

Substituting into equation (**1**) gives:

$$-3\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -3y - \frac{\mathrm{d}y}{\mathrm{d}t} + 5y$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0$$

Solving the auxiliary equation

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \mathrm{i}$$

So $y = \mathrm{e}^{-t}(A\cos t + B\sin t)$

Then differentiating with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{e}^{-t}(A\cos t + B\sin t) + \mathrm{e}^{-t}(-A\sin t + B\cos t)$$

Substituting in equation (**2**) gives:

$$-\mathrm{e}^{-t}(A\cos t + B\sin t) + \mathrm{e}^{-t}(-A\sin t + B\cos t) = -3(\mathrm{e}^{-t}(A\cos t + B\sin t)) - x$$

$$\Rightarrow x = \mathrm{e}^{-t}(A\cos t + B\sin t + A\sin t - B\cos t - 3A\cos t - 3B\sin t)$$

$$\Rightarrow x = -\mathrm{e}^{-t}((2A + B)\cos t + (2B - A)\sin t)$$

So the general solutions are

$$x = -\mathrm{e}^{-t}((2A + B)\cos t + (2B - A)\sin t) \text{ and } y = \mathrm{e}^{-t}(A\cos t + B\sin t)$$

Note that the problem can also be solved by rearranging equation (**1**) to obtain $5y = -\frac{\mathrm{d}x}{\mathrm{d}t} + x$

Then differentiating with respect to t and substituting in equation (**2**) and following a similar methodology gives the mathematically equivalent results:

$$x = \mathrm{e}^{-t}(A\cos t + B\sin t) \text{ and } y = \frac{1}{5}\mathrm{e}^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

b At $t = 0$, $x = 1$ and $y = 2$ so

From the equation for x: $-(2A + B) = 1 \Rightarrow B = -1 - 2A$

From the equation for y: $A = 2 \Rightarrow B = -5$

Substituting for A and B gives the particular solutions

$$x = \mathrm{e}^{-t}(\cos t + 12\sin t)$$

$$y = \mathrm{e}^{-t}(2\cos t - 5\sin t)$$

3 The system of equations is

$$\frac{dx}{dt} = 2x - 3y - 2 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = x + y - 1 \qquad (\mathbf{2})$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = \frac{dy}{dt} - y + 1$$

$$\frac{dx}{dt} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substituting into equation (**1**) gives:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2\frac{dy}{dt} - 2y + 2 - 3y - 2$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y = 0$$

Solving the auxiliary equation

$$m^2 - 3m + 5 = 0 \Rightarrow m = \frac{3 \pm \sqrt{9-20}}{2} = \frac{3}{2} \pm \frac{\sqrt{11}}{2}$$

So $y = e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right)$

Then differentiating with respect to t

$$\frac{dy}{dt} = \frac{3}{2}e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) + e^{\frac{3t}{2}}\left(-\frac{\sqrt{11}}{2}A\sin\frac{\sqrt{11}}{2}t + \frac{\sqrt{11}}{2}B\cos\frac{\sqrt{11}}{2}t\right)$$

$$= e^{\frac{3t}{2}}\left(\left(\frac{3}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{3}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right)$$

Substituting into equation (**2**) gives

$$e^{\frac{3t}{2}}\left(\left(\frac{3}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{3}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right) = x + e^{\frac{3t}{2}}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) - 1$$

$$\Rightarrow x = e^{\frac{3t}{2}}\left(\left(\frac{1}{2}A + \frac{\sqrt{11}}{2}B\right)\cos\frac{\sqrt{11}}{2}t + \left(\frac{1}{2}B - \frac{\sqrt{11}}{2}A\right)\sin\frac{\sqrt{11}}{2}t\right) + 1$$

Using the initial conditions at $t = 0$, $x = 0$ and $y = 1$ gives:

$$\frac{1}{2}A + \frac{\sqrt{11}}{2}B + 1 = 0 \Rightarrow A = -\sqrt{11}B - 2 \quad \text{and } A = 1, \text{ so } B = -\frac{3}{\sqrt{11}}$$

