## Modelling with differential equations – mixed exercise 8

**1 a** $a = \frac{dv}{dt} = 20t\ e^{-t^2}$

$v = \int 20t\ e^{-t^2} dt = -10e^{-t^2} + c$

When $t = 0,\ v = 8$ so $-10 + c = 8 \Rightarrow c = 18$

So $v = 18 - 10e^{-t^2}$

**b** As $t \to \infty$, $e^{-t} \to 0$ and $v \to 18$

The limiting velocity of $P$ is $18\,\text{m s}^{-1}$

**2** $a = \frac{dv}{dt} = \frac{18}{(2t+3)^3} = 18(2t+3)^{-3}$

$v = \int 18(2t+3)^{-3} dt = \frac{18}{-2 \times 2}(2t+3)^{-2} + c$

$= c - \frac{9}{2(2t+3)^2}$

When $t = 0,\ v = 0$ so $c - \frac{9}{2 \times 3^2} = 0 \Rightarrow c = \frac{1}{2}$

So formula for velocity is $v = \frac{1}{2} - \frac{9}{2(2t+3)^2}$

When $v = 0.48$

$v = \frac{1}{2} - \frac{9}{2(2t+3)^2} = 0.48 \Rightarrow \frac{9}{2(2t+3)^2} = 0.02$

$(2t+3)^2 = \frac{9}{2 \times 0.02} = 225$

As $t \geqslant 0$, $2t + 3 = +\sqrt{225} = 15$

So $t = \frac{15-3}{2} = 6$

**3 a** $a = \frac{dv}{dt} = \frac{100}{(2t+5)^2} = 100(2t+5)^{-2}$

$v = \int 100(2t+5)^{-2}\, dt = \frac{100}{2 \times -1}(2t+5)^{-1} + c$

$= c - \frac{50}{2t+5}$

When $t = 0,\ v = 0$ so $c - \frac{50}{5} = 0 \Rightarrow c = 10$

So $v = 10 - \frac{50}{2t+5}$

**3 b** $x=\int\left(10-\frac{50}{2t+5}\right)\mathrm{d}t=10t-25\ln(2t+5)+c$

When $t=0,\ x=0$ so $-25\ln 5+c=0\Rightarrow c=25\ln 5$

So formula for distance travelled is $x=10t-25\ln(2t+5)+25\ln 5$

When $t=10$

$x=100-25\ln 25+25\ln 5=100-25\ln\frac{25}{5}$

$=100-25\ln 5=59.8\,\mathrm{m}$ (1 d.p.)

The distance moved by the car in the first 10 seconds of its motion is 59.8 metres.

**4 a** $a=\frac{\mathrm{d}v}{\mathrm{d}t}=\cos^2 t=\frac{1}{2}+\frac{1}{2}\cos 2t$ using the trigonometric identity $\cos 2t=2\cos^2 t-1$

$v=\int\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)\mathrm{d}t=\frac{1}{2}t+\frac{1}{4}\sin 2t+c$

When $t=0,\ v=0\Rightarrow A=0$

So formula for velocity is $v=\frac{1}{2}t+\frac{1}{4}\sin 2t$

When $t=\pi$ $v=\frac{\pi}{2}+\frac{1}{4}\sin 2\pi=\frac{\pi}{2}+0=\frac{\pi}{2}$

The speed of P when $t=\pi$ is $\frac{\pi}{2}\,\mathrm{m\,s^{-1}}$

**b** The distance of $P$ from $O$ when $t=\frac{\pi}{4}$ is given by

$x=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)\mathrm{d}t=\left[\frac{1}{4}t^2-\frac{1}{8}\cos 2t\right]_0^{\frac{\pi}{4}}$

$=\left(\frac{\pi^2}{64}-\cancel{\frac{1}{8}\cos\frac{\pi}{2}}\right)-\left(0-\frac{1}{8}\right)$

$=\frac{\pi^2}{64}+\frac{1}{8}=\frac{1}{64}(\pi^2+8)$

The distance of $P$ from $O$ when $t=\frac{\pi}{4}$ is $\frac{1}{64}(\pi^2+8)\,\mathrm{m}$, as required.

**5 a** $a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{2t+3}{t+1}=\frac{2(t+1)+1}{t+1}=2+\frac{1}{t+1}$

$v=2t+\ln(t+1)+c$

When $t=0,\ v=0$ so $0+c=0\Rightarrow c=0$

So $v=2t+\ln(t+1)$

**5 b** $x = \int 2t + \ln(t+1)\,\mathrm{d}t$

Using integration by parts

$$\int \ln(t+1)\,\mathrm{d}t = \int 1\ln(t+1)\,\mathrm{d}t = t\ln(t+1) - \int \frac{t}{t+1}\,\mathrm{d}t$$

$$= t\ln(t+1) - \int\left(1 - \frac{1}{t+1}\right)\mathrm{d}t = t\ln(t+1) - t + \ln(t+1)$$

$$= (t+1)\ln(t+1) - t$$

So $x = \int 2t + \ln(t+1)\,\mathrm{d}t = t^2 + (t+1)\ln(t+1) - t + c$

The distance of $P$ from $O$ when $t = 2$ is given by $x = \int_0^2 (2t + \ln(t+1))\,\mathrm{d}t$

Hence $x = \left[t^2 - t + (t+1)\ln(t+1)\right]_0^2 = 4 - 2 + 3\ln 3 = 2 + 3\ln 3$

The distance of $P$ from $O$ when $t = 2$ is $(2 + 3\ln 3)$ m.

