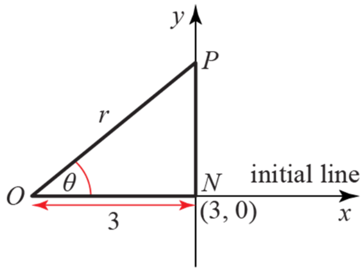
Review exercise 2

1 a $r = 2$

> You can just write the answer to part **a** down. The equation $r = k$ is the equation of a circle centre O and radius k, for any positive k.

b



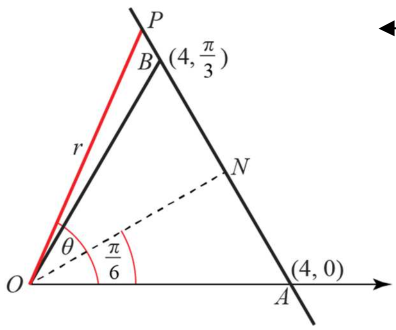
For any point P on the line

$$\frac{3}{r} = \cos\theta$$

$$r = \frac{3}{\cos\theta} = 3\sec\theta$$

> If the point (3, 0) is labelled N, trigonometry on the right-angled triangle ONP gives the polar equation of the line.

c



> In this diagram, the point (4, 0) is labelled A, the point $\left(4, \frac{\pi}{3}\right)$ is labelled B and the foot of the perpendicular from O to AB is labelled N. The triangle OAB is equilateral and $\angle AON = \frac{1}{2}60^\circ = 30^\circ = \frac{\pi}{6}$ radians.

In the triangle ONA

$$\frac{ON}{OA} = \frac{ON}{4} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$ON = 2\sqrt{3}$$

In the triangle ONP,

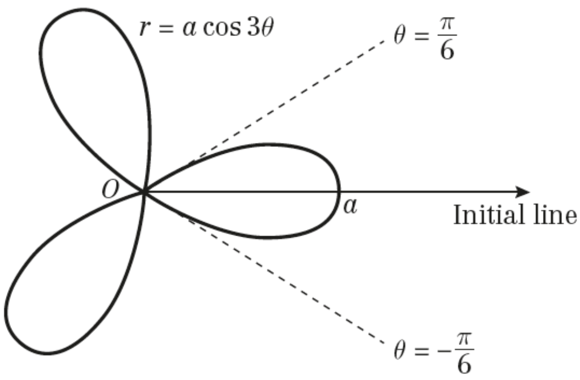
$$\frac{ON}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$\frac{2\sqrt{3}}{t} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$r = 2\sqrt{3}\sec\left(\theta - \frac{\pi}{6}\right)$$

> This relation is true for any point P on the line and, as $OP = r$ this gives you the polar equation of the line.

2 a



At $\theta = -\frac{\pi}{6}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $a\cos 6\theta$ has its greatest value of a. Then, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{6}$. Between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, $\cos 6\theta$ is negative and, as $r \geqslant 0$, the curve does not exist. The pattern repeats itself in the other intervals where the curve exists.

b $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2 \,\mathrm{d}\theta$

$$\frac{1}{2}\int a^2\cos^2 3\theta \,\mathrm{d}\theta = \frac{a^2}{2}\int\left(\frac{1}{2}\cos 6\theta + \frac{1}{2}\right)\mathrm{d}\theta$$

$$= \frac{a^2}{4}\left(\frac{\sin 6\theta}{6} + \theta\right)$$

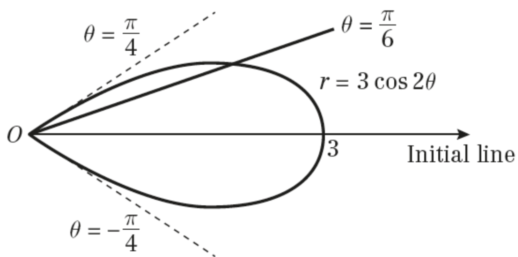
$$A = \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{a^2}{4}\left[\frac{1}{6}(0-0) + \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{a^2}{4} \times \frac{\pi}{3} = \frac{\pi}{12}a^2$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 3\theta$.

$\sin\left(6 \times \frac{\pi}{6}\right) = \sin\pi = 0$

3 a



At $\theta = -\frac{\pi}{4}$, $r = 0$, As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $3\cos 2\theta$ has its greatest value of 3. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

3 b $A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta$

$$\frac{1}{2}\int r^2 \mathrm{d}\theta = \frac{1}{2}\int 9\cos^2 2\theta \mathrm{d}\theta$$

$$= \frac{9}{2}\int\left(\frac{\cos 4\theta}{2}+\frac{1}{2}\right)\mathrm{d}\theta = \frac{9}{4}\int(\cos 4\theta + 1)\,\mathrm{d}\theta$$

$$= \frac{9}{4}\left[\frac{\sin 4\theta}{4}+\theta\right]$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$A = \frac{9}{4}\left[\frac{\sin 4\theta}{4}+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{9}{4}\left[\frac{1}{4}\left(0-\frac{\sqrt{3}}{2}\right)+\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right]$$

$$= -\frac{9\sqrt{3}}{32}+\frac{3\pi}{16} = \frac{3}{32}(2\pi - 3\sqrt{3})$$

$\sin\left(4\times\frac{\pi}{6}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

c Let $y = r\sin\theta = 3\cos 2\theta\sin\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -6\sin 2\theta\sin\theta + 3\cos 2\theta\cos\theta = 0$$

$$2\sin 2\theta\sin 2\theta = \cos 2\theta\cos\theta$$

$$\frac{\sin 2\theta\sin\theta}{\cos 2\theta\cos\theta} = \tan 2\theta\tan\theta = \frac{1}{2}$$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of such a point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

$$\frac{2\tan^2\theta}{1-\tan^2\theta} = \frac{1}{2}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.

$$4\tan^2\theta = 1-\tan^2\theta$$

$$5\tan^2\theta = 1$$

$$\tan\theta = \frac{1}{\sqrt{5}}$$

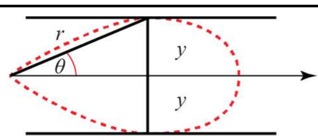
One value of $\tan\theta$ is sufficient to complete the question. r is not needed.

The distance between the two tangents is given by

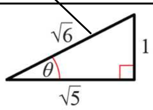
$$2y = 2r\sin\theta = 6\cos 2\theta\sin\theta = 6(2\cos^2\theta - 1)\sin\theta$$

$$= 6\times\left(2\times\frac{5}{6}-1\right)\times\frac{1}{\sqrt{6}} = 6\times\frac{2}{3}\times\frac{1}{\sqrt{6}}$$

$$= \frac{2\sqrt{6}}{3}$$



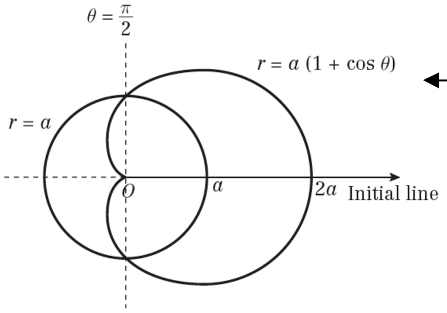
This sketch shows you that the distance between the two tangents parallel to the initial line is given by $2y = 2r\sin\theta$.



As $(\sqrt{5})^2 + 1^2 = (\sqrt{6})^2$, if $\tan\theta = \frac{1}{\sqrt{5}}$,

then $\sin\theta = \frac{1}{\sqrt{6}}$ and $\cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$.

4 a



$r = a(1+\cos\theta)$ is a cardioid and $r = a$ is a circle centre O, radius a.

b Let $y = r\sin\theta = a(1+\cos\theta)\sin\theta$

$$= a\sin\theta + a\cos\theta\sin\theta = a\sin\theta + \frac{a}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = a\cos\theta + a\cos 2\theta = 0$$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinates θ of such points by finding the values of θ for which $y = r\sin\theta$ has stationary values.

$$\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$$

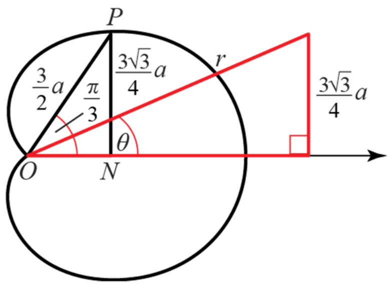
$$\cos\theta = \frac{1}{2},\ \cos\theta = -1$$

$$\theta = \pm\frac{\pi}{3},\ \theta = \pi$$

At $\theta = \frac{\pi}{3}$,

$$r = a\left(1+\cos\frac{\pi}{3}\right) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$$

And $y = r\sin\frac{\pi}{3} = \frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a$



You find the distance (labelled PN in the diagram above) from the point where the tangent meets the curve to the initial line.

The polar equation of the tangent is given by

$$r\sin\theta = \frac{3\sqrt{3}}{4}a$$

$$r = \frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$$

The polar equation is found by trigonometry in the triangle marked in red on the diagram above.

Similarly at $\theta = -\frac{\pi}{3}$, the equation of the

tangent is $r = -\frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$.

At $\theta = \pi$, the equation of the tangent is

$\theta = \pi$.

It is easy to overlook this case. The half-line $\theta = \pi$ does touch the cardioid at the pole.

4 c The circle and the cardioids meet when

$a = a(1+\cos\theta) \Rightarrow \cos\theta = \theta$

$\theta = \pm\frac{\pi}{2}$

To find the area of the cardioid between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$A = 2\times\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,\mathrm{d}\theta$

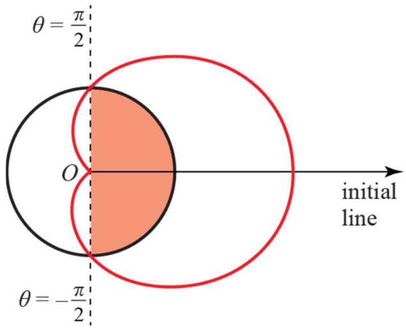
The total area is twice the area above the initial line.

$$\int r^2\,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2\,\mathrm{d}\theta = \int a^2(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$



$$A = a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{3\pi}{4}+2\right)$$

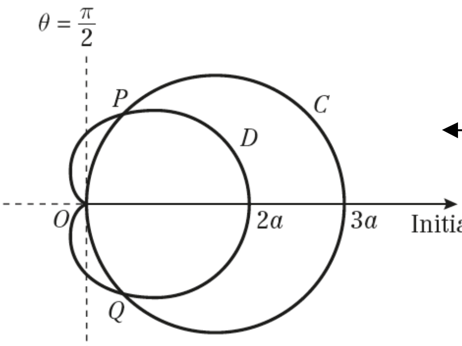
The required area is A less half of the circle

$$\left(\frac{3\pi}{4}+2\right)a^2 - \frac{1}{2}\pi a^2 = \frac{1}{4}\pi a^2 + 2a^2$$

$$= \left(\frac{\pi+8}{4}\right)a^2, \text{ as required.}$$

The area you are asked to find is inside the cardioid and outside the circle. You find it by subtracting the shaded semi-circle from the area of the cardioid bounded by the half-lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

5 a



The curve C is a circle of diameter $3a$ and the curve D is a cardioid.
The points of intersection of C and D have been marked on the diagram. The question does not specify which is P and which is Q. They could be interchanged. This would make no substantial difference to the solution of the question.

5 b The points of intersection of C and D are given by

$$3\not{a}\cos\theta = \not{a}(1+\cos\theta)$$

$$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

In this question $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$.

Where $\cos\theta = \frac{1}{2}$

$$r = 3a\cos\frac{\pi}{3} = 3a \times \frac{1}{2} = \frac{3}{2}a$$

$$P: \left(\frac{3}{2}a, \frac{\pi}{3}\right), \; Q: \left(\frac{3}{2}a, -\frac{\pi}{3}\right)$$

c The area between D, the initial line and OP is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2\,\mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

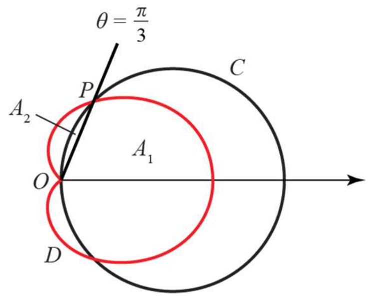
$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$



By the symmetry of the figure, to find the area inside C but outside D, you subtract two areas A_1 and two areas A_2 from the area inside C. C is a circle of radius $\frac{3a}{2}$.

d Let the smaller area enclosed by C and the half-line $\theta = \frac{\pi}{3}$ be A_2.