Substituting for A and B gives the particular solutions

$$x = e^{\frac{3t}{2}}\left(\left(\frac{1}{2} - \frac{3\sqrt{11}}{2\sqrt{11}}\right)\cos\frac{\sqrt{11}}{2}t + \left(-\frac{3}{2\sqrt{11}} - \frac{\sqrt{11}}{2}\right)\sin\frac{\sqrt{11}}{2}t\right) + 1 = e^{\frac{3t}{2}}\left(-\cos\frac{\sqrt{11}}{2}t - \frac{7}{\sqrt{11}}\sin\frac{\sqrt{11}}{2}t\right) + 1$$

$$y = e^{\frac{3t}{2}}\left(\cos\frac{\sqrt{11}}{2}t - \frac{3}{\sqrt{11}}\sin\frac{\sqrt{11}}{2}t\right)$$

4 a The system of equations is

$$\frac{dx}{dt} = 0.2x + 0.2y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -0.5x + 0.4y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 5\frac{dx}{dt} - x \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = 5\frac{d^2x}{dt^2} - \frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$5\frac{d^2x}{dt^2} - \frac{dx}{dt} = -0.5x + 2\frac{dx}{dt} - 0.4x$$

$$5\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 0.9x = 0$$

Dividing through by 5 gives:

$$\frac{d^2x}{dt^2} - 0.6\frac{dx}{dt} + 0.18x = 0$$

b To find x, first solve the auxiliary equation

$$m^2 - 0.6m + 0.18 = 0$$

$$m = \frac{0.6 \pm \sqrt{0.36 - 0.72}}{2} = \frac{0.6 \pm \sqrt{-0.36}}{2} = 0.3 \pm 0.3\mathrm{i}$$

So $x = e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$

c To find y, first differentiate the equation for x from part **b**

$$\frac{dx}{dt} = e^{0.3t}(-0.3A\sin 0.3t + 0.3B\cos 0.3t) + 0.3e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$$

$$= 0.3e^{0.3t}(B - A)\sin 0.3t + 0.3e^{0.3t}(A + B)\cos 0.3t$$

Then substituting into equation (**3**)

$$y = 5\frac{dx}{dt} - x = 5\left(0.3e^{0.3t}(B - A)\sin 0.3t + 0.3e^{0.3t}(A + B)\cos 0.3t\right) - e^{0.3t}(A\cos 0.3t + B\sin 0.3t)$$

$$= e^{0.3t}\left((1.5A + 1.5B - A)\cos 0.3t + (1.5B - 1.5A - B)\sin 0.3t\right)$$

$$= 0.1e^{0.3t}\left((5A + 15B)\cos 0.3t + (5B - 15A)\sin 0.3t\right)$$

4 d At $t=0$, there were 3 sand foxes and 111 meerkats on the island, i.e. $x=3$ and $y=111$. So:

$e^0 A\cos 0 + B\sin 0 = 3 \Rightarrow A = 3$

$0.1e^0(5A+15B)\cos 0 = 111 \Rightarrow 0.1(15+15B) = 111$

$\Rightarrow 111 = 1.5 + 1.5B \Rightarrow B = 73$

So need to find t for which $y = 0$

$$y = 0.1e^{0.3t}(1110\cos(0.3t) + 320\sin(0.3t))$$
$$0 = 0.1e^{0.3t}(1110\cos(0.3t) + 320\sin(0.3t))$$
$$0 = 1110\cos(0.3t) + 320\sin(0.3t)$$
$$1110\cos(0.3t) = -320\sin(0.3t)$$
$$-\frac{1110}{320} = \tan(0.3t)$$
$$0.3t = 1.8515...$$
$$t = 6.17 \text{ years (3 s.f.)}$$

So, since the first measurement was taken in 2012, the meerkats will die out some time in 2018.

e In the year when the meerkats die out, $t=6$, so

$$x = e^{0.3t}(3\cos(0.3t) + 73\sin(0.3t))$$
$$= e^{0.3\times 6.17}(3\cos(0.3\times 6.17) + 73\sin(0.3\times 6.17))$$
$$= 441 \text{ (3 s.f.)}$$

The number of foxes has to be a natural number, so round to 441 foxes.

f The model seems reasonable for the first few years, where both the number of meerkats and the foxes are sensible. When after 6 years (see part **d**) the meerkats die out, the model becomes unsuitable. Considering larger values of t shows that the number of meerkats becomes negative. For instance, consider $t = 10$, we have

$$y = 0.1e^3(1110\cos 3 + 320\sin 3) \approx -2096.5$$

which is not a feasible number of animals. Similarly, for the number of foxes, if we consider $t = 15$, we get:

$$x = e^{0.3\times 15}\left(3\cos(0.3\times 15) + 73\sin(0.3\times 15)\right) \approx -6480.521$$

This behaviour can also be seen from the equations for x and y – as the coefficient of cos and sin are very large, they will lead to large negative expressions inside the brackets. Since the exponential on front of the bracket is always positive, the expressions will periodically become negative, making this model unsuitable for modelling the number of animals in the long term.