**6 a** $\frac{\mathrm{d}V}{\mathrm{d}t} = 2t + 3V + 5$ can be rearranged as $\frac{\mathrm{d}V}{\mathrm{d}t} - 3V = 2t + 5$

The integrating factor is $\mathrm{e}^{\int -3\mathrm{d}t} = \mathrm{e}^{-3t}$

Multiplying the equation by the integrating factor

$$\mathrm{e}^{-3t}\frac{\mathrm{d}V}{\mathrm{d}t} - 3\mathrm{e}^{-3t}V = 2t\mathrm{e}^{-3t} + 5\mathrm{e}^{-3t}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-3t}V) = 2t\mathrm{e}^{-3t} + 5\mathrm{e}^{-3t}$$

$$\Rightarrow \mathrm{e}^{-3t}V = 2\int t\mathrm{e}^{-3t}\mathrm{d}t + 5\int \mathrm{e}^{-3t}\mathrm{d}t$$ use integration by parts for the first integral

$$\Rightarrow \mathrm{e}^{-3t}V = -\frac{2}{3}t\mathrm{e}^{-3t} - \frac{2}{9}\mathrm{e}^{-3t} - \frac{5}{3}\mathrm{e}^{-3t} + c$$

$$\Rightarrow \mathrm{e}^{-3t}V = -\frac{1}{3}\mathrm{e}^{-3t}\left(2t + \frac{17}{3}\right) + c$$

So $V = -\frac{2}{3}t - \frac{17}{9} + c\mathrm{e}^{3t}$

When $t = 0, V = 1$ so $-\frac{17}{9} + c\mathrm{e}^0 = 1 \Rightarrow c = \frac{26}{9}$

So solution is $V = -\frac{2}{3}t - \frac{17}{9} + \frac{26}{9}\mathrm{e}^{3t}$

Which is of the required form with $A = -\frac{2}{3}, B = -\frac{17}{9}, C = \frac{26}{9}$

**b** When $t = 2$, $V = -\frac{2}{3}\times 2 - \frac{17}{9} + \frac{26}{9}\mathrm{e}^{3\times 2} = 1162.2\,\mathrm{cm}^3$ (1 d.p.)

**c** Because of the exponential term, this model predicts that the bacteria will reproduce infinitely many times and the colony will grow without limit. A decay factor could be added to make the model more realistic.

**7 a** The rate at which the contaminant enters the reservoir is constant, 4g in every litre of fluid, which flows at the rate of 300 l/day. So $4\times300=1200\,\text{g}$ of contaminant enters the reservoir every day.

Let $x$ be the grams of contaminant in the reservoir

So the proportion of contaminant in the reservoir is $\dfrac{x}{10000+100t}$

The denominator is the total amount of liquid in the reservoir on day $t$: the initial 10 000 litres, minus the 200 litres which leaks every day, plus the 300 litres which enters the container each day.

So the amount of contaminant that leaks from the reservoir each day is $\dfrac{x}{10000+100t}\times200$

Thus $\dfrac{\mathrm{d}x}{\mathrm{d}t}=1200-\dfrac{200x}{10000+100t}=1200-\dfrac{2x}{100+t}$

**b** Rearrange the differential equation as $\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{2x}{100+t}=1200$

The integrating factor is $\mathrm{e}^{\int\frac{2}{100+t}\mathrm{d}t}=\mathrm{e}^{2\ln(100+t)}=(t+100)^2$

Multiplying the equation by the integrating factor

$(t+100)^2\dfrac{\mathrm{d}x}{\mathrm{d}t}+x=(t+100)^2\times1200$

$\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}t}\left((t+100)^2x\right)=1200t^2+240\,000t+12\,000\,000$

$\Rightarrow x(t+100)^2=400t^3+120\,000t^2+12\,000\,000t+c$

$\Rightarrow x=400\left(\dfrac{t^3+300t^2+30\,000t+c'}{(t+100)^2}\right)$

When $t=0, x=0$ (the reservoir initially contains fresh water so $\dfrac{400c'}{100}=0\Rightarrow c'=0$

So $x=400\left(\dfrac{t^3+300t^2+30\,000t}{(t+100)^2}\right)$

When $t=7,\ x=400\left(\dfrac{343+300\times49+30\,000\times7}{107^2}\right)=7860\,\text{g}$ (3 s.f.)