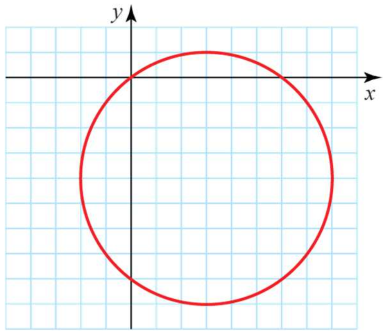
$$R = \pi\left(\frac{3a}{2}\right)^2 - 2A_1 - 2A_2$$

$$= \frac{9a^2\pi}{4} - \frac{2a^2}{16}(4\pi+9\sqrt{3}) - \frac{6a^2}{16}(2\pi-3\sqrt{3})$$

This is twice the area you are given in the question.

$$= \frac{9a^2\pi}{4} - \frac{\pi a^2}{2} - \frac{9\sqrt{3}a^2}{8} - \frac{3\pi a^2}{4} + \frac{9\sqrt{3}a^2}{8} = \pi a^2\text{, as required.}$$

6 a The locus is a circle of centre $3-4i$ and radius 5, so the Argand diagram is the following:



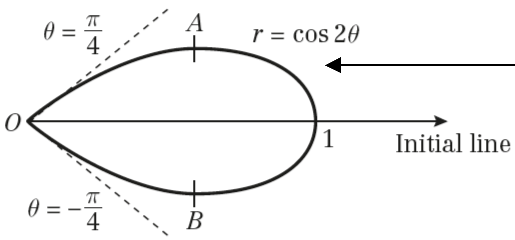
b Suppose z is such that $|z-3+4i|=5$. Then assuming $z=r(\cos\theta+i\sin\theta)$ we get that $|r\cos\theta-3+i(r\sin\theta+4)|=5$; but the magnitude of this complex number is given by $\sqrt{(r\cos\theta-3)^2+(r\sin\theta+4)^2}$, so we get the following:

$$r^2\cos^2\theta-6r\cos\theta+9+r^2\sin^2\theta+8r\sin\theta+16=25$$
$$r^2+25-6r\cos\theta+8r\sin\theta=25$$
$$r=6\cos\theta-8\sin\theta$$

c The area of A is the area of the circle minus the areas that are enclosed in the fourth Cartesian quadrant. Now consider the 63circular sector enclosed between the radii that intersect the circle on the real line. The intersections between the circle and the real line are the origin and 6. Then since the circle has centre $3-4i$, if we interpret this in the Cartesian plane we can find the angle between the radii: it is $\arccos\left(\frac{16-9}{25}\right)$, as the cosine of the angle is given by the ratio between the inner product and the product of the magnitudes of the two radii, seen as vector. Then the area of the circular sector is $\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}$. From this we subtract the area of the triangle formed by the two real points and the circle, which is 12. With the same procedure applied to the complex line we find that the arc between the origin and $-8i$ encloses an area of $\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12$. Then the area of A is:

$$25\pi-\left(\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12\right)-\left(\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}-12\right)$$
$$=78.5-11.2-4.1=63.3$$

7 a



At $\theta=-\frac{\pi}{4}$, $r=0$. As θ increases, r increases until $\theta=0$. For $\theta=0$, $\cos 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to 0 at $\theta=\frac{\pi}{4}$.

7 b $y = r\sin\theta = \cos 2\theta \sin\theta$

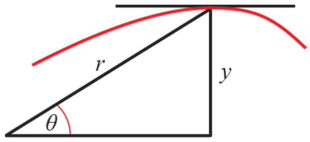
$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$

$-4\sin\theta\cos\theta\sin\theta + (1-2\sin^2\theta)\cos\theta = 0$

$\cos\theta(-4\sin^2\theta + 1 - 2\sin^2\theta) = 0$

At A and B, $\cos\theta \neq 0$

$6\sin^2\theta = 1$



Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. The diagram above shows that $y = r\sin\theta$. You find the polar coordinates θ of the points by finding the values of θ for which $r\sin\theta$ has a maximum or minimum value.

$\sin\theta = \pm\frac{1}{\sqrt{6}}$

$\theta = \pm 0.420\,534\ldots$

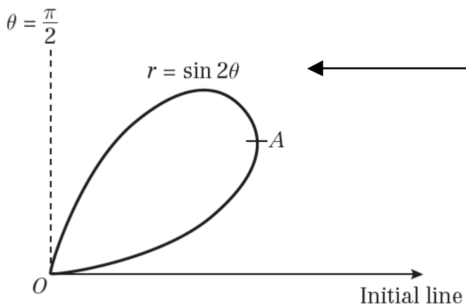
$r = \cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{6} = \frac{2}{3}$

To 3 significant figures, the polar coordinates of A and B are

(0.667, 0.421) and (0.667, −0.421).

r has an exact value but the question specifically asks for 3 significant figures. Unless the question specifies otherwise, in polar coordinates, you should always give the value of the angle in radians.

8 a



At $\theta = 0$, $r = 0$. As θ increases, r increases until $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{4}$, $\sin 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to $\sin\left(2 \times \frac{\pi}{2}\right) = \sin\pi = 0$ at $\theta = \frac{\pi}{2}$.

8 b $x = r\cos\theta = \sin 2\theta\cos\theta$

$$\frac{dx}{d\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$$
$$= 4\cos^3\theta - 2\cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 6\cos^3\theta - 4\cos\theta = 0$$
$$2\cos\theta(3\cos^2\theta - 2) = 0$$

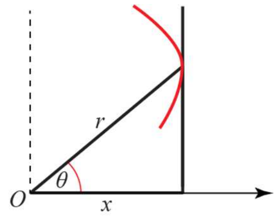
At A, $\cos\theta \neq 0$

$$\cos^2\theta = \frac{2}{3}$$
$$\cos\theta = \left(\frac{2}{3}\right)^{\frac{1}{2}}, \text{ for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$
$$\theta = 0.615\,479\ldots$$

By calculator

$r = \sin 2\theta = 0.942\,809\ldots$

To 3 significant figures, the coordinates of A are (0.943, 0.615)

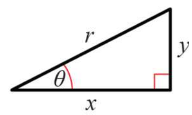


Where the tangent at a point is parallel to $\theta = \frac{\pi}{2}$ (which is the same as being perpendicular to the initial line), the distance x from the half line $\theta = \frac{\pi}{2}$ has a stationary value. The diagram above shows that $x = r\cos\theta$. You find the polar coordinates θ of such points by finding the values of θ for which $r\cos\theta$ has a maximum or minimum value.

9 a $r = 6\cos\theta$

Multiplying the equation by r

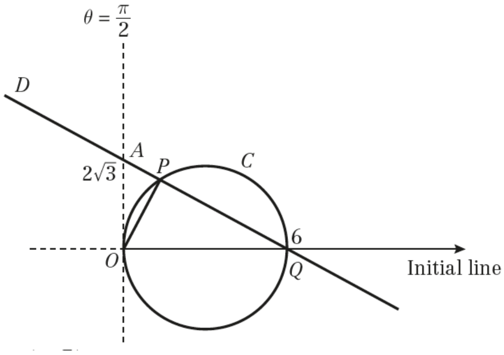
$$r^2 = 6r\cos\theta$$
$$x^2 + y^2 = 6x$$
$$x^2 - 6x + 9 + y^2 = 0$$
$$(x-3)^2 + y^2 = 9$$
$$r = 3\sec\left(\frac{\pi}{3} - \theta\right)$$
$$3 = r\cos\left(\frac{\pi}{3} - \theta\right) = r\cos\frac{\pi}{3}\cos\theta + r\sin\frac{\pi}{3}\sin\theta$$
$$= \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$$
$$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$
$$x + \sqrt{3}y = 6$$



This diagram illustrates the relations between polar and Cartesian coordinates. The relations you need to solve the question are $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.

This is an acceptable answer but putting the equation into a form which shows that the curve is a circle, centre (3, 0) and radius 3, helps you to draw the sketch in part **b**.

9 b



The initial line is the positive x-axis and the half-line $u = \frac{\pi}{2}$ is the positive y-axis.

At $x = 0$, $x + \sqrt{3}y = 6$ gives $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

c By inspection, the polar coordinates of Q are (6, 0)

The question does not say which point is P and which is Q. You can choose which is which.

$\angle OPQ = 90^\circ$

The angle in a semi-circle is a right angle.

In the triangle OAQ

$$\tan AQO = \frac{OA}{OQ} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \Rightarrow \angle AQO = 30^\circ$$

In the triangle OPQ

$$OP = OQ \sin PQO = 6 \sin 30^\circ = 3$$

$$\angle POQ = 180^\circ - 90^\circ - 30^\circ = 60^\circ = \frac{\pi}{3}$$

Hence the polar coordinates of P are

$$(OP, \angle POQ) = \left(3, \frac{\pi}{3}\right)$$

10 $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \,\mathrm{d}\theta$

You need to know the formula for the area of polar curves $A = \frac{1}{2}\int r^2 \,\mathrm{d}\theta$. In this question, the diagram shows that the limits are 0 and $\frac{\pi}{2}$.

$$\frac{1}{2}\int r^2 \,\mathrm{d}\theta = \frac{1}{2}\int a^2 \sin 2\theta \,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]$$

$$A = \frac{a^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}[1 - (-1)]$$

$$= \frac{1}{2}a^2$$

$\cos\left(2 \times \frac{\pi}{2}\right) = \cos \pi = -1$ and $\cos 0 = 1$.

$$A = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta$$

The lower limit, $\frac{\pi}{8}$, is given by the polar equation of m. The upper limit, $\frac{\pi}{4}$, can be identified from the domain of definition, $0 \leqslant \theta \leqslant \frac{\pi}{4}$ given in the question and the diagram.

11 $\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 16a^2\cos^2 2\theta \mathrm{d}\theta$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 2\theta \mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \left(\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$= 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$A = 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= 4a^2\left[\left(\frac{\pi}{4}-\frac{\pi}{8}\right)+\left(0-\frac{1}{4}\right)\right]$$

$\sin\left(4\times\frac{\pi}{4}\right) = \sin\pi = 0$ and $\sin\left(4\times\frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$.

$$= 4a^2\left[\frac{\pi}{8}-\frac{1}{4}\right]$$

$$= \frac{1}{2}a^2(\pi - 2)$$

12 $A = 2\times\frac{1}{2}\int_0^{\pi} r^2 \mathrm{d}\theta$

The method used here is to find twice the area above the initial line.

$$= \int_0^{\pi} a^2\left(1+\frac{1}{2}\cos\theta\right)^2 \mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(1+\cos\theta+\frac{1}{4}\cos^2\theta\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= a^2\int_0^{\pi}\left(1+\cos\theta+\frac{1}{8}\cos 2\theta+\frac{1}{8}\right)\mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(\frac{9}{8}+\cos\theta+\frac{1}{8}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{9}{8}\theta+\sin\theta+\frac{\sin 2\theta}{16}\right]_0^{\pi}$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{9}{8}\theta$, which has a non-zero value at π.

$$= a^2\times\frac{9}{8}\pi = \frac{9}{8}\pi a^2$$

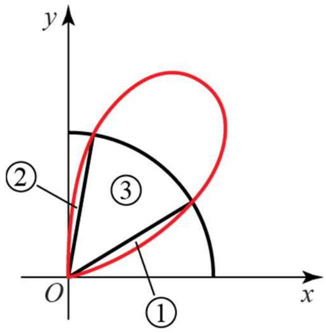
13 a The curves intersect at

$$\frac{1}{2} = \sin 2\theta$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

b



The shaded area can be broken up into three parts. You can find the small areas labelled (**1**) and (**2**), which are equal in area, by integration. The larger area is a sector of a circle and you find this using $A = \frac{1}{2}r^2\theta$, where θ is in radians.

The area of the sector (**3**) is given by

$$A_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \frac{\pi}{3} = \frac{\pi}{24}$$

The radius of the sector is $\frac{1}{2}$ and the angle is $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.

The area of (**1**) is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{12}} r^2 \, d\theta$$

Using $\cos 2A = 1 - 2\sin^2 A$ with $A = 2\theta$.

$$\frac{1}{2}\int \sin^2 2\theta \, d\theta = \frac{1}{2}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$A_1 = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - 0 - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$\sin\left(4 \times \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

The area of the shaded region is given by

$$2 \times A_1 + A_3 = \frac{1}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \frac{\pi}{24} = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

14 a Let $y = r\sin\theta$

> Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of the point by finding the value θ for which $y = r\sin\theta$ has a stationary value.

$$y = a(3+\sqrt{5}\cos\theta)\sin\theta$$

$$= 3a\sin\theta + \sqrt{5}a\cos\theta\sin\theta = 3a\sin\theta + \frac{\sqrt{5}a}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = 3a\cos\theta + \sqrt{5}a\cos 2\theta = 0$$

$$3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) = 0$$

$$2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$$

$$\cos\theta = -3 \pm \frac{\sqrt{(9+40)}}{4\sqrt{5}}$$

> As $|\cos\theta| \leqslant 1$, you reject the value $-\frac{10}{4\sqrt{5}} \approx -1.118$.

$$= \frac{-3+7}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

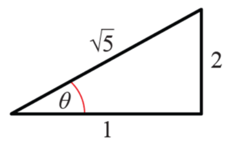
By calculator
$\theta = \pm 1.107$ (3 d.p.)