5 a The system of equations is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + 2y \qquad (\mathbf{1})$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x + y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 1.5x + 0.5\frac{\mathrm{d}x}{\mathrm{d}t} \qquad (\mathbf{3})$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5\frac{\mathrm{d}x}{\mathrm{d}t} + 0.5\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Substituting into equation (**2**)

$$1.5\frac{\mathrm{d}x}{\mathrm{d}t} + 0.5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -2x + 1.5x + 0.5\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

Solving the auxiliary equation

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1$$

So $x = (A + Bt)\mathrm{e}^{-t}$

Then differentiating with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = B\mathrm{e}^{-t} - (A + Bt)\mathrm{e}^{-t} = \mathrm{e}^{-t}(B - A - Bt)$$

Substituting into equation (**3**) gives:

$$y = 1.5(A + Bt)\mathrm{e}^{-t} + 0.5\mathrm{e}^{-t}(B - A - Bt)$$

$$= (A + 0.5B + Bt)\mathrm{e}^{-t}$$

Using the initial conditions, at $t = 0$, $x = 1$ and $y = 2$. Hence:

$$(A + B \times 0)\mathrm{e}^0 = 1 \Rightarrow A = 1$$

$$(A + 0.5B + B \times 0)\mathrm{e}^0 = 2 \Rightarrow B = 2$$

Substituting for A and B gives the particular solutions

$x = (1 + 2t)\mathrm{e}^{-t} = P\mathrm{e}^{-t}$ where $P(t) = 1 + 2t$

$y = (2 + 2t)\mathrm{e}^{-t} = Q\mathrm{e}^{-t}$ where $Q(t) = 2 + 2t$

b At $t = 2$, $x = 5\mathrm{e}^{-2} = 0.677$ litres (3 d.p.) and $y = 6\mathrm{e}^{-2} = 0.812$ litres (3 d.p.)

c As t gets large, e^t becomes much larger than t, so $t\mathrm{e}^{-t} \to 0$. So the amounts of both chemicals in the tank tend to zero.

6 a The system of equations is

$$\frac{dx}{dt} = -4y \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = 4x \qquad (\mathbf{2})$$

Rearranging equation (**2**) and differentiating with respect to t gives:

$$x = 0.25\frac{dy}{dt} \qquad (\mathbf{3})$$

$$\frac{dx}{dt} = 0.25\frac{d^2y}{dt^2}$$

Substituting into equation (**1**) gives:

$$0.25\frac{d^2y}{dt^2} = -4y$$

$$\frac{d^2y}{dt^2} = -16y \qquad (\mathbf{4})$$

This equation describes simple harmonic motion for y.

b Solving the auxiliary equation for equation (**3**)

$m^2 + 16 = 0 \Rightarrow m = \pm 4\mathrm{i}$

So $y = A\cos 4t + B\sin 4t$

Then differentiating with respect to t

$$\frac{dy}{dt} = -4A\sin 4t + 4B\cos 4t$$

Then substituting into equation (**3**)

$x = B\cos 4t - A\sin 4t$

Using the initial conditions, at $t = 0$, $x = 4$ and $y = 5$. So:

$B\cos 0 - A\sin 0 = 4 \Rightarrow B = 4$

$A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

Substituting for A and B gives the particular solutions

$x = 4\cos 4t - 5\sin 4t$

$y = 5\cos 4t + 4\sin 4t$

7 a The system of equations is

$$\frac{dx}{dt} = -0.03x + 0.01y + 50 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = 0.01x - 0.03y \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = 100\frac{dx}{dt} + 3x - 5000 \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = 100\frac{d^2x}{dt^2} + 3\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$100\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = 0.01x - 3\frac{dx}{dt} - 0.09x + 150$$

$$\frac{d^2x}{dt^2} + 0.06\frac{dx}{dt} + 0.0008x = 1.5$$

b Solving the auxiliary equation

$$m^2 + 0.06m + 0.0008 = 0$$

$$(m + 0.04)(m + 0.02) = 0$$

$$m = -0.04 \text{ or } -0.02$$

So the complementary function is $x = Ae^{-0.04t} + Be^{-0.02t}$

Try a constant for the particular integral, $x = \lambda$, so $\dot{x} = \ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + 0.006\frac{dx}{dt} + 0.0008x = 1.5$ gives:

$$0.0008\lambda = 1.5 \Rightarrow \lambda = 1875$$

So $x = Ae^{-0.04t} + Be^{-0.02t} + 1875$

Differentiating with respect to t

$$\frac{dx}{dt} = -0.04Ae^{-0.04t} - 0.02Be^{-0.02t}$$

Then substituting into equation (**3**)

$$y = 100(-0.04Ae^{-0.04t} - 0.02Be^{-0.02t}) + 3(Ae^{-0.04t} + Be^{-0.02t} + 1875) - 5000$$

$$= Be^{-0.02t} - Ae^{-0.04t} + 625$$

So the general solutions are

$$x = Ae^{-0.04t} + Be^{-0.02t} + 1875$$

$$y = Be^{-0.02t} - Ae^{-0.04t} + 625$$

c Since the exponential terms tend to zero in the expressions for both x and y, the total amount of toxin in the blood will tend to $x_{\text{lim}} = 1875\,\text{mg}$ and in the organs to $y_{\text{lim}} = 625\,\text{mg}$.

8 a The system of equations is

$$\frac{dx}{dt} = -2x + y + 1 \qquad (\mathbf{1})$$

$$\frac{dy}{dt} = -4x + y + 2 \qquad (\mathbf{2})$$

Rearranging equation (**1**) and differentiating with respect to t gives:

$$y = \frac{dx}{dt} + 2x - 1 \qquad (\mathbf{3})$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} + 2\frac{dx}{dt}$$

Substitute into equation (**2**) gives:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 4x + \frac{dx}{dt} + 2x - 1 + 2$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 1$$

b Solving the auxiliary equation

$$m^2 + m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2 \text{ or } -3$$

So the complementary function is $x = Ae^{2t} + Be^{-3t}$

Try a constant for the particular integral, $x = \lambda$, so $\dot{x} = \ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 1$ gives:

$$-6\lambda = 1 \Rightarrow \lambda = -\frac{1}{6}$$

So $x = Ae^{2t} + Be^{-3t} - \frac{1}{6}$

Differentiating with respect to t

$$\frac{dx}{dt} = 2Ae^{2t} - 3Be^{-3t}$$

Then substituting into equation (**3**)

$$y = 2Ae^{2t} - 3Be^{-3t} + 2\left(Ae^{2t} + Be^{-3t} - \frac{1}{6}\right) - 1$$

$$= 4Ae^{2t} - Be^{-3t} - \frac{4}{3}$$

c Note that both x, y are dominated by the positive exponential term, so as t gets large, both quantities will grow to infinity. Thus the model does not seem to be well suited to describe the amounts of nutrients.

Challenge

$$\frac{dy}{dt} = 2y - \frac{y^2}{6000} - 60x$$
$$\frac{dx}{dt} = 0.02y - x$$

The populations are stable if $\frac{dx}{dt} = \frac{dy}{dt} = 0$. So:

$$2y - \frac{y^2}{6000} - 60x = 0 \qquad (\mathbf{1})$$
$$0.02y - x = 0 \Rightarrow x = 0.02y \qquad (\mathbf{2})$$

Substituting for x from equation (**2**) into equation (**1**) gives

$$2y - \frac{y^2}{6000} - 60 \times 0.02y = 0$$
$$\Rightarrow \frac{y^2}{6000} - 0.8y = 0$$
$$\Rightarrow y^2 - 4800y = 0$$
$$\Rightarrow y(y - 4800) = 0$$

So $y = 0$ or 4800

When $y = 0$, $x = 0$; when $y = 4800$, $x = 0.02(4800) = 96$
So ignoring the trivial case when both populations are 0, the solution is 96 owls and 4800 field mice.