**c** The model assumes that the contaminant mixes uniformly and instantaneously, which is not a realistic representation of this situation.

**8 a** The particle moves with simple harmonic motion.

**b** Given $\ddot{x}=-49x$, solve the auxiliary equation

$m^2+49=0\Rightarrow m=\pm7\mathrm{i}$

So the general solution is $x=A\cos7t+B\sin7t$

$\dot{x}=-7A\sin7t+7B\cos7t$

At time $t=0$ the particle is at rest, $\dot{x}=0$ so $-7A\sin0+7B\cos0=0\Rightarrow B=0$

At time $t=0$ the particle is at point B. Suppose the distance from $B$ to $O$ is equal to $d$.

Then $d=A\cos0\Rightarrow A=d$, so the solution is $x=d\cos7t$

To find the period of motion, solve $7t=2\pi\Rightarrow$ the period of motion is $\dfrac{2\pi}{7}$ seconds

**9 a** The particle moves with simple harmonic motion.

**b** Given $\ddot{x} = -\frac{50}{3}x$, solve the auxiliary equation

$m^2 + \frac{50}{3} = 0 \Rightarrow m = \pm\sqrt{-\frac{50}{3}} = \pm\frac{5\text{i}\sqrt{2}}{\sqrt{3}} = \pm\frac{5\text{i}\sqrt{6}}{3}$

So $x = A\cos\frac{5\sqrt{6}}{3}t + B\sin\frac{5\sqrt{6}}{3}t$

At time $t = 0$, $x = 4$ so $A\cos\frac{5\sqrt{6}}{3}0 + B\sin\frac{5\sqrt{6}}{3}0 = 4 \Rightarrow A = 4$

$\dot{x} = -\frac{5\sqrt{6}}{3}A\sin\frac{5\sqrt{6}}{3}t + \frac{5\sqrt{6}}{3}B\cos\frac{5\sqrt{6}}{3}t$

At time $t = 0$ the particle is at rest, so $\dot{x} = 0$ and $-\frac{5\sqrt{6}}{3}A\sin\frac{5\sqrt{6}}{3}0 + \frac{5\sqrt{6}}{3}B\cos\frac{5\sqrt{6}}{3}0 = 0 \Rightarrow B = 0$

Substituting for $A$ and $B$ gives the solution:

$x = 4\cos\frac{5\sqrt{6}}{3}t$ or $x = 4\cos 5\sqrt{\frac{2}{3}}t$ as $\frac{\sqrt{6}}{3} = \frac{\sqrt{2}\sqrt{3}}{3} = \frac{\sqrt{2}}{\sqrt{3}}$

To find the period of motion, solve $\frac{5\sqrt{6}}{3}t = 2\pi$

So the period of motion is $\frac{6\pi}{5\sqrt{6}} = \frac{\sqrt{6}\pi}{5} = 1.54\text{ s}$ (3 s.f.)

The amplitude of motion will be described by the oscillation of the particle around the natural length of the spring, so $a = 4 - 2.5 = 1.5$ m

**10 a** $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -0.25x$, so solving the auxiliary equation

$m^2 + 0.25 = 0 \Rightarrow m = \pm 0.5\text{i}$

So $x = A\cos 0.5t + B\sin 0.5t$

Writing $x$ in the form $x = R\sin(0.5t + \alpha) = R\sin\alpha\cos 0.5t + R\cos\alpha\sin 0.5t$

So $R\sin\alpha = A$ and $R\cos\alpha = B$

And hence $R^2 = A^2 + B^2$, $\tan\alpha = \frac{A}{B}$

Maximum displacement occurs when $\sin(0.5t + \alpha) = \pm 1 \Rightarrow R = 4$

It occurs when $t = 2$, so $1 + \alpha = 0.5\pi + k\pi \Rightarrow \alpha = 0.5\pi + k\pi - 1$

So $\tan\alpha = \tan(0.5\pi + k\pi - 1) = \tan(k\pi + (0.5\pi - 1)) = \tan(0.5\pi - 1) = \cot 1$

So $\frac{A}{B} = \cot 1 \Rightarrow A = B\cot 1$ and $R^2 = A^2 + B^2 = 4^2 = 16$

Thus $B^2\cot^2 1 + B^2 = 16 \Rightarrow B^2 = \frac{16}{\cot^2 1 + 1} = 11.3292 \Rightarrow B = 3.366$ (3 d.p.)

And $A = B\cot 1 = 2.161$ (3 d.p.)

So $x = 2.161\cos 0.5t + 3.366\sin 0.5t$

**10 b** The float travels between its highest and lowest point in half of the time it takes it to complete the oscillation (the period of the motion).
To find the period of motion, solve $0.5t = 2\pi$, so the period of motion is $4\pi$
So it takes the boat $2\pi$ seconds.