At $\cos\theta = \frac{1}{\sqrt{5}}$

$$r = a(3+\sqrt{5}\cos\theta) = a\left(3+\sqrt{5}\times\frac{1}{\sqrt{5}}\right) = 4a$$

The polar coordinates are

P: $(4a, 1.107)$, Q: $(4a, -1.107)$

> 
>
> As $1^2 + 2^2 = (\sqrt{5})^2$, the diagram illustrates that if $\cos\theta = \frac{1}{\sqrt{5}}$ then $\sin\theta = \frac{2}{\sqrt{5}}$.

b $PQ = 2y = 2r\sin\theta$

$$= 2\times 4a \times \frac{2}{\sqrt{5}} = \frac{16}{\sqrt{5}}a = 20\text{m, given}$$

$$a = \frac{20\sqrt{5}}{16}\text{m} = \frac{5\sqrt{5}}{4}\text{m}$$

14 c Total area$=2\times\frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$

$r = a(3+\sqrt{5}\cos\theta)$

$$\text{so area} = a^2\int_0^{\pi}(3+\sqrt{5}\cos\theta)^2\,\mathrm{d}\theta$$
$$= a^2\int_0^{\pi}(9+6\sqrt{5}\cos\theta+5\cos^2\theta)\mathrm{d}\theta$$
$$= a^2\left[9\theta+6\sqrt{5}\sin\theta+\frac{5}{4}\sin 2\theta+\frac{5}{2}\theta\right]_0^{\pi}$$
$$= a^2\left[\frac{23}{2}\theta+6\sqrt{5}\sin\theta+\frac{5}{4}\sin 2\theta\right]_0^{\pi}$$
$$= a^2\left[\frac{23}{2}\pi\right]$$
$$= \frac{23}{2}\pi a^2$$

$$a = \frac{5\sqrt{5}}{4}$$

$$\text{so area} = \frac{23}{2}\left(\frac{5\sqrt{5}}{4}\right)^2\pi$$
$$= \frac{2875}{32}\pi\ \text{m}^2$$

15 a Area $=\frac{1}{2}\int_{-\pi}^{\pi} r^2\,\mathrm{d}\theta$

$$= \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2\mathrm{d}\theta$$
$$= \frac{a^2}{2}\int_{-\pi}^{\pi}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$
$$= \frac{a^2}{2}\int_{-\pi}^{\pi}\mathrm{d}\theta + a^2\int_{-\pi}^{\pi}\cos\theta\,\mathrm{d}\theta+\frac{a^2}{2}\int_{-\pi}^{\pi}\cos^2\theta\,\mathrm{d}\theta$$
$$= a^2\pi + a^2\left[\sin\theta\right]_{-\pi}^{\pi}+\frac{a^2}{4}\int_{-\pi}^{\pi}\mathrm{d}\theta+\frac{a^2}{4}\int_{-\pi}^{\pi}\cos 2\theta\,\mathrm{d}\theta \quad \text{using the identity } \cos 2\theta\equiv 2\cos^2\theta-1$$
$$= a^2\pi+\frac{a^2\pi}{2}+\frac{a^2}{4}\left[\frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$$
$$= \frac{3\pi a^2}{2}$$

15 b At A and B, $\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta)=0$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(r\cos\theta)&=\frac{\mathrm{d}}{\mathrm{d}\theta}(a\cos\theta(1+\cos\theta))\\&=a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta+\cos^2\theta)\\&=-a\sin\theta+a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos^2\theta)\\&=-a\sin\theta-2a\cos\theta\sin\theta,\text{ using the product rule}\\&=-a\sin\theta(1+2\cos\theta)\end{aligned}$$

Setting this equal to 0 gives:

$\sin\theta(1+2\cos\theta)=0$

$\sin\theta=0$ or $\cos\theta=-\frac{1}{2}$

$\therefore\ \theta=0$ or $\pm\frac{2\pi}{3}$

But $\theta=0$ is where C intersects the initial line so $\theta=\pm\frac{2\pi}{3}$ at A and B.

$\theta=\pm\frac{2\pi}{3}\Rightarrow r=a\left(1+\cos\frac{2\pi}{3}\right)=\frac{a}{2}$

So the polar coordinates of A and B are A: $\left(\frac{a}{2},\frac{2\pi}{3}\right)$ and B: $\left(\frac{a}{2},-\frac{2\pi}{3}\right)$.

c When C intersects the initial line, $r=2a$.

Therefore, length $\mathrm{WX}=2a$ + length of x–component of vector $\overrightarrow{\mathrm{OA}}$

$=2a+\frac{2}{a}\cos\frac{\pi}{3}=2a+\frac{a}{4}=\frac{9a}{4}$.

d Area $\mathrm{WXYZ}=\frac{9a}{4}\times\frac{3a\sqrt{3}}{2}=\frac{27\sqrt{3}a^2}{8}$.

e Area wasted = Area WXYZ − Area inside C

$$=\frac{27\sqrt{3}a^2}{8}-\frac{3\pi a^2}{2}$$

So when $a=10$ cm, the area of card wasted is

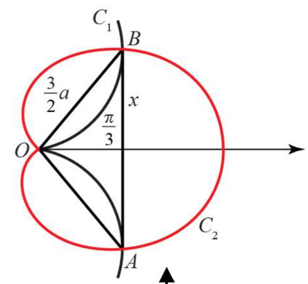
$\frac{2700\sqrt{3}}{8}-\frac{300\pi}{2}=113\text{ cm}^2$ (to 3 s.f.)

16 a C_1 and C_2 intersect where

$$3\cancel{a}(1-\cos\theta)=\cancel{a}(1+\cos\theta)$$
$$3-3\cos\theta=1+\cos\theta$$
$$4\cos\theta=2\Rightarrow\cos\theta=\frac{1}{2}$$
$$\theta=\pm\frac{\pi}{3}$$

Where $\cos\theta=\frac{1}{2}$

$$r=a(1+\cos\theta)=a\left(1+\frac{1}{2}\right)=\frac{3}{2}a$$
$$A:\left(\frac{3}{2}a,-\frac{\pi}{3}\right),\ B:\left(\frac{3}{2}a,\frac{\pi}{3}\right)$$

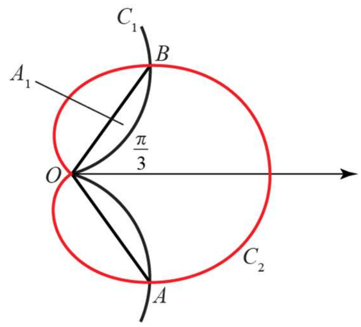


Referring to the diagram, $\frac{x}{\frac{3}{2}a}=\sin\frac{\pi}{3}\Rightarrow x=\frac{3}{2}a\sin\frac{\pi}{3}$ and $AB=2x$.

b $AB=2\times\frac{3}{2}a\sin\frac{\pi}{3}=3a\times\frac{\sqrt{3}}{2}$

$=\frac{3\sqrt{3}}{2}a$, as required.

16 c The area A_1 enclosed by OB and C_1 is given by



$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int 9a^2(1-\cos\theta)^2 \mathrm{d}\theta = \int 9a^2(1-2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= 9a^2\int\left(1-2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

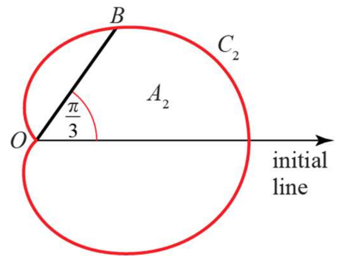
$$= 9a^2\int\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{2}a^2\left[\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{9a^2}{16}(4\pi-7\sqrt{3})$$

The area A_2 enclosed by the initial line, C_2 and OB is given by



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_2 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$

The required area R is given by

$$R = 2(A_2 - A_1)$$

$$= 2\left[\frac{a^2}{16}(4\pi+9\sqrt{3})-\frac{9a^2}{16}(4\pi-7\sqrt{3})\right]$$

$$= \frac{2a^2}{16}\left[4\pi+9\sqrt{3}-(36\pi-63\sqrt{3})\right]$$

$$= \frac{a^2}{8}\left[72\sqrt{3}-32\pi\right] = (9\sqrt{3}-4\pi)a^2$$

16 d $\frac{3\sqrt{3}}{2}a = 4.5\,\text{cm}$

$a = \frac{9}{3\sqrt{3}}\,\text{cm} = \sqrt{3}\,\text{cm}$

The area of the badge is

$(9\sqrt{3} - 4\pi)a^2 = (9\sqrt{3} - 4\pi) \times 3\,\text{cm}^2$

$= 9.07\,\text{cm}^2$ (3 s.f.)

> You use the result from part **b** to find a and substitute the value of a into the result of part **c**.

17 a $A: (5a,\ 0),\ B: (3a,\ 0)$

> For A, at $\theta = 0$, $r = a(3 + 2\cos 0) = a(3+2) = 5a$.
> For B, at $\theta = 0$, $r = a(5 - 2\cos 0) = a(5-2) = 3a$.

b The curves intersect where

$\not{a}(3 + 2\cos\theta) = \not{a}(5 - 2\cos\theta)$

$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

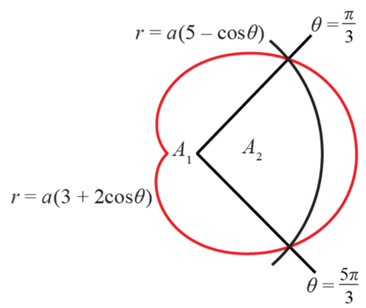
> In this question $0 \le \theta < 2\pi$.

Where $\cos\theta = \frac{1}{2}$

$r = a(3 + 2\cos\theta) = a\left(3 + 2 \times \frac{1}{2}\right) = 4a$

$C: \left(4a, \frac{5\pi}{3}\right),\ D: \left(4a, \frac{\pi}{3}\right)$

17 c The area A1 enclosed by $r = a(3 + 2\cos\theta)$ and the half-lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ is given by



The shaded area in the question is the sum of the two areas A_1 and A_2 shown in the diagram above. It is important that you carefully distinguish which curve is which.

$$A_1 = 2 \times \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2 \mathrm{d}\theta$$

$$\begin{aligned}\int r^2 \mathrm{d}\theta &= \int a^2(3+2\cos\theta)^2 \mathrm{d}\theta \\ &= a^2\int (9 + 12\cos\theta + 4\cos^2\theta)\mathrm{d}\theta \\ &= a^2\int (9 + 12\cos\theta + 2\cos 2\theta + 2)\mathrm{d}\theta \\ &= a^2\int (11 + 12\cos\theta + 2\cos 2\theta)\mathrm{d}\theta \\ &= a^2\left[11\theta + 12\sin\theta + \sin 2\theta\right]\end{aligned}$$

$$\begin{aligned}A_1 &= a^2\left[11\theta + 12\sin\theta + \sin 2\theta\right]_{\frac{\pi}{3}}^{\pi} \\ &= a^2\left[11\left(\pi - \frac{\pi}{3}\right) + 12\left(0 - \frac{\sqrt{3}}{2}\right) + \left(0 - \frac{\sqrt{3}}{2}\right)\right] \\ &= a^2\left[\frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right]\end{aligned}$$

The area A_2 enclosed by $r = a(5 - 2\cos\theta)$ and the half-lines $\theta = \frac{5\pi}{3}$ and $\theta = \frac{\pi}{3}$ is given by

$$A_2 = 2 \times \frac{1}{2}\int_{0}^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\begin{aligned}\int r^2 \mathrm{d}\theta &= \int a^2(5-2\cos\theta)^2 \mathrm{d}\theta = a^2\int (25 - 20\cos\theta + 4\cos^2\theta)\mathrm{d}\theta \\ &= a^2\int (25 - 20\cos\theta + 2\cos 2\theta + 2)\mathrm{d}\theta \\ &= a^2\int (27 - 20\cos\theta + 2\cos 2\theta)\mathrm{d}\theta \\ &= a^2\left[27\theta - 20\sin\theta + \sin 2\theta\right]\end{aligned}$$

The double angle formulae, here $\cos 2\theta = 2\cos^2\theta - 1$, are used in all questions involving the areas of cardioids.

$$\begin{aligned}A_2 &= a^2\left[27\theta - 20\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{3}} \\ &= a^2\left[27 \times \frac{\pi}{3} - 20 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] \\ &= a^2\left[\frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right]\end{aligned}$$

The area of the overlapping region is given by

$$\begin{aligned}A_1 + A_2 &= a^2\left(\frac{22\pi}{3} - \frac{13\sqrt{3}}{2} + \frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right) \\ &= a^2\left(\frac{49\pi}{3} - 16\sqrt{3}\right) \\ &= \frac{a^2}{3}(49\pi - 48\sqrt{3}),\text{ as required.}\end{aligned}$$

18 $2\tanh x - 1 = 0$

$$2\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1$$

$$2e^x - 2e^{-x} = e^x + e^{-x}$$

$$e^x = 3e^{-x}$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

You multiply both sides of this equation by e^x.

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

19

$$5 = 3\cosh x$$

$$5 = 3\left(\frac{e^x + e^{-x}}{2}\right)$$

$$10 = 3e^x + 3e^{-x}$$

$$3e^x - 10 + 3e^{-x} = 0$$

$$3e^{2x} - 10e^x + 3 = 0$$

$$(3e^x - 1)(e^x - 3) = 0$$

$$e^x = \frac{1}{3},\ e^x = 3$$

$$x = \ln\left(\frac{1}{3}\right),\ \ln 3$$

The wording of the question requires you to use the definition $\cosh x = \frac{e^x + e^{-x}}{2}$.

You multiply this equation throughout by e^x.

You may find it helpful to substitute $y = e^x$ and then, factorise, $3y^2 - 10y + 3 = (3y - 1)(y - 3) = 0$. This gives $y = \frac{1}{3}$ and $y = 3$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 3$. With practice, the substitution can be omitted.

The answer $x = -\ln 3$ would also be acceptable.

20 The curves intersect when

$$5\sinh x = 4\cosh x$$
$$5\left(\frac{e^x - e^{-x}}{2}\right) = 4\left(\frac{e^x + e^{-x}}{2}\right)$$
$$5e^x - 5e^{-x} = 4e^x + 4e^{-x}$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$e^x = 9e^{-x}$$
$$e^{2x} = 9$$
$$2x = \ln 9$$
$$x = \frac{1}{2}\ln 9 = \ln\sqrt{9} = \ln 3$$

Using the law of logarithms $n\ln a = \ln a^n$ with $n = \frac{1}{2}$ and $a = 9$.

$$y = 5\sinh(\ln 3) = 5\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) = \frac{5}{2}\times\left(3 - \frac{1}{3}\right)$$
$$= \frac{5}{2}\times\frac{8}{3} = \frac{20}{3}$$
$$p = 3,\ q = \frac{20}{3}$$

$e^{\ln 3} = 3$ and $e^{-\ln 3} = e^{\ln 1 - \ln 3} = e^{\ln\frac{1}{3}} = \frac{1}{3}$, using $\ln 1 = 0$ and the law of logarithms $\ln a - \ln b = \ln\frac{a}{b}$.

21 $5\cosh x - 2\sinh x = 11$

$$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

$$5e^x + 5e^{-x} - 2e^x + 2e^{-x} = 22$$
$$3e^x - 22 + 7e^{-x} = 0$$

You multiply this equation throughout by e^x.

$$3e^{2x} - 22e^x + 7 = 0$$
$$(3e^x - 1)(e^x - 7) = 0$$

You may find it helpful to substitute $y = e^x$ and then, factorising $3y^2 - 22y + 7 = (3y - 1)(y - 7) = 0$. This gives $y = \frac{1}{3}$ and $y = 7$ and, hence, $e^x = \frac{1}{3}$ and $e^x = 7$. With practice, the substitution can be omitted.

$$e^x = \frac{1}{3},\ 7$$
$$x = \ln\frac{1}{3},\ \ln 7$$

22 $6\sinh 2x + 9\cosh 2x = 7$

$$6\left(\frac{e^{2x} - e^{-2x}}{2}\right) + 9\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 7$$

$$6e^{2x} - 6e^{-2x} + 9e^{2x} + 9e^{-2x} = 14$$

$$15e^{2x} - 14 + 3e^{-2x} = 0$$

$$15e^{4x} - 14e^{2x} + 3 = 0$$

$$(3e^{2x} - 1)(5e^{2x} - 3) = 0$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ replacing x by $2x$.

You multiply this equation throughout by e^{2x}.

$$e^{2x} = \frac{1}{3}, \frac{3}{5}$$

$$2x = \ln\frac{1}{3}, \ln\frac{3}{5}$$

$$x = \frac{1}{2}\ln\frac{1}{3}, \frac{1}{2}\ln\frac{3}{5}$$

$$p = \frac{1}{3}, \frac{3}{5}$$

You take the logarithms of both sides of this equation and use the property that $\ln e^{2x} = 2x$.

23 a $\sinh x + 2\cosh x = k$

$$\frac{e^x - e^{-x}}{2} + 2\left(\frac{e^x + e^{-x}}{2}\right) = k$$

$$e^x - e^{-x} + 2e^x + 2e^{-x} = 2k$$

$$3e^x - 2k + e^{-x} = 0$$

$$3e^{2x} - 2ke^x + 1 = 0$$

You use the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Let $y = e^x$

$$3y^2 - 2ky + 1 = 0$$

$$y = \frac{2k \pm \sqrt{(4k^2 - 12)}}{6}$$

$$= \frac{k \pm \sqrt{(k^2 - 3)}}{3} \quad \textbf{(1)}$$

Using the quadratic formula $y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

For real y

$k^2 - 3 \geqslant 0 \Rightarrow k \geqslant \sqrt{3},\ k \leqslant -\sqrt{3}$

As $y = e^x > 0$ for all real x, $k \leqslant -\sqrt{3}$ is rejected.

$k \geqslant \sqrt{3}$.

If $x \leqslant -\sqrt{3}$, then both $\frac{k + \sqrt{k^2 - 3}}{3}$ and $\frac{k - \sqrt{(k^2 - 3)}}{3}$ are negative.

b Using **(1)** above with $k = 2$

$$y = e^x = \frac{2 \pm \sqrt{(4 - 3)}}{3} = \frac{2 \pm 1}{3}$$

$$e^x = 1, \frac{1}{3} \Rightarrow x = \ln 1, \ln\frac{1}{3} = 0, -\ln 3$$

You could solve the equation in part **b** without using part **a** but it is efficient to use the work you have already done.

24 a $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$$

$$= \frac{4}{4} = 1, \text{ as required.}$$

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Rewriting the equation in terms of the exponential definitions of the hyperbolic functions, it becomes:

$$\frac{1}{\frac{e^x - e^{-x}}{2}} - \frac{2}{\frac{e^x - e^{-x}}{e^x + e^{-x}}} = 2$$

$$\frac{2}{e^x - e^{-x}} - \frac{2(e^x + e^{-x})}{e^x - e^{-x}} = 2$$

$$2 - 2e^x - 2e^{-x} = 2e^x - 2e^{-x}$$

$$4e^x = 2$$

$$x = \ln\frac{1}{2} = -\ln 2$$

25 a $2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$

$$= 2 \times \frac{e^{2x} + 2 + e^{-2x}}{4} - 1$$

$$= \frac{2e^{2x}}{4} + \frac{4}{4} + \frac{2e^{-2x}}{4} - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x, \text{ as required}$$

$(e^x + e^{-x})^2 = (e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2$
$= e^{2x} + 2 + e^{-2x}$

b Using the result in part **a**

$$\cosh 2x - 5\cosh x = 2$$

$$2\cosh^2 x - 1 - 5\cosh x = 2$$

$$2\cosh^2 x - 5\cosh x - 3 = 0$$

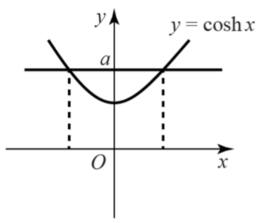
$$(2\cosh x + 1)(\cosh x - 3) = 0$$

$$\cosh x = -\frac{1}{2}, \ \cosh x = 3$$

$\cosh x = -\frac{1}{2}$ is impossible

$x = \pm \operatorname{arcosh} 3 = \pm \ln(3 + \sqrt{8})$

If $\operatorname{arcosh} x > 0$ then $\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$.
However if $\cosh x = a$, where $a > 0$, then there are two answers $x = \pm \ln(a + \sqrt{(a^2 - 1)})$



These answers can also be written as $x = \ln(3 \pm \sqrt{8})$.

26 a $4\cosh^3 x - 3\cosh x = 4\left(\frac{e^x + e^{-x}}{2}\right)^3 - 3\left(\frac{e^x + e^{-x}}{2}\right)$

$= \frac{e^x + 3e^x + 3e^{-x} + e^{-3x}}{2} - \frac{3e^x + 3e^{-x}}{2}$

$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x$, as required.

Using the binomial expansion
$(e^x + e^{-x})^3$
$= (e^x)^3 + 3(e^x)^2 \cdot e^{-x}$
$+ 3e^x(e^{-x})^2 + (e^{-x})^3$
$= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}$.

b $\cosh 3x = 5\cosh x$

Using the result in part **a**

$4\cosh^3 x - 3\cosh x = 5\cosh x$

$4\cosh^3 x - 8\cosh x = 0$

$4\cosh x(\cosh^2 x - 2) = 0$

As for all x, $\cosh x \geqslant 1$,

$\cosh x = \sqrt{2}$

$x = \pm\ln(\sqrt{2} + 1)$

$-\ln(\sqrt{2}+1) = \ln\left(\frac{1}{\sqrt{2}+1}\right) = \ln\left(\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$

$= \ln\left(\frac{\sqrt{2}-1}{1}\right) = \ln(\sqrt{2}-1)$

There are 3 possible answers to this cubic, $\cosh x = 0$, $\cosh x = -\sqrt{2}$ and $\cosh x = \sqrt{2}$. As for all real x, $\cosh x \geqslant 1$ only the last of the three gives real values of x.

Using $\text{arcosh}\, x = \ln\left(x + \sqrt{(x^2 - 1)}\right)$

The solutions of $\cosh 3x = 5\cosh x$, as natural logarithms, are $x = \ln(\sqrt{2} \pm 1)$.

27 a $\cosh A \cosh B - \sinh A \sinh B$

$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$

$= \frac{1}{4}\left(e^{A+B} + e^{-A+B} + e^{A-B} + e^{-A-B} - e^{A+B} + e^{-A+B} + e^{A-B} - e^{-A-B}\right)$

$= \frac{1}{4}\left(2e^{-A+B} + 2e^{A-B}\right) = \frac{e^{A-B} + e^{-(A-B)}}{2}$

$= \cosh(A - B)$, as required.

When multiplying out the brackets you must be careful to obtain all eight terms with the correct signs.

You use the definition $\cosh x = \frac{e^x + e^{-x}}{2}$ with $x = A + B$.

b $\cosh x \cosh 1 - \sinh x \sinh 1 = \sinh x$

$\cosh x \cosh 1 = \sinh x(1 + \sinh 1)$

$\tanh x = \frac{\cosh 1}{1 + \sinh 1}$

You expand $\cosh(x - 1)$ using the result of part **a**.

Divide both sides of this equation by $\cosh x(1 + \sinh 1)$ and use $\tanh x = \frac{\sinh x}{\cosh x}$.

$\tanh x = \frac{\frac{e+e^{-1}}{2}}{1 + \frac{e-e^{-1}}{2}} = \frac{e + e^{-1}}{2 + e - e^{-1}} = \frac{e^2 + 1}{e^2 + 2e - 1}$, as required.

28 a Let $y = \text{arsinh}\, x$

Then $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \dfrac{2x + \sqrt{(4x^2 + 4)}}{2}$

$= \dfrac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$

Taking the natural logarithms of both sides,

$y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

b $\text{arsinh}(\cot\theta) = \ln\left[\cot\theta + \sqrt{(1+\cot^2\theta)}\right]$

$= \ln(\cot\theta + \text{cosec}\,\theta)$

$= \ln\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{1}{\sin\theta}\right) = \ln\left(\dfrac{\cos\theta + 1}{\sin\theta}\right)$

$= \ln\left(\dfrac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \ln\left(\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) = \ln\left(\cot\dfrac{\theta}{2}\right)$, as required.

Using the result of part **a** with $x = \cot\theta$.

Using $\text{cosec}^2\theta = 1 + \cot^2\theta$.

You use both double angle formulae $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x\cos x$ with $2x = \theta$.

29 a Let $y = \text{artanh}\, x$

$x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$= \dfrac{e^y - e^{-y}}{e^y + e^{-y}} \times \dfrac{e^y}{e^y}$

$= \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$xe^{2y} + x = e^{2y} - 1$

$e^{2y}(1 - x) = 1 + x$

$e^{2y} = \dfrac{1+x}{1-x}$

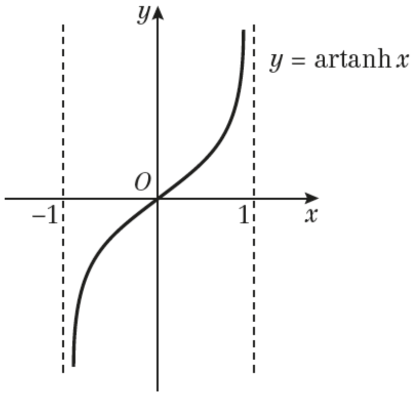
$2y = \ln\left(\dfrac{1+x}{1-x}\right)$

$y = \text{artanh}\, x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ for $|x| < 1$ as required.

You have been asked to prove a standard result in this question. You should learn the proof of this and other similar results as part of your preparation for the examination.

Make e^{2y} the subject of the formulas and then take logarithms.

29 b



You need to be able to sketch the graphs of the hyperbolic and inverse hyperbolic functions. When you sketch a graph you should show any important features of the curve. In this case, you should show the asymptotes $x=-1$ and $x=1$ of the curve.

c $x=\tanh\left[\ln\sqrt{(6x)}\right]$

$$\ln\sqrt{(6x)}=\operatorname{artanh}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

You use the result in part **a**.

$$\ln\sqrt{(6x)}=\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$\sqrt{(6x)}=\sqrt{\left(\frac{1+x}{1-x}\right)}$$

As you have squared this equation, you might have introduced an incorrect solution. It would be sensible to check on your calculator that $x=\frac{1}{2},\frac{1}{3}$ are solutions of $x=\tanh\left[\ln\sqrt{(6x)}\right]$. In this case, both are correct.

Squaring

$$6x=\frac{1+x}{1-x}$$

$$6x-6x^2=1+x$$

$$6x^2-5x+1=(3x-1)(2x-1)=0$$

$$x=\frac{1}{2},\frac{1}{3}$$

30 a $\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)+\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$

$$=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$

There are a number of different ways of starting this question. The method used here begins by using the log rule $\log a+\log b=\log ab$.

$$=\ln\frac{1-(1-x^2)}{x^2}=\ln\frac{x^2}{x^2}=\ln 1=0$$

This is the difference of two squares $a^2-b^2=(a-b)(a+b)$ with $a=1$ and $b=\sqrt{(1-x^2)}$.

Hence

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$$, as required.

30 b Let $y = \text{arcosh}\,\dfrac{1}{x} = \text{arsech}\,x$

$$\text{sech}\,y = x$$

$$\frac{2}{e^y + e^{-y}} = x$$

$$2 = xe^y + xe^{-y}$$

Multiply throughout by e^y and treat the result as a quadratic in e^y.

$$xe^{2y} - 2e^y + x = 0$$

$$e^y = \frac{2 \pm \sqrt{(4 - 4x^2)}}{2x} = \frac{1 \pm \sqrt{(1 - x^2)}}{x}$$

$$y = \ln\left(\frac{1 + \sqrt{(1 - x^2)}}{x}\right),\ \ln\left(\frac{1 - \sqrt{(1 - x^2)}}{x}\right)$$

$$= \pm\ln\left(\frac{1 + \sqrt{(1 - x^2)}}{x}\right), \text{ using the result of a}$$

Either of the two answers is possible but it is conventional to take $0 < \text{arsech}\,x \leqslant 1$.

$$y = \text{arsech}\,x = \ln\left(\frac{1 + \sqrt{(1 - x^2)}}{x}\right), \text{ as required.}$$

c Using $\text{sech}^2 x = 1 - \tanh^2 x$

$$3\tanh^2 x - 4\,\text{sech}\,x + 1 = 0,$$

$$3 - 3\,\text{sech}^2 x - 4\,\text{sech}\,x + 1 = 0$$

$$3\,\text{sech}^2 x + 4\,\text{sech}\,x - 4 = 0$$

$$(3\,\text{sech}\,x - 2)(\text{sech}\,x + 2) = 0$$

$\text{sech}\,x = -2$ is impossible with real values of x.

$$\text{sech}\,x = \frac{2}{3}$$

$$x = \pm\ln\left(\frac{1 + \sqrt{(1 - \frac{4}{9})}}{\frac{2}{3}}\right) = \pm\ln\left(\frac{3 + \sqrt{5}}{2}\right)$$

$x = \ln\left(\dfrac{3 \pm \sqrt{5}}{2}\right)$ is another correct form of the answer.