**c** This model does not account for any changes in the height of the waves, i.e. the amplitude of the motion over time.

**11 a** $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\dfrac{\mathrm{d}x}{\mathrm{d}t} + n^2x = 0$

The auxiliary equation is $m^2 + 2km + n^2 = 0$

$m^2 + 2km + n^2 = 0$

$m = \dfrac{-2k \pm \sqrt{4k^2 - 4n^2}}{2} = -k \pm \sqrt{k^2 - n^2}$

As $0 < k < n \Rightarrow k^2 - n^2 < 0$ so $m = -k \pm \mathrm{i}\sqrt{n^2 - k^2}$

The general solution is $x = \mathrm{e}^{-kt}\left(A\cos t\sqrt{n^2 - k^2} + B\sin t\sqrt{n^2 - k^2}\right)$

The general solution can also be written in the form $x = A'\mathrm{e}^{-kt}\cos(\omega t + \varepsilon)$, where $\omega = \sqrt{(n^2 - k^2)}$

**b** To find the period of motion, solve $t\sqrt{n^2 - k^2} = 2\pi$

So the period of motion is $\dfrac{2\pi}{\sqrt{n^2 - k^2}}$

**12 a** The auxiliary equation is

$m^2 + 2km + 2k^2 = 0$

$m = \dfrac{-2k \pm \sqrt{(4k^2 - 8k^2)}}{2} = \dfrac{-2k \pm \sqrt{(-4k^2)}}{2} = -k \pm k\mathrm{i}$

So $x = \mathrm{e}^{-kt}(A\cos kt + B\sin kt)$

Using the initial conditions, when $t = 0,\ x = 0 \Rightarrow 0 = A$

As $A = 0,\ \dot{x} = -k\mathrm{e}^{-kt}B\sin kt + B\mathrm{e}^{-kt}k\cos kt$

$t = 0,\ \dot{x} = U$ so $Bk = U \Rightarrow B = \dfrac{U}{k}$

So the general solution is $x = \mathrm{e}^{-kt}\dfrac{U}{k}\sin kt$

$\dot{x} = -k\mathrm{e}^{-kt}\dfrac{U}{k}\sin kt + \dfrac{U}{k}\mathrm{e}^{-kt}k\cos kt$

When $\dot{x} = 0,\quad U\mathrm{e}^{-kt}(\sin kt - \cos kt) = 0$

$\Rightarrow \sin kt = \cos kt$

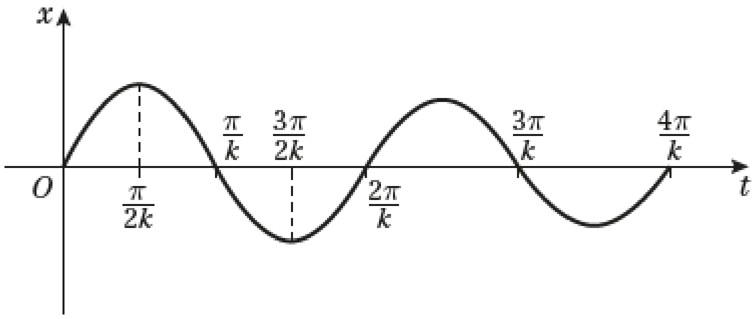
$\Rightarrow \tan kt = 1$

$\Rightarrow kt = \dfrac{\pi}{4} + n\pi$

So $kt = \left(n + \dfrac{1}{4}\right)\pi,\ n \in \mathbb{N}$ as required

**12 b** From c, the maxima and minima occur when $kt = \left(n + \frac{1}{4}\right)\pi$

Multiplying $\sin kt$ by $e^{-kt}$ causes the amplitude to decrease as $t$ increases.




**13 a** The auxiliary equation is

$m^2 + n^2 = 0 \Rightarrow m = \pm\, in$

So the complementary function is $x = A\cos nt + B\sin nt$

The form of the particular integral is $x = \mu t + \lambda$, so $\dot{x} = \mu$ and $\ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + n^2 x = n^2 Ut$ gives:

$n^2(\mu t + \lambda) = n^2 Ut$

$\Rightarrow \mu = U$ and $\lambda = 0$

The general solution is $x = A\cos nt + B\sin nt + Ut$

Using the initial conditions when $t = 0, x = 0 \Rightarrow A = 0$

$\dot{x} = Bn\cos nt + U$

When $t = 0,\ \dot{x} = 0$ so $Bn + U = 0 \Rightarrow B = -\frac{U}{n}$

Substituting for $A$ and $B$ gives the solution:

$$x = Ut - \frac{U}{n}\sin nt$$

**b** Using the result in part **a**, $\dot{x} = U - U\cos nt$

When $\dot{x} = 0,\quad 1 - \cos nt = 0$

$\Rightarrow \cos nt = 1$

$\Rightarrow nt = 0,\ 2\pi,\ 4\pi, \ldots$

So smallest positive value of $T$ is $\frac{2\pi}{n}$

**c** When $t = \frac{2\pi}{n}$, $x = U\frac{2\pi}{n} - \frac{U}{n}\sin 2\pi = \frac{2U\pi}{n}$

$P$ has moved a distance $\frac{2U\pi}{n}$ when it first comes to rest.