31 a $\cosh 3\theta = \cosh(2\theta + \theta)$

$= \cosh 2\theta \cosh\theta + \sinh 2\theta \sinh\theta$

$\cosh\theta = c$ and $\sinh\theta = s$

$\cosh 3\theta = (2c^2 - 1)c + 2sc \times s$

$= 2c^3 - c + 2s^2 c$

$= 2c^3 - c + 2(c^2 - 1)c$

$= 2c^3 - c + 2c^2 - 2c$

$= 4\cosh^3\theta - 3\cosh\theta$

$\cosh 5\theta = \cosh(3\theta + 2\theta) = \cosh 3\theta \cosh 2\theta + \sinh 3\theta \sinh 2\theta$

$\cosh 3\theta \cosh 2\theta = (4c^3 - 3c)(2c^2 - 1)$

$= 8c^5 - 10c^3 + 3c$

$\sinh 3\theta \sinh 2\theta = \sin(2\theta + \theta)\sinh 2\theta$

$= (\sinh 2\theta \cosh\theta + \cosh 2\theta \sinh\theta)\sinh 2\theta$

$= (2sc \times c + (2c^2 - 1)s)2sc$

$= 2(4c^2 - 1)s^2 c$

$= 2(4c^2 - 1)(c^2 - 1)c$

$= 8c^5 - 10c^3 + 2c$

Combining the results

$\cosh 5\theta = 8c^2 - 10c^3 + 3c + 8c^5 - 10c^3 + 2c$

$= 16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta$

In a complicated calculation like this, it is sensible to use the abbreviated notation suggested here but, if you intend to use a notation like this, you should state the notation in the solution so that the marker knows what you are doing.

You use the 'double angle' for hyperbolics $\cosh 2\theta = 2\cosh^2\theta - 1$ and $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$. The signs in these formulae can be worked out using Osborn's rule.

b $2\cosh 5x + 10\cosh 3x + 20\cosh x = 243,$

Letting $\cosh x = c$ and using the results in **a**

$32c^5 - 40c^3 + 10c + 40c^3 - 30c + 20c = 243$

$$c^5 = \frac{243}{32} \Rightarrow c = \frac{3}{2}$$

$$x = \pm\text{arcosh}\frac{3}{2} \approx \pm 0.96$$

You can use an inverse hyperbolic button on your calculator to find $\text{arcosh}\frac{3}{2}$.

32 a $\cosh 2x = \dfrac{e^{2x} + e^{-2x}}{2} = \dfrac{e^{2\ln k} + e^{-2\ln k}}{2}$

$$= \frac{e^{\ln k^2} + e^{\ln\frac{1}{k^2}}}{2} = \frac{1}{2}\left(k^2 + \frac{1}{k^2}\right)$$

$$= \frac{1}{2}\left(\frac{k^4 + 1}{k^2}\right) = \frac{k^4 + 1}{2k^2}, \text{ as requred.}$$

Using the law of logarithms $n\ln x = \ln x^n$, with $n = -2$, $-2\ln k = \ln k^{-2} = \ln\frac{1}{k^2}$.

32 b $f(x) = px - \tanh 2x$

For a stationary value

$f'(x) = p - 2\text{sech}^2 2x = 0$

$$p = 2\text{sech}^2 2x = \frac{2}{\cosh^2 2x}$$

Using the result of part **a** with $k = 2$

If $x = \ln 2$

$$\cosh 2x = \frac{2^4 + 1}{2 \times 2^2} = \frac{17}{8}$$

Hence

$$p = \frac{2}{\left(\frac{17}{8}\right)^2} = \frac{128}{289}$$

There is no 'hence' in this question but using the result in part **a** shortens the working. The question requires an exact fraction for the answer and you should not use a calculator other than, possibly, for multiplying and dividing fractions.

33 a $y = -x + \tanh 4x$

$$\frac{dy}{dx} = -1 + 4\text{sech}^2 4x = 0$$

$$\text{sech}^2 4x = \frac{1}{4} \Rightarrow \cosh^2 4x = 4$$

$\cosh 4x = 2$

$$4x = \text{arcosh}\, 2 = \ln(2 + \sqrt{3})$$

$$x = \frac{1}{4}\ln(2 + \sqrt{3})$$

As $\cosh x \geqslant 1$ for all real x, $\cosh 4x = -2$ is impossible.

For $x \geqslant 0$, there is only one value of x which gives a stationary value. The question tells you that the curve has a maximum point so, in this question, you need not show that this point is a maximum by, for example, examining the second derivative.

b $\tanh^2 4x = 1 - \text{sech}^2 4x = 1 - \frac{1}{4} = \frac{3}{4}$

As $x \geqslant 0$, $\tanh 4x = \frac{\sqrt{3}}{2}$

At $x = \frac{1}{4}\ln(2 + \sqrt{3})$

$$y = -x + \tanh 4x = -\frac{1}{4}\ln(2 + \sqrt{3}) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - \ln(2 + \sqrt{3})\right\}, \text{ as required.}$$

You need a value for $\tanh 4x$ and this is easiest found using the hyperbolic identity $\text{sech}^2 x = 1 - \tanh^2 x$.

34 a Deriving the function we get $f'(x) = 2\cosh^2 2x + 2\sinh^2 2x$; thus $f'(0) = 2$ is the first non-zero term. Deriving $f'(x)$ we find $f''(x) = 2[4\cosh 2x \sinh 2x + 4\sinh 2x \cosh 2x] = 16f(x)$

So the corresponding term in the Maclaurin series will be zero, but on the other hand we already know that $f^{(3)}(x) = (16f(x))'$ which is $16f'(x)$ by linearity of the derivative operator, so $f^{(3)}(0) = 32$. Similarly, we find that $f^{(4)}(0) = 0$ and $f^{(5)}(0) = 512$, so the first terms of the series expansion are as follows:

$$f(x) = 2x + \tfrac{16}{3}x^3 + \tfrac{64}{15}x^5 + \ldots$$

34 b From **a** it is easy to see that $f^{(n)}(x)=16^{n-2}f'(x)$ for n odd and $f^{(n)}(x)=16^{n-2}f(x)$ for n even; however, all the even terms of the Maclaurin series are zero. So a formula for the nth non-zero term is $\frac{2^{4(n-1)}}{n!}f'(0)x^{2n-1}$; since $f'(0)=2$ we can rewrite this as $\frac{2^{4n-3}}{(2n-1)!}x^{2n-1}$.

35 a First of all we note that $f(0)=1$, so this is the first non-zero term of the series expansion. The first derivative of the function is given by $\sinh x\cos 2x-2\sin 2x\cosh x$; of course, since one summand features the term $\sinh x$, and the other one features $\sin x$, $f'(0)=0$. The second derivative is given by $-4\sinh x\sin 2x-3\cosh x\cos 2x$, so $f''(0)=-3$: then the beginning of the series expansion is $1-\frac{3}{2}x^2$.

b This expression gives $f(0.1)\sim 0.985$. The correct value of the function is about 0.9849710. Then the error is given by $\frac{0.9849710-0.985}{0.9849710}\sim 0.000029$, which corresponds to an error of the 0.0029%.

36 $x=\frac{a}{\sinh\theta}=a(\sinh\theta)^{-1}$

$$\frac{dx}{d\theta}=-a(\sinh\theta)^{-2}\cosh\theta=-\frac{a\cosh\theta}{\sinh^2\theta}$$

When substituting remember to substitute for the dx as well as the rest of the integral.

$$\int\frac{1}{x\sqrt{(x^2+a^2)}}dx=\int\frac{1}{\frac{a}{\sinh\theta}\sqrt{\left(\frac{a^2}{\sinh^2\theta}+a^2\right)}}\times\frac{dx}{d\theta}d\theta$$

$$=\int\frac{\frac{-a\cosh\theta}{\sinh^2\theta}}{\frac{a^2\sqrt{1+\sinh^2\theta}}{\sinh^2\theta}}d\theta=\frac{-1}{a}\int\frac{\cosh\theta}{\cos\theta}d\theta$$

Use $1+\sinh^2\theta=\cosh^2\theta$ to simplify this expression.

$$=-\frac{1}{a}\int\frac{\cosh\theta}{\cosh\theta}d\theta=-\frac{1}{a}\int 1\,d\theta$$

$$=-\frac{1}{a}\theta+\text{constant}$$

$$=-\frac{1}{a}\text{arsinh}\left(\frac{a}{x}\right)+\text{constant, as required.}$$

As $x=\frac{a}{\sinh\theta}$, then $\sinh\theta=\frac{a}{x}$ and $\theta=\text{arsinh}\left(\frac{a}{x}\right)$.

37 a Let $y=\text{artanh}\,x$

$\tanh y=x$

Differentiate implicitly with respect to x

To differentiate a function $f(y)$ with respect to x you use a version of the chain rule $\frac{d}{dx}(f(y))=f'(y)\times\frac{dy}{dx}$.

$$\text{sech}^2 y\frac{dy}{dx}=1\Rightarrow\frac{dy}{dx}=\frac{1}{\text{sech}^2 y}=\frac{1}{1-\tanh^2 y}$$

$$=\frac{1}{1-x^2}\text{, as required}$$

37 b Using integration by parts and the result in part **a**

$$\int \text{artanh}\, x \,dx = \int 1 \times \text{artanh}\, x \,dx$$

$$= x\,\text{artanh}\, x - \int \frac{x}{1-x^2}\,dx$$

$$= x\,\text{artanh}\, x + \frac{1}{2}\ln(1-x^2) + A$$

You use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with $u = \text{artanh}\, x$ and $\frac{dv}{dx} = 1$. You know $\frac{du}{dx}$ from part **a**.

This solution uses the result $\int \frac{f'(x)}{f(x)}dx = \ln f(x)$. So $\int \frac{-2x}{1-x^2}dx = \ln(1-x^2)$ and you multiply this by $-\frac{1}{2}$ to complete the solution. This is a question where there are a number of possible alternative forms of the answer.

38 a Let $y = \text{arsinh}\, x$ then $x = \sinh y = \frac{e^y - e^{-y}}{2}$

$$2x = e^y - e^{-y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x + \sqrt{(4x^2 - 4)}}{2}$$

$$= \frac{2x + 2\sqrt{(x^2+1)}}{2} = x + \sqrt{(x^2+1)}$$

You multiply this equation throughout by e^y and treat the result as a quadratic in e^y.

The quadratic formula has $\pm$ in it. However $x - \sqrt{(x^2+1)}$ is negative for all real x and does not have a real logarithm, so you can ignore the negative sign.

Taking the natural logarithms of both sides, $y = \ln\left[x + \sqrt{(x^2+1)}\right]$, as required.

b $y = \text{arsinh}\, x$

$\sinh y = x$

Differentiating implicitly with respect to x

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2 \Rightarrow \cosh y = \sqrt{(1+x^2)}$$

Hence $\frac{d}{dx}(\text{arsinh}\, x) = \frac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}}$, as required.

arsinh x is an increasing function of x for all x. So its gradient is always positive and you need not consider the negative square root.

38 c $y = (\text{arsinh}\,x)^2$

$$\frac{dy}{dx} = 2\text{arsinh}\,x\,(1+x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2(1+x^2)^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + 2\text{arsinh}\,x \times \left(-\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{3}{2}}$$

$$= 2(1+x^2)^{-1} - 2x\text{arsinh}\,x\,(1+x^2)^{-\frac{3}{2}}$$

You use the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 1\ 2\text{arsinh}\,x$ and $v = (1+x^2)^{-\frac{1}{2}}$.

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2$$

$$= (1+x^2)\left(2(1+x^2)^{-1} - 2\text{arsinh}\,x(1+x^2)^{-\frac{3}{2}}\right) + x \times 2\text{arsinh}\,x(1+x^2)^{-\frac{1}{2}} - 2$$

$$= 2 - 2x\text{arsinh}\,x(1+x^2)^{-\frac{1}{2}} + 2x\text{arsinh}\,x(1+x^2)^{-\frac{1}{2}} - 2$$

$= 0$, as required.

d $\int_0^1 \text{arsinh}\,x\,dx = \int_0^1 1 \times \text{arsinh}\,x\,dx$

$$= \left[x\text{arsinh}\,x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{(1+x^2)}}\,dx$$

$$= \text{arsinh}\,1 - \left[\sqrt{(1+x^2)}\right]_0^1$$

$$= \ln(1+\sqrt{2}) - \sqrt{2} + 1$$

$\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) = \frac{1}{2} \times 2x \times (1+x^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{(1+x^2)}}$

so $\int \frac{x}{\sqrt{(1+x^2)}}\,dx = \sqrt{(1+x^2)}$.

39 a $4x^2 + 4x + 5 = (px+q)^2 + r$

$$= p^2x^2 + 2pqx + q^2 + r$$

Equating coefficients of x^2

$4 = p^2 \Rightarrow p = 2$

The conditions of the question allow $p = -2$ as an answer, but the negative sign would make the integrals following awkward, so choose the positive root.

Equating coefficients of x

$4 = 2pq = 4q \Rightarrow q = 1$

Equating constant coefficients

$5 = q^2 + r = 1 + r \Rightarrow r = 4$

$p = 2, q = 1, r = 4$

39 b $\int \frac{1}{4x^2+4x+5}\,dx = \int \frac{1}{(2x+1)^2+4}\,dx$

Let $2x+1 = 2\tan\theta$

$$2\frac{d}{d\theta} = 2\sec^2\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$$

$$\int \frac{1}{(2x+1)^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{1}{4\cancel{\sec^2\theta}}(\cancel{\sec^2\theta})\,d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\arctan\left(\frac{2x+1}{2}\right) + C$$

If you know a formula of the type $\int \frac{1}{a^2x^2+b^2}\,dx = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right)$, or you are confident at writing down integrals by inspection, you may be able to find this integral without working. It is, however, very easy to make errors with the constant and get, for example, the common error $\frac{1}{2}\arctan\left(\frac{2x+1}{2}\right)+C$.

c $\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx$

As in part **b**, you may be able to write down this integral without working.

Let $2x+1 = 2\sinh\theta$

$$2\frac{dx}{d\theta} = 2\cosh\theta \Rightarrow \frac{dx}{d\theta} = \cosh\theta$$

$$\int \frac{2}{\sqrt{((2x+1)^2+4)}}\,dx = \int \frac{2}{\sqrt{(4\sinh^2\theta+4)}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{2}{2\cosh\theta}(\cosh\theta)\,d\theta = \int 1\,d\theta$$

$$= \theta + C = \text{arsinh}\left(\frac{2x+1}{2}\right) + C$$

Using $\text{arsinh}\,x = \ln\left(x+\sqrt{(x^2+1)}\right)$

$$\int \frac{2}{\sqrt{(4x^2+4x+5)}}\,dx = \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{(2x+1)^2}{4}+1\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{\left(\frac{4x^2+4x+1+4}{4}\right)}\right]+C$$

$$= \ln\left[\left(\frac{2x+1}{2}\right)+\frac{1}{2}\sqrt{(4x^2+4x+5)}\right]+C$$

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right] - \ln 2 + C$$

$-\ln 2 +$ an arbitrary constant is another arbitrary constant.

$$= \ln\left[(2x+1)+\sqrt{(4x^2+4x+5)}\right] + k, \text{ as required.}$$

40 Solve the integral as follows:

$$\int \frac{x+2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$
$$=\int \frac{x}{\sqrt{4x^2+9}}\,\mathrm{d}x+\int \frac{2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$

Then the first integral is a standard integral of the form $k\mathrm{f}'(x)(\mathrm{f}(x))^n$, and we can easily see that its value is $\frac{\sqrt{4x^2+9}}{4}$, as the derivative of $\sqrt{4x^2+9}$ is $\frac{4x}{\sqrt{4x^2+9}}$. For the second integral we make a substitution: if $t=\frac{2}{3}x$, then:

$$\int \frac{2}{\sqrt{4x^2+9}}\,\mathrm{d}x$$
$$=\int \frac{2}{\sqrt{4\frac{9}{4}t^2+9}}\,\tfrac{3}{2}\mathrm{d}t$$
$$=\int \frac{1}{\sqrt{t^2+1}}\,\mathrm{d}t=\operatorname{arsinh} t=\operatorname{arsinh}\tfrac{2}{3}x$$

Then we can conclude that the integral of the function is $\frac{4x^2+9}{4}+\operatorname{arsinh}\frac{2}{3}x+C$.

41 Since $(x-2)^2=x^2-4x+4$, we can rewrite the integral as $=\int_2^5 \frac{1}{2\sqrt{\frac{(x-2)^2}{4}+1}}\,\mathrm{d}x$. So if we put

$t=\frac{x-2}{2}$ we get that the integral is:

$$\int_0^{\frac{3}{2}} \frac{1}{2\sqrt{t^2+1}}2\,\mathrm{d}t$$
$$=\int_0^{\frac{3}{2}} \frac{1}{\sqrt{t^2+1}}\,\mathrm{d}t$$
$$=\operatorname{arsinh}\tfrac{3}{2}$$

42 $\int x \operatorname{arcosh} x \,dx = \frac{x^2}{2}\operatorname{arcosh} x - \int \frac{x^2}{2\sqrt{(x^2-1)}}dx$

This solution uses integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, with $u = \operatorname{arcosh}x$ and $\frac{dv}{dx} = x$. There are other possible approaches to this question, for example, substituting $u = \operatorname{arcosh}x$.

To find the remaining integral, let $x = \cosh\theta$.

$$\frac{dx}{d\theta} = \sinh\theta$$

$$\int \frac{x^2}{2\sqrt{(x^2-1)}}dx = \int \frac{\cosh^2\theta}{2\sqrt{\cosh^2\theta - 1}}\left(\frac{dx}{d\theta}\right)d\theta$$

$$= \int \frac{\cosh^2\theta}{2\sinh\theta}\sinh\theta d\theta = \frac{1}{2}\int \cosh^2\theta d\theta$$

$$= \frac{1}{4}\int(\cosh 2\theta + 1)d\theta$$

Using the identity $\cosh 2\theta = 2\cosh^2\theta - 1$.

$$= \frac{\sinh 2\theta}{8} + \frac{\theta}{4} = \frac{\sinh\theta\cosh\theta}{4} + \frac{\theta}{4}$$

$$= \frac{\left[\sqrt{(x^2-1)}\right]x}{4} + \frac{1}{4}\operatorname{arcosh}x$$

$\sinh\theta = \sqrt{(\cosh^2\theta - 1)} = \sqrt{(x^2-1)}$

Hence the area, A, of R is given by

$$A = \left[\frac{x^2}{2}\operatorname{arcosh}x - \frac{1}{4}x\sqrt{(x^2-1)} - \frac{1}{4}\operatorname{arcosh}x\right]_1^2$$

$$= \left[\left(\frac{x^2}{2} - \frac{1}{4}\right)\operatorname{arcosh}x - \frac{1}{4}x\sqrt{(x^2-1)}\right]_1^2$$

$$= \left[\frac{7}{4}\operatorname{arcosh}2 - \frac{\sqrt{3}}{2}\right] - [0]$$

As $\operatorname{arcosh} 1 = 0$ and $\sqrt{(1^2-1)} = 0$, both terms are zero at the lower limit.

$$= \frac{7}{4}\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}, \text{ as required.}$$

43 Calculate the area of the section of the greenhouse with the integral:

$$\int_{-2}^{2}\frac{10}{\sqrt{x^2+9}}\,dx$$

$$= 10\int_{-2}^{2}\frac{1}{3\sqrt{\left(\frac{x}{3}\right)^2+1}}\,dx$$

Substitute $t = \frac{x}{3}$ to get:

$$10\int_{-\frac{2}{3}}^{\frac{2}{3}}\frac{1}{\sqrt{t^2+1}}\,dx$$

$$= 10\left[\operatorname{arsinh}\tfrac{2}{3} - \left(\operatorname{arsinh}\left(-\tfrac{2}{3}\right)\right)\right]$$

$$= 10\left[2\operatorname{arsinh}\tfrac{2}{3}\right] = 20\operatorname{arsinh}\tfrac{2}{3}$$

Then the volume of the greenhouse is given by $20\operatorname{arsinh}\frac{2}{3}\cdot 55$ which, to the third significant figure, is 688 m^3.

44 The integrating factor is

$e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$

Multiply the equation throughout by x^4

$x^4 \frac{dy}{dx} + 4x^3 y = 6x^5 - 5x^4$

$\frac{d}{dx}(x^4 y) = 6x^5 - 5x^4$

$x^4 y = \int (6x^5 - 5x^4)dx = x^6 - x^5 + C$

$y = x^2 - x + \frac{C}{x^4}$

If the differential equation has the form $\frac{dy}{dx} + Py = Q$, the integrating factor is $e^{\int P dx}$.

For any function $f(x), e^{\ln f(x)} = f(x)$.

It is important that you remember to add the constant of integration. When you divide by x^4, the constant becomes a function of x and its omission would be a significant error.

45 The integrating factor is

$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiply the equation throughout by $\frac{1}{x}$

$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x$

$\frac{d}{dx}\left(\frac{y}{x}\right) = x$

$\frac{y}{x} = \frac{x^2}{2} + C$

$y = \frac{x^3}{2} + Cx$

For all n, $n\ln x = \ln x^n$, so for $n = -1$, $-\ln x = \ln x^{-1} = \ln \frac{1}{x}$.

The product rule for differentiating, in this case $\frac{d}{dx}\left(y \times \frac{1}{x}\right) = \frac{dy}{dx} \times \frac{1}{x} + y \times -\frac{1}{x^2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to x.

46 $(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}$

$\frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x(x+1)}$

If the equation is in the form $R\frac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(x + 1)$, before finding the integrating factor.

The integrating factor is

$e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$

Multiply throughout by $(x + 1)^2$

$(x+1)^2 \frac{dy}{dx} + 2(x+1)y = \frac{x+1}{x}$

$\frac{d}{dx}\left((x+1)^2 y\right) = 1 + \frac{1}{x}$

$(x+1)^2 y = \int \left(1 + \frac{1}{x}\right)dx = x + \ln x + C$

$y = \frac{x + \ln x + C}{(x+1)^2}$

To integrate $\frac{x+1}{x}$, write $\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$.

You divide throughout by $(x + 1)^2$ to obtain the equation in the form $y = f(x)$. This is required by the wording of the question.

47 The integrating factor is $e^{\int \tan x \, dx}$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

Here we have that $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $\int \frac{-\sin x}{\cos x} dx = \ln \cos x$.

Hence

$$e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$$

Multiply the differential equation throughout by sec x

$$\sec x \frac{dy}{dx} + y \sec x \tan x = e^{2x} \sec x \cos x = e^{2x}$$

$\sec x \cos x = \frac{1}{\cos x} \times \cos x = 1$

$$\frac{d}{dx}(y \sec x) = e^{2x}$$

$$y \sec x = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

Multiply throughout by cos x

$$y = \left(\frac{e^{2x}}{2} + C\right) \cos x$$

$y = 2$ at $x = 0$

$$2 = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

The condition $y = 2$ at $x = 0$ enables you to evaluate the constant of integration and find the particular solution of the differential equation for these values.

$$y = \frac{1}{2}\left(e^{2x} + 3\right) \cos x$$

48 The integrating factor is $e^{\int 2\cot 2x \, dx}$

$$\int 2\cot 2x \, dx = \int \frac{2\cos 2x}{\sin 2x} dx = \ln \sin 2x$$

Hence

$$e^{\int 2\cot 2x \, dx} = e^{\ln \sin 2x} = \sin 2x$$

Multiply the differential equation throughout by sin $2x$

$$\sin 2x \frac{dy}{dx} + 2y \cos 2x = \sin x \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$.

$$\frac{d}{dx}(y \sin 2x) = 2\sin^2 x \cos x$$

$$y \sin 2x = \frac{2\sin^3 x}{2} + C$$

$$y = \frac{2\sin^3 x}{3 \sin 2x} + \frac{C}{\sin 2x}$$

As $\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$, then $\int \sin^2 x \cos dx = \frac{\sin^3 x}{3}$. It saves time to find integrals of this type by inspection. However, you can use the substitution $\sin^y = s$ if you find inspection difficult.

49 $(1+x)\dfrac{dy}{dx}-xy=xe^{-x}$

$$\frac{dy}{dx}-\frac{xy}{1+x}=\frac{xe^{-x}}{1+x} \qquad \textbf{(1)}$$

If the equation is in the form $R\dfrac{dy}{dx}+Sy=T$, you must begin by dividing throughout by R, in this case $(1+x)$ before finding the integrating factor.

The integrating factor is $e^{\int -\frac{x}{1+x}dx}$

$$\frac{x}{1+x}=\frac{1+x-1}{1+x}=1-\frac{1}{1+x}$$

To integrate an expression in which the degree of the numerator is greater or equal to the degree of the denominator, you must transform the expression into one with a proper fraction. This can be done using partial fractions, long division or, as here, using decomposition.

Hence

$$\int\frac{x}{1+x}dx=x-\ln(1+x)$$

and the integrating factor is

$$e^{-x+\ln(1+x)}=e^{-x}e^{\ln(1+x)}=e^{-x}+(1+x)$$

Multiplying **(1)** throughout by $(1+x)e^{-x}$

$$(1+x)e^{-x}\frac{dy}{dx}-xe^{-x}y=xe^{-2x}$$

$$\frac{d}{dx}\left(y(1+x)e^{-x}\right)=xe^{-2x}$$

$$y(1+x)e^{-x}=\int xe^{-2x}dx$$

You integrate $x\,e^{-2x}$ using integration by parts.

$$=-\frac{xe^{-2x}}{2}+\int\frac{e^{-2x}}{2}dx=-\frac{xe^{-2x}}{2}-\frac{e^{-2x}}{4}+C$$

$$\frac{e^{-2x}}{e^{-x}}=e^{-2x-(-x)}=e^{-2x+x}=e^{-x}$$

$$y=-\frac{xe^{-2x}}{2(1+x)e^{-x}}-\frac{e^{-2x}}{4(1+x)e^{-x}}+\frac{C}{(1+x)e^{-x}}$$

$$=-\frac{xe^{-x}}{2(1+x)}-\frac{e^{-x}}{4(1+x)}+\frac{Ce^{x}}{(1+x)}$$

$y=1$ at $x=0$

$$1=0-\frac{1}{4}+C\Rightarrow C=\frac{5}{4}$$

$$y=\frac{5e^{x}}{4(1+x)}-\frac{xe^{-x}}{2(1+x)}-\frac{e^{-x}}{4(1+x)}$$

This expression could be put over a common denominator but, other than requiring that y is expressed in terms of x, the question asks for no particular form and this is an acceptable answer.

50 a Dividing throughout by cos x

$$\frac{dy}{dx}+\frac{\sin x}{\cos x}y=\cos^2 x \qquad \textbf{(1)}$$

$$\int\frac{\sin x}{\cos x}dx=-\int\frac{-\sin x}{\cos x}dx=-\ln\cos x=\ln\frac{1}{\cos x}=\ln\sec x$$

Hence the integrating factor is $e^{\ln\sec x}=\sec x$

Multiply (**1**) by sec x

$$\sec x\frac{dy}{dx}+\sec x\frac{\sin x}{\cos x}y=\cos^2 x\sec x$$

$$\sec x\frac{dy}{dx}+(\sec x\tan x)y=\cos x$$

$$\frac{d}{dx}(y\sec x)=\cos x$$

$$y\sec x=\int\cos x\,dx=\sin x+C$$

Multiplying throughout by cos x

$$y=\sin x\cos x+C\cos x$$

Here we use $\int\frac{f'(x)}{f(x)}dx=\ln f(x)$. As $-\sin x$ is the derivative if cos x, $-\int\frac{-\sin x}{\cos x}dx=-\ln\cos x$.