**14 a** $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=150\cos t$

Try a form of the particular integral as $x=\lambda\cos t+\mu\sin t$

Then $\frac{dx}{dt}=-\lambda\sin t+\mu\cos t$ and $\frac{d^2x}{dt^2}=-\lambda\cos t-\mu\sin t$

Substituting into the equation gives

$-\lambda\cos t-\mu\sin t-4\lambda\sin t+4\mu\cos t+3\lambda\cos t+3\mu\sin t=150\cos t$

$\Rightarrow(-\lambda+4\mu+3\lambda)\cos t+(3\mu-\mu-4\lambda)\sin t=150\cos t$

$\Rightarrow(2\lambda+4\mu)\cos t+(2\mu-4\lambda)\sin t=150\cos t$

Equating coefficients of $\cos t$: $2\lambda+4\mu=150\Rightarrow\lambda=75-2\mu$

Equating coefficients of $\sin t$: $2\mu-4\lambda=0\Rightarrow\ \mu=2\lambda$

So $5\lambda=75\Rightarrow\lambda=15$

And $\mu=30$

So a particular solution is $x=15\cos t+30\sin t$

**b** To find the general solution, solve the auxiliary equation

$m^2+4m+3=0$

$(m+3)(m+1)=0$

$m=-3$ or $-1$

So the general solution is

$x=Ae^{-3t}+Be^{-t}+15\cos t+30\sin t$

**c** Using the initial conditions to find the particular solution.

At time $t=0$, $x=0$ so $A+B+15=0\Rightarrow A=-B-15$ **(1)**

$\frac{dx}{dt}=-3Ae^{-3t}-Be^{-t}-15\sin t+30\cos t$

At time $t=0$, $\frac{dx}{dt}=0$ so $-3A-B+30=0\Rightarrow B=30-3A$ **(2)**

Substituting equation (**2**) in equation (**1**) gives

$A=-30+3A-15\Rightarrow 2A=45\Rightarrow A=22.5$

$B=30-3A=30-67.5=-37.5$

So $x=22.5e^{-3t}-37.5e^{-t}+15\cos t+30\sin t$

When $t=10$ $x=22.5e^{-30}-37.5e^{-10}+15\cos 10+30\sin 10=-29$ m (2 s.f.)

**15 a** First solving the corresponding homogeneous equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+10x=0$

The auxiliary equation is

$m^2+2m+10=0$

$m=\frac{-2\pm\sqrt{4-40}}{2}=-1\pm 3\mathrm{i}$

So the complementary function is $x=\mathrm{e}^{-t}(A\cos 3t+B\sin 3t)$

Try a particular integral of the form $x=\lambda\cos t+\mu\sin t$

$\frac{dx}{dt}=-\lambda\sin t+\mu\cos t$ and $\frac{d^2x}{dt^2}=-\lambda\cos t-\mu\sin t$

Substituting into $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+10x=27\cos t-6\sin t$ gives:

$-\lambda\cos t-\mu\sin t-2\lambda\sin t+2\mu\cos t+10\lambda\cos t+10\mu\sin t=27\cos t-6\sin t$

$\Rightarrow(2\mu+9\lambda)\cos t+(9\mu-2\lambda)\sin t=27\cos t-6\sin t$

Equating coefficients of $\cos t$: $2\mu+9\lambda=27\Rightarrow\ 18\lambda=54-4\mu$ **(1)**

Equating coefficients of $\sin t$: $9\mu-2\lambda=-6\Rightarrow\ 18\lambda=81\mu+54$ **(2)**

Substituting equation (**2**) in equation (**1**) gives:

$85\mu=0\Rightarrow\mu=0$

So $18\lambda=54\Rightarrow\lambda=3$

So the particular integral is $3\cos t$

The general solution is $x=\mathrm{e}^{-t}(A\cos 3t+B\sin 3t)+3\cos t$

**b** When $t=0, x=3$ so $A+3=0\Rightarrow A=0$

So $x=3\cos t+B\mathrm{e}^{-t}\sin 3t$

$\frac{dx}{dt}=-3\sin t+3B\mathrm{e}^{-t}\cos 3t-B\mathrm{e}^{-t}\sin 3t$

When $t=0,\frac{dx}{dt}=6$ so $3B=6\Rightarrow B=2$

Substituting for $A$ and $B$ gives the solution:

$x=2\mathrm{e}^{-t}\sin 3t+3\cos t$

**c** After a week $t\approx 7$ days, $\mathrm{e}^{-t}\approx 0$, so $x\approx 3\cos t$

Therefore, the distance between highest and lowest water level is $3-(-3)=6\,\mathrm{m}$

**16 a** The system of equations is

$$\frac{dx}{dt} = 2x + y \qquad \textbf{(1)}$$

$$\frac{dy}{dt} = -2x + 4y \qquad \textbf{(2)}$$

Rearranging equation (**1**) and differentiating with respect to $t$ gives:

$$y = \frac{dx}{dt} - 2x \qquad \textbf{(3)}$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} - 2\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = -2x + 4\frac{dx}{dt} - 8x$$

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 10x = 0$$

**b** Solving the auxiliary equation

$$m^2 - 6x + 10 = 0$$

$$m = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm \mathrm{i}$$

So $x = e^{3t}(A\cos t + B\sin t)$, which is of the required form with $\alpha = 3,\ \beta = 1$

**c** $$\frac{dx}{dt} = 3e^{3t}(A\cos t + B\sin t) + e^{3t}(-A\sin t + B\cos t)$$

So substituting in $y = \frac{dx}{dt} - 2x$, i.e. equation (**3**) from part **a**

$$y = 3e^{3t}(A\cos t + B\sin t) + e^{3t}(-A\sin t + B\cos t) - 2e^{3t}(A\cos t + B\sin t)$$

$$= e^{3t}\big((3A + B - 2A)\cos t + (3B - A - 2B)\sin t\big)$$

$$= e^{3t}\big((A + B)\cos t + (B - A)\sin t\big)$$

**d** At $t = 0$, $x = 10$ and $y = 20$. Thus

$$e^0 A\cos 0 + B\sin 0 = 10 \Rightarrow A = 10$$

$$e^0\big((A + B)\cos 0 + (B - A)\sin 0\big) = 20 \Rightarrow A + B = 20 \Rightarrow B = 10$$

Substituting for $A$ and $B$ gives:

$x = 10e^{3t}(\cos t + \sin t)$ and $y = 20e^{3t}\cos t$

So find $t$ for which $y = 0$, $20e^{3t}\cos t = 0 \Rightarrow t = \frac{\pi}{2} \approx 1.5$

So, since the study began at the start of 2008, the slugs die out some time in 2009.

**e** At $t = \frac{\pi}{2}$, $x = 10e^{\frac{3\pi}{2}}\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 10e^{\frac{3\pi}{2}} = 1113$ to nearest whole number

**f** The model predicts a very large number of hedgehogs when the slugs die out, and a very large increase in only 1.5 years, so perhaps it is not a very good model.

**17 a** The system of equations is

$$\frac{dx}{dt} = -4x + 3y \qquad \textbf{(1)}$$

$$\frac{dy}{dt} = -3x + 2y \qquad \textbf{(2)}$$

Rearranging equation (**1**) and differentiating with respect to $t$ gives:

$$y = \frac{1}{3}\frac{dx}{dt} + \frac{4}{3}x \qquad \textbf{(3)}$$

$$\frac{dy}{dt} = \frac{1}{3}\frac{d^2x}{dt^2} + \frac{4}{3}\frac{dx}{dt}$$

Substituting into equation (**2**) gives:

$$\frac{1}{3}\frac{d^2x}{dt^2} + \frac{4}{3}\frac{dx}{dt} = -3x + \frac{2}{3}\frac{dx}{dt} + \frac{8}{3}x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

Solve the auxiliary equation

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1$$

So $x = Ate^{-t} + Be^{-t}$

Differentiating with respect to $t$ and substituting in equation (**3**) gives:

$$\frac{dx}{dt} = Ae^{-t} - Ate^{-t} - Be^{-t} = -Ate^{-t} + (A-B)e^{-t}$$

$$y = -\frac{1}{3}Ate^{-t} + \frac{1}{3}(A-B)e^{-t} + \frac{4}{3}Ate^{-t} + \frac{4}{3}Be^{-t}$$

$$= Ate^{-t} + \left(B + \frac{1}{3}A\right)e^{-t}$$

Using the initial conditions, at $t = 0$, $x = 10$ and $y = 20$, so:

$$Be^0 = 10 \Rightarrow B = 10$$

$$\left(B + \frac{1}{3}A\right)e^0 = 20 \Rightarrow \frac{1}{3}A = 20 - B = 10 \Rightarrow A = 30$$

Substituting for $A$ and $B$ gives the particular solutions

$$x = 30te^{-t} + 10e^{-t} \quad \text{and} \quad y = 30te^{-t} + 20e^{-t}$$

Which is of the required form with $A = 30, B = 10, C = 30$ and $D = 20$

**b** At time $t = 2.1$

$x = 30 \times 2.1e^{-2.1} + 10e^{-2.1} = 9$ to the nearest organism

$y = 30 \times 2.1e^{-2.1} + 20e^{-2.1} = 10$ to the nearest organism

**c** Since both expressions are dominated by the exponential, which decays to zero as $t$ gets large, both organisms will eventually die out.