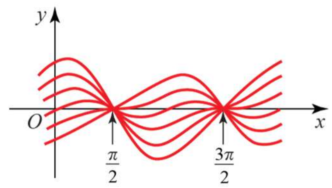
As ln 1 = 0, $-\ln\cos x=\ln 1-\ln\cos x=\ln\frac{1}{\cos x}$, using the log law $\ln a-\ln b=\ln\frac{a}{b}$.

b Where $\cos x=0$ and $0\leqslant x\leqslant 2\pi$

$$x=\frac{\pi}{2},\frac{3\pi}{2}$$

The points $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$ lie on all of the solution curves of the differential equation.

In general, for a given value of x, different values of c give different values of y. However, if cos x = 0, the c will have no effect and y will be zero for any value of c.

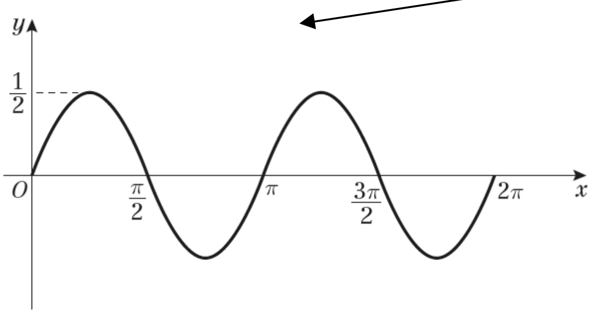


c $y=\sin x\cos x+\cos x$

At $x=0$, $y=0$

$$0=0+C\Rightarrow C=0$$

$$y=\sin x\cos x=\frac{1}{2}\sin 2x$$



Using the identity sin $2x$ = 2sin x cos x. sin $2x$ is a function with period π. So the curve makes two complete oscillations in the interval $0\leqslant x\leqslant 2\pi$

51 a The integrating factor is

$$e^{\int 2\mathrm{d}x} = e^{2x}$$

Multiplying the differential equation throughout by e^{2x}

$$e^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2e^{2x}y = xe^{2x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(ye^{2x}) = xe^{2x}$$

$$ye^{2x} = \int xe^{2x}\mathrm{d}x$$

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

Integrate by parts.

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

b $y = 1$ at $x = 0$

$$1 = 0 - \frac{1}{4} + C \Rightarrow C = \frac{5}{4}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{5e^{-2x}}{4}$$

This is the particular solution of the differential equation for $y = 1$ at $x = 0$. You are asked to sketch this in part **c**.

For a minimum $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - \frac{5e^{-2x}}{2} = 0 \Rightarrow 5e^{-2x} = 1 \Rightarrow e^{2x} = 5$$

$$\ln e^{2x} = \ln 5 \Rightarrow 2x = \ln 5$$

$$x = \frac{1}{2}\ln 5$$

At the minimum, the differential equation reduces to

$$2y = x$$

The differential equation is $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x$. At the minimum, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so $2y = x$. If you did not see this you could, of course, substitute $x = \frac{1}{2}\ln 5$ into the particular solution and find y. This would take longer but would gain full marks.

Hence

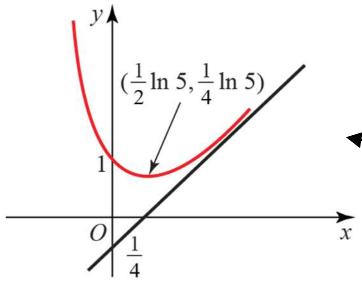
$$y = \frac{1}{2}x = \frac{1}{4}\ln 5$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 5e^{-2x} > 0 \text{ for any real } x$$

This confirms the point is a minimum.

The coordinates of the minimum are $\left(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5\right)$.

51 c



As x increases, $e^{-2x} \to 0$ and so $\frac{x}{2} - \frac{1}{4} + \frac{5e^{-2x}}{4} \to \frac{x}{2} - \frac{1}{4}$. This means that $y = \frac{x}{2} - \frac{1}{4}$ is an asymptote of the curve.
This has been drawn on the graph. It is not essential to do this, but if you recognise that this line is an asymptote, it helps you to draw the correct shape of the curve.

52 a Rewrite the equation as $\frac{dy}{y} = \sinh x \, dx$. Integrate this to get $\ln y = \cosh x + C$, and therefore $y = Ce^{\cosh x}$ is the general solution of the equation.

b When x = 0, the function takes the value Ce. Then the particular solution that satisfies $f(0) = e^3$ is $y = e^{2+\cosh x}$.

53 The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m+2)^2 = -1$$
$$m = -2 \pm i$$

The general solution is

$$\theta = e^{-2t}(A\cos t + B\sin t)$$

If the solutions to the auxiliary equation are $\alpha \pm i\beta$, you may quote the result that the general solution of the differential equation is $e^{\alpha t}.(A\cos \beta t + B\sin \beta t)$.

$$t = 0, \theta = 3$$
$$3 = A$$

Using $\sin 0 = 0$ and $\cos 0 = 1$.

$$\frac{d\theta}{dt} = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$$
$$t = 0, \frac{d\theta}{dt} = -6$$
$$-6 = -2A + B$$
$$B = 2A - 6 = 0$$

As $A = 3$

The particular solution is

$$\theta = 3e^{-2t}\cos t$$

54 a $y = 3x\sin 2x \Rightarrow \frac{dy}{dx} = 3\sin 2x + 6x\cos 2x$

$\frac{d^2y}{dx^2} = 6\cos 2x + 6\cos 2x - 12x\sin 2x$

$= 12\cos 2x - 12x\sin 2x$

Use the product rule for differentiating.

Substituting into the differential equation

$12\cos 2x - \cancel{12x\sin 2x} + \cancel{12x\sin 2x} = k\cos 2x$

Hence

$k = 12$

b The auxiliary equation is

$m^2 + 4 = 0$

$m = \pm 2i$

The complementary function is given by

$y = A\cos 2x + B\sin 2x$

If the solutions to the auxiliary equation are $m = \pm\alpha i$, you may quote the result that the complementary function is $A\cos\alpha x + B\sin\alpha x$.

From **a**, the general solution is

$y = A\cos 2x + B\sin 2x + 3x\sin 2x$

Part **a** of the question gives you that $3x\sin 2x$ is a particular integral of the differential equation and general solution = complementary function + particular integral.

$x = 0, y = 2$

$2 = A$

$x = \frac{\pi}{4}, y = \frac{\pi}{2}$

$\frac{\pi}{2} = A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2} + 3\times\frac{\pi}{4}\sin\frac{\pi}{2}$

$\frac{\pi}{2} = B + \frac{3\pi}{4} \Rightarrow B = -\frac{\pi}{4}$

Use $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

The particular solution is

$y = 2\cos 2x - \frac{\pi}{4}\sin 2x + 3x\sin 2x$

55 a $y = a + bx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = b$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$

Substituting into the differential equation

$0 - 4b + 4a + 4bx = 16 + 4x$

Equating the coefficients of x

$4b - 4 \Rightarrow b = 1$

Equating the constant coefficients

$-4b + 4a = 16$

$-4 + 4a = 16 \Rightarrow a = 5$ ← Use $b = 1$.

$a = 5, b = 1$

b The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$, repeated

The complementary function is given by

$y = \mathrm{e}^{2x}(A + Bx)$ ← If the auxiliary equation has a repeated root α, then the complementary function is $\mathrm{e}^{\alpha x}(A + Bx)$. You can quote this result.

The general solution is

$y = \mathrm{e}^{2x}(A + Bx) + 5 + x$ ← general solution = complementary function + particular integral.

$y = 8,\ x = 0$

$8 = A + 5 \Rightarrow A = 3$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x}(A + Bx) + B\mathrm{e}^{2x} + 1$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 9, x = 0$

$9 = 2A + B + 1 \Rightarrow B = 8 - 2A = 2$ ← Use $A = 3$

The Particular solution is

$y = \mathrm{e}^{2x}(3 + 2x) + 5 + x$

56 a The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m+2)^2 = -1$$
$$m = -2 \pm \mathrm{i}$$

The complementary function is given by
$y = e^{-2x}(A\cos x + B\sin x)$

For a particular integral, let $y = p\cos 2x + q\sin 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2p\sin 2x + 2q\cos 2x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4p\cos 2x - 4q\sin 2x$$

If the right hand side of the second order differential equation is a sine or cosine function, then you should try a particular integral of the form $p\cos\omega x + q\sin\omega x$, with an appropriate ω. Here $\omega = 2$.

Substituting into the differential equation

$$-4p\cos 2x - 4q\sin 2x - 8p\sin 2x + 8q\cos 2x + 5p\cos 2x + 5q\sin 2x = 65\sin 2x$$
$$(-4p + 8q + 5p)\cos 2x + (-4q - 8p + 5q)\sin 2x = 65\sin 2x$$

Equating the coefficients of cos $2x$ and sin $2x$

$\cos 2x$: $\quad -4p + 8q + 5p = 0 \Rightarrow p + 8q = 0 \quad$ **(1)**

$\sin 2x$: $\quad -4q - 8p + 5q = 65 \Rightarrow -8p + q = 65 \quad$ **(2)**

$8p + 64q = 0 \quad$ **(3)**

$65q = 65 \Rightarrow q = 1$

The coefficients of cos$2x$ and sin $2x$ can be equated separately. The coefficient of cos$2x$ on the right hand side of this equation is zero.

Multiply (**1**) by 8 and add the result to (**2**).

Substitute $q = 1$ into (**1**)

$$p + 8 = 0 \Rightarrow p = -8$$

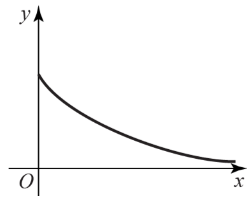
A particular integral is $-8\cos 2x + \sin 2x$

The general solution is

$$y = e^{-2x}(A\cos x + B\sin x) + \sin 2x - 8\cos 2x$$

56 b As $x \to \infty, \mathrm{e}^{-2x} \to 0$ and, hence,

$y \to \sin 2x - 8\cos 2x$



The graph of e^{-2x} against x has this shape. As x becomes larger e^{-2x} is close to zero, so e^{-2x} (A cos x + B sin x) is also small.

Let

$\sin 2x - 8\cos 2x = R\sin(2x - \alpha)$

$= R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$

Equating the coefficients of cos $2x$ and sin $2x$

$1 = R\cos\alpha \ldots$ **(4)**

$8 = R\sin\alpha \ldots$ **(5)**

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 8^2 = 65$

$R^2 = 65 \Rightarrow R\sqrt{65}$

Add **(4)** squared to **(5)** squared and use the identity $\cos^2\alpha + \sin^2\alpha = 1$.

$\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{8}{1} \Rightarrow \tan\alpha = 8$

Divide **(5)** by **(4)**.

Hence, for large x, y can be approximated by the sine function $\sqrt{65}$ sin ($2x - a$), where tan $\alpha = 8$ ($a \approx 82.9^\circ$)

57 a The auxiliary equation is

$m^2 + 2m + 2 = 0$

$m^2 + 2m + 1 = -1$

$(m+1)^2 = -1$

$m = -1 \pm \mathrm{i}$

The complementary function is

$y = \mathrm{e}^{-t}(A\cos t + B\sin t)$

Try a particular integral $y = k\,\mathrm{e}^{-t}$

If the right hand side of the differential equation is $\lambda\,\mathrm{e}^{at+b}$, where λ is any constant, then a possible form of the particular integral is $k\,\mathrm{e}^{at+b}$.

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = -k\,\mathrm{e}^{-t}, \ \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = k\,\mathrm{e}^{-t}$

Substituting into the differential equation

$k\,\mathrm{e}^{-t} - 2k\,\mathrm{e}^{-t} + 2k\,\mathrm{e}^{-t} = 2\mathrm{e}^{-t}$

Divide throughout by e^{-t}.

$k - 2k + 2k = 2 \Rightarrow k = 2$

A particular integral is $2\mathrm{e}^{-t}$

The general solution is

$y = \mathrm{e}^{-t}(A\cos t + B\sin t) + 2\mathrm{e}^{-t}$

57 b $y = 1, \quad t = 0$

$1 = A + 2 \Rightarrow A = -1$

Substitute the boundary condition $y = 1$, $t = 0$ into the general solution gives you an equation for one arbitrary constant.

$\frac{dy}{dt} = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t) - 2e^{-t}$

Use the product rule for differentiating.

$\frac{dy}{dt} = 1, \; t = 0$

$1 = -A + B - 2 \Rightarrow B = 3 + A = 2$

As $A = -1$.

The particular solution is

$y = e^{-t}(2\sin t - \cos t) + 2e^{-t}$

58 a The auxiliary equation is

$m^2 + 2m + 5 = 0$

$m^2 + 2m + 1 = -4$

$(m+1)^2 = -4$

$m = -1 \pm 2i$

You may use any appropriate method to solve the quadratic. Completing the square works efficiently when the coefficient of *m* is given.

The general solution is

$x = e^{-t}(A\cos 2t + B\sin 2t)$

b $x = 1, \; t = 0$

$1 = A$

$\frac{dx}{dt} = -e^{-t}(A\cos 2t + B\sin 2t) + 2e^{-t}(-A\sin 2t + B\cos 2t)$

Use the product rule for differentiation.

$\frac{dx}{dt} = 1, \; t = 0$

$1 = -A + 2B \Rightarrow 2B = A + 1 = 2 \Rightarrow B = 1$

The particular solution is

$x = e^{-t}(\cos 2t + \sin 2t)$

Both *A* and *B* are 1.