**18 a** The system of equations is

$$\frac{dx}{dt} = 2 + \frac{1}{3}y - \frac{1}{2}x \qquad \textbf{(1)}$$

$$\frac{dy}{dt} = 1 + \frac{1}{2}x - \frac{2}{3}y \qquad \textbf{(2)}$$

Rearranging equation (**2**) and differentiating with respect to $t$ gives:

$$x = 2\frac{dy}{dt} + \frac{4}{3}y - 2 \qquad \textbf{(3)}$$

$$\frac{dx}{dt} = 2\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt}$$

Substituting into equation (**1**) gives:

$$2\frac{d^2y}{dt^2} + \frac{4}{3}\frac{dy}{dt} = 2 + \frac{1}{3}y - \frac{dy}{dt} - \frac{2}{3}y + 1$$

$$\Rightarrow 2\frac{d^2y}{dt^2} + \frac{7}{3}\frac{dy}{dt} + \frac{1}{3}y = 3$$

$$\Rightarrow 6\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + y = 9 \qquad \text{as required}$$

**b** Solving the auxiliary equation

$$6m^2 + 7m + 1 = 0$$

$$(6m+1)(m+1) = 0$$

$$m = -1 \text{ or } -\frac{1}{6}$$

So $y = Ae^{-t} + Be^{-\frac{1}{6}t}$

Use a particular integral in the form $y = \lambda$, so $\dot{y} = \ddot{y} = 0$

Substituting into $6\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + y = 9$ gives $\lambda = 9$

$$y = Ae^{-t} + Be^{-\frac{1}{6}t} + 9$$

Differentiating with respect to $t$ and substituting into equation (**3**)

$$\frac{dy}{dt} = -Ae^{-t} - \frac{1}{6}Be^{-\frac{1}{6}t}$$

$$x = -2Ae^{-t} - \frac{1}{3}Be^{-\frac{1}{6}t} + \frac{4}{3}Ae^{-t} + \frac{4}{3}Be^{-\frac{1}{6}t} + 12 - 2 = -\frac{2}{3}Ae^{-t} + Be^{-\frac{1}{6}t} + 10$$

Now use the initial conditions, at time $t = 0$, $x = y = 8$, so:

$$-\frac{2}{3}Ae^0 + Be^0 + 10 = 8 \Rightarrow 3B - 2A = -6 \qquad \textbf{(4)}$$

$$Ae^0 + Be^0 + 9 = 8 \Rightarrow -A = B + 1 \qquad \textbf{(5)}$$

Substituting equation (**5**) into equation (**1**) gives:

$$5B + 2 = -6 \Rightarrow B = -\frac{8}{5} \text{ and so } A = \frac{3}{5}$$

Substituting for A and B gives the particular solutions

$$x = -\frac{2}{5}e^{-t} - \frac{8}{5}e^{-\frac{1}{6}t} + 10 \quad \text{and} \quad y = \frac{3}{5}e^{-t} - \frac{8}{5}e^{-\frac{1}{6}t} + 9$$

**18 c** As $t$ becomes large $e^{-t} \to 0$, so there will be approximately 10 litres in tank $A$ and 9 litres in tank $B$.

**Challenge**

**a i** From the question $\frac{dX}{dt} = -X$ so rearranging gives

$\frac{dX}{X} = -dt$

$\Rightarrow \ln X = -t + c$

$\Rightarrow X = e^{-t+c} = Ce^{-t}$

Using the initial condition at $t = 0, X = 300$ so $C = 300$

So $X = 300e^{-t}$ as required.

**ii** The amount of water in $Y$ can be modelled as $\frac{dY}{dt} = X - Y$

Where $X$ is the amount of water coming in from the tank above, and $Y$ is the rate at which the water escapes the tank.

So $\frac{dY}{dt} = 300e^{-t} - Y$, which can be rearranged as $\frac{dY}{dt} + Y = 300e^{-t}$

Multiplying by the integrating factor, $e^{\int 1 dt} = e^t$, gives:

$e^t \frac{dY}{dt} + e^t Y = 300$

$\Rightarrow \frac{d}{dt}\left(e^t Y\right) = 300$

$\Rightarrow e^t Y = 300t + c$

$\Rightarrow Y = 300te^{-t} + ce^{-t}$

Using the initial condition at $t = 0, Y = 200 \Rightarrow c = 200$

When $X = Y$

$e^{-t}(300t + 200) = 300e^{-t}$

$\Rightarrow 300t + 200 = 300$

$\Rightarrow t = \frac{100}{300} = \frac{1}{3} = 20$ minutes

**iii** As no water escapes tank $Z$, and the rate at which the water is added is equal to the rate at which it leaves container $Y$. So the amount of water in tank $Z$ can be modelled as $\frac{dZ}{dt} = Y$

$\frac{dZ}{dt} = Y = e^{-t}(300t + 200)$

$Z = 300\int te^{-t}dt + 200\int e^{-t}dt$

The second integral can be solved directly and the first can be solved by integration by parts

$Z = -300te^{-t} - 300e^{-t} - 200e^{-t} + c = -300te^{-t} - 500e^{-t} + c$

Using the initial condition at $t = 0, Z = 100$ so $-500e^{-0} + c = 100 \Rightarrow c = 600$

So $Z = -300te^{-t} - 500e^{-t} + 600$, which can be written as $Z = 600 - 100e^{-t}(3t + 5)$

**Challenge**

**b i** In this scenario $\frac{\mathrm{d}X}{\mathrm{d}t} = -2X$ as the second tap in tank $X$ doubles the rate of flow

$\frac{\mathrm{d}X}{X} = -2\mathrm{d}t$

$\Rightarrow \ln X = -2t + c$

$\Rightarrow X = \mathrm{e}^{-2t+c} = C\mathrm{e}^{-2t}$

Suppose initially tank $X$ contains $V$ gallons of water. Hence $V = C\mathrm{e}^0 \Rightarrow C = V$ and $X = V\mathrm{e}^{-2t}$

The amount of water in tank $Y$ can be now modelled as $\frac{\mathrm{d}Y}{\mathrm{d}t} = 2X - Y$

$\frac{\mathrm{d}Y}{\mathrm{d}t} = 2V\mathrm{e}^{-2t} - Y$

$\frac{\mathrm{d}Y}{\mathrm{d}t} + Y = 2V\mathrm{e}^{-2t}$

Multiplying by the integrating factor, $\mathrm{e}^{\int 1\mathrm{d}t} = \mathrm{e}^t$, gives:

$\mathrm{e}^t\frac{\mathrm{d}Y}{\mathrm{d}t} + \mathrm{e}^t Y = 2V\mathrm{e}^{-t}$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t Y\right) = 2V\mathrm{e}^{-t}$

$\Rightarrow \mathrm{e}^t Y = -2V\mathrm{e}^{-t} + c$

$\Rightarrow Y = -2V\mathrm{e}^{-2t} + c\mathrm{e}^{-t}$

Initially the tank is empty, so $-2V + c = 0 \Rightarrow c = 2V$

And thus $Y = 2V(-\mathrm{e}^{-2t} + \mathrm{e}^{-t})$

To find the maximum amount of water in tank $Y$, differentiate with respect to $t$ and set to zero

$\frac{\mathrm{d}Y}{\mathrm{d}t} = 2V(2\mathrm{e}^{-2t} - \mathrm{e}^{-t})$

When $\frac{\mathrm{d}Y}{\mathrm{d}t} = 0, 2V(2\mathrm{e}^{-2t} - \mathrm{e}^{-t}) = 0 \Rightarrow 2\mathrm{e}^{-2t} = \mathrm{e}^{-t} \Rightarrow \mathrm{e}^{-t} = \frac{1}{2}$

So water in tank Y is at a maximum when $t = -\ln\frac{1}{2} = 0.6931\ldots = 42$ mins (to the nearest minute)

**Challenge**

**b ii** The amount of water in tank $Z$ can be modelled as before using $\frac{\mathrm{d}Z}{\mathrm{d}t} = Y$

$\frac{\mathrm{d}Z}{\mathrm{d}t} = 2V(-\mathrm{e}^{-2t} + \mathrm{e}^{-t}) \Rightarrow Z = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t}) + c$

Initially the tank is empty, so $2V(0.5\mathrm{e}^{0} - \mathrm{e}^{0}) + c = 0 \Rightarrow c = V$

So $Z = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t} + 0.5)$

When $Y = Z$

$2V(-\mathrm{e}^{-2t} + \mathrm{e}^{-t}) = 2V(0.5\mathrm{e}^{-2t} - \mathrm{e}^{-t} + 0.5)$

$\Rightarrow 1.5\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} + 0.5 = 0$

$\Rightarrow 3\mathrm{e}^{-2t} - 4\mathrm{e}^{-t} + 1 = 0$

$\Rightarrow (3\mathrm{e}^{-t} - 1)(\mathrm{e}^{-t} - 1) = 0$

$\Rightarrow \mathrm{e}^{-t} = \frac{1}{3}$ or 1

$\mathrm{e}^{-t} = 1 \Rightarrow t = 0$, which is the trivial case because initially both tanks are empty

$\mathrm{e}^{-t} = \frac{1}{3} \Rightarrow t = \ln 3$

So the tanks have the same amount of water after $\ln 3$ hours.