58 c The curve crosses the t-axis where

$e^{-t}(\cos 2t + \sin 2t) = 0$

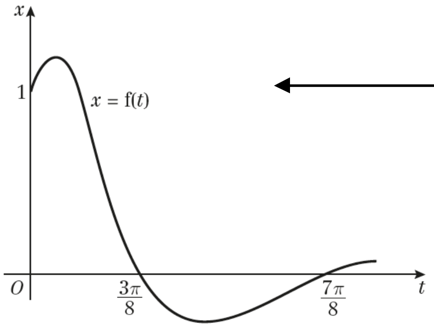
$\cos 2t + \sin 2t = 0$

$\sin 2t = -\cos 2t$

$\tan 2t = -1$

$2t = \frac{3\pi}{4}, \frac{7\pi}{4}$

$t = \frac{3\pi}{8}, \frac{7\pi}{8}$



e^{-t} can never be zero.

Divide both sides by $\cos 2t$ and use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

The boundary conditions give you that at $t = 0$, $x = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the two points where $t = \frac{3\pi}{8}$ and $\frac{7\pi}{8}$.

59 a The auxiliary equation is

$2m^2 + 7m + 3 = 0$

$(2m+1)(m+3) = 0$

$m = -\frac{1}{2}, -3$

The complementary function is given by

$y = Ae^{-\frac{1}{2}t} + Be^{-3t}$

If the auxiliary equation has two real solutions α and β, the complementary function is $y = Ae^{\alpha t} + Be^{\beta t}$. You can quote this result.

For a particular integral, try $y = at^2 + bt + c$

$\frac{dy}{dt} = 2at + b, \frac{d^2y}{dt^2} = 2a$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is a quadratic, so you try the general quadratic $at^2 + bt + c$.

Substitute into the differential equation

$4a + 14at + 7b + 3at^2 + 3bt + 3c = 3t^2 + 11t$

$3at^2 + (14a + 3b)t + 4a + 7b + 3c = 3t^2 + 11t$

Equating the coefficients of t^2

$3a = 3 \Rightarrow a = 1$

Equating the coefficients of t

Use $a = 1$.

$14a + 3b = 11 \Rightarrow 3b = 11 - 14a = -3 \Rightarrow b = -1$

Equating the constant coefficients

Use $a = 1$ and $b = -1$.

$4a + 7b + 3c = 0 \Rightarrow 3c = -4a - 7b = 3 \Rightarrow c = 1$

A particular integral is $t^2 - t + 1$.

The general solution is $y = Ae^{-\frac{1}{2}t} + Be^{-3t} + t^2 - t + 1$.

59 b $y = 1,\ t = 0$

$1 = A + B + 1 \Rightarrow A + B = 0 \quad \textbf{(1)}$

$\frac{dy}{dt} = -\frac{1}{2}Ae^{-\frac{1}{2}t} - 3Be^{-3t} + 2t - 1$

Differentiate the general solution in part **a** with respect to *t*.

$\frac{dy}{dt} = 1,\ t = 0$

$1 = -\frac{1}{2}A - 3B - 1 \Rightarrow \frac{1}{2}A + 3B = -2 \quad \textbf{(2)}$

$A + 6B = -4 \quad \textbf{(3)}$

$5B = -4 \Rightarrow B = -\frac{4}{5}$

Multiply (**2**) by 2 and then subtract (**1**) from (**3**).

Substituting $B = \frac{4}{5}$ into **(1)**

$A - \frac{4}{5} = 0 \Rightarrow A = \frac{4}{5}$

The particular solution is $y = \frac{4}{5}\left(e^{-\frac{1}{2}t} - e^{-3t}\right) + t^2 - t + 1.$

c When $t = 1,\ y = \frac{4}{5}\left(e^{-\frac{1}{2}} - e^{-3}\right) + 1 = 1.45$ (3 s.f.)

60 a Let $y = \lambda x \cos 3x$

$\frac{dy}{dx} = \lambda \cos 3x - 3\lambda x \sin 3x$

$\frac{d^2y}{dx^2} = -3\lambda \sin 3x - 3\lambda \sin 3x - 9\lambda x \cos 3x$

$= -6\lambda \sin 3x - 9\lambda x \cos 3x$

Use the product rule for differentiation

$\frac{d}{dx}(x \sin 3x) = \frac{d}{dx}(x)\sin 3x + x\frac{d}{dx}(\sin 3x)$

$= \sin 3x + 3x \cos 3x$

Substituting into the differential equation

$-6\lambda \sin 3x - \cancel{9\lambda x \cos 3x} + \cancel{9\lambda x \cos 3x} = -12 \sin 3x$

Hence

$\lambda = 2$

b The auxiliary equation is

$m^2 + 9 = 0 \Rightarrow m^2 = -9$

$m = \pm 3i$

The complementary function is given by

$y = A\cos 3x + B\sin 3x$

The general solution is

$y = A\cos 3x + B\sin 3x + 2x\cos 3x$

Part **a** shows that $2x\cos 3x$ is a particular integral of the differential equation and general solution = complementary function + particular integral

60 c $y = 1,\ x = 0$

$1 = A$

$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos 3x - 6x\sin 3x$

Differentiate the general solution in part **b** with respect to x.

$\frac{dy}{dx} = 2,\ x = 0$

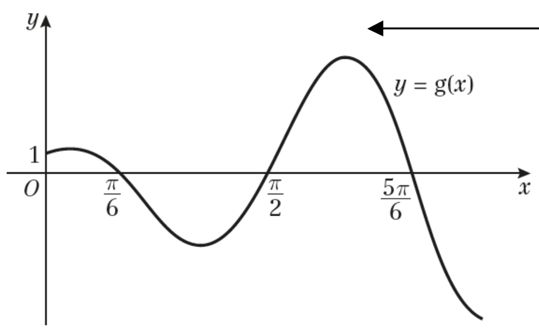
$2 = 3B + 2 \Rightarrow B = 0$

The particular solution is

$y = \cos 3x + 2x\cos 3x = (1+2x)\cos 3x$

d For $x > 0$, the curve crosses the x-axis at $\cos 3x = 0$

$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



The boundary conditions give you that at $x = 0,\ y = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the three points where $x = \frac{\pi}{6}, \frac{\pi}{2}$ and $\frac{5\pi}{6}$.
The $(1+2x)$ factor in the general solution means that the size of the oscillations increases as x increases.

61 a If $y = Kt^2 e^{3t}$

$\frac{dy}{dt} = 2Kte^{3t} + 3Kt^2e^{3t}$

$\frac{d^2y}{dt^2} = 2Kte^{3t} + 3Kt^2e^{3t} + 6Kte^{3t} + 9Kt^2e^{3t}$

$= 2Ke^{3t} + 12Kte^{3t} + 9Kt^2e^{3t}$

Substituting into the differential equation

$2Ke^{3t} + \cancel{12Kte^{3t}} + \cancel{9Kt^2e^{3t}} - \cancel{12Kte^{3t}} - \cancel{18Kte^{3t}} + \cancel{9Kte^{3t}} = 4e^{3t}$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

Hence

$2K = 4 \Rightarrow K = 2$

$2t^2e^{3t}$ is a particular integral of the differential equation.

61 b The auxiliary equation is

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3$, repeated

The complementary function is given by

$y = e^{3t}(A + Bt)$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$y = e^{3t}(A + Bt) + 2t^2 e^{3t} = (A + Bt + 2t^2)e^{3t}$

c $y = 3,\ t = 0$

$3 = A$

$\frac{dy}{dt} = (B + 4t)e^{3t} + 3(A + Bt + 2t^2)e^{3t}$

$\frac{dy}{dt} = 1,\ t = 0$

$1 = B + 3A \Rightarrow B = 1 - 3A \Rightarrow B = -8$

As $A = 3$.

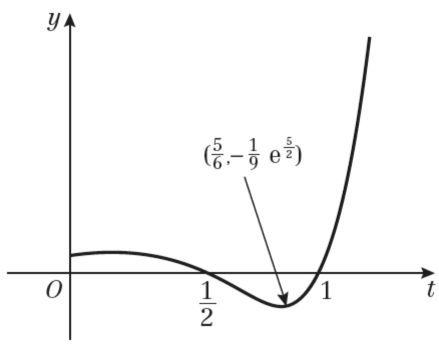
The particular solution is

$y = (3 - 8t + 2t^2)e^{3t}$

d This particular solution crosses the t-axis where

$1 - 3t + 2t^2 = (1 - 2t)(1 - t) = 0$

$t = \frac{1}{2}, 1$



For a minimum $\frac{dy}{dt} = 0$

$(-3 + 4t)e^{3t} + (1 - 3t + 2t^2)3e^{3t} = 0$

$-3 + 4t + 3 - 9t + 6t^2 = 0$

$6t^2 - 5t = t(6t - 5) = 0 \Rightarrow t = 0, \frac{5}{6}$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

61 d From the diagram $t=\frac{5}{6}$ gives the minimum

It is clear from the diagram that there is a minimum point between $t=\frac{1}{2}$ and $t=1$.
You do not have to consider the second derivative to show that it is a minimum.

At $t=\frac{5}{6}$

$$y=\left(1-3\times\frac{5}{6}+2\times\left(\frac{5}{6}\right)^2\right)e^{3\times\frac{5}{6}}=-\frac{1}{9}e^{\frac{5}{2}}$$

The coordinates of the minimum point are

$\left(\frac{5}{6},-\frac{1}{9}e^{\frac{5}{2}}\right)$.

62 a The auxiliary equation is

$$2m^2+5m+2=0$$
$$(2m+1)(m+2)=0$$
$$m=-\frac{1}{2},-2$$

If the auxiliary equation has two real solutions α and β, the complementary function is $x=Ae^{\alpha t}+Be^{\beta t}$. You can quote this result.

The complementary function is given by

$$x=Ae^{-\frac{1}{2}t}+Be^{-2t}$$

For a particular integration, try $x=pt+q$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is linear, so you try the general linear function $pt+q$.

$$\frac{dx}{dt}=p,\ \frac{d^2x}{dt^2}=0$$

Substituting into the differential equation

$$0+5p+2pt+2q=2t+9$$

Equating the coefficients of t

$$2p=2\Rightarrow p=1$$

Equating the constant coefficients

$$5p+2q=9\Rightarrow q=\frac{9-5p}{2}\Rightarrow q=2$$

A particular integral is $t+2$

The general solution is

$$x=Ae^{-\frac{1}{2}t}+Be^{-2t}+t+2$$

62 b $x = 3,\ t = 0$

$3 = A + B + 2 \Rightarrow A + B = 1 \qquad$ **(1)**

$\frac{dx}{dt} = -\frac{1}{2}Ae^{-\frac{1}{2}t} - 2Be^{-2t} + 1$

Differentiating the general solution in part **a**.

$\frac{dx}{dt} = -1,\ t = 0$

$-1 = -\frac{1}{2}A - 2B + 1 \Rightarrow \frac{1}{2}A + 2B = 2 \qquad$ **(2)**

$A + 4B = 4 \qquad$ **(3)**

$3B = 3 \Rightarrow B = 1$

Multiplying (**2**) by 2 and subtracting (**1**) from (**3**).

Substituting $B = 1$ into (**1**)

$A + 1 = 1 \Rightarrow A = 0$

The particular solution is

$x = e^{-2t} + t + 2$

c For a minimum

$\frac{dx}{dt} = -2e^{-2t} + 1 = 0$

$e^{-2t} = \frac{1}{2}$

You take logarithms of both sides of this equation and use $e^{\ln f(x)} = f(x)$.

$-2t = \ln\frac{1}{2} = -\ln 2$

$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, as $\ln 1 = 0$.

$t = \frac{1}{2}\ln 2$

$\frac{d^2x}{dt^2} = 4e^{-2t} > 0$, for any real t

Hence the stationary value is a minimum value

When $t = \frac{1}{2}\ln 2$

$e^{-\ln 2} = e^{\ln 1 - \ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

$x = e^{-\ln 2} + \frac{1}{2}\ln 2 + 2 = \frac{1}{2} + \frac{1}{2}\ln 2 + 2 = \frac{5}{2} + \frac{1}{2}\ln 2$

The minimum distance is $\frac{1}{2}(5 + \ln 2)$ m, as required.

63 a If $x = At^2 e^{-t}$

$$\frac{dx}{dt} = 2At\,e^{-t} - At^2\,e^{-t}$$

$$\frac{d^2x}{dt^2} = 2A\,e^{-t} - 2At\,e^{-t} - 2At\,e^{-t} + At^2\,e^{-t}$$
$$= 2A\,e^{-t} - 4At\,e^{-t} + At^2 e^{-t}$$

Substituting into the differential equation

$$2A\,e^{-t} - \cancel{4At\,e^{-t}} + \cancel{At^2\,e^{-t}} + \cancel{4At^2\,e^{-t}} - \cancel{2At^2\,e^{-t}} + \cancel{At^2\,e^{-t}} = e^{-t}$$

e^{-t} cannot be zero, so you can divide throughout by e^{-t}.

Hence

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

b The auxiliary equation is

$$m^2 + 2m + 1 = (m+1)^2 = 0$$
$$m = -1, \text{ repeated}$$

The complementary function is given by

$$x = e^{-t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$x = e^{-t}(A + Bt) + \frac{1}{2}t^2 e^{-t} = \left(A + Bt + \frac{1}{2}t^2\right)e^{-t}$$

From part **a**, $\frac{1}{2}t^2 e^{-t}$ is a particular integral of the differential equation.

$x = 1,\ t = 0$

$$1 = A$$

$$\frac{dx}{dt} = (B + t)e^{-t} - \left(A + Bt + \frac{1}{2}t^2\right)e^{-t}$$

$$\frac{dx}{dt} = 0,\ t = 0$$

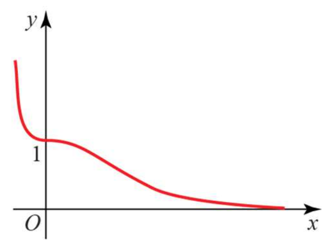
$$0 = B - A \Rightarrow B = A = 1$$

The particular solution is

$$x = \left(1 + t + \frac{1}{2}t^2\right)e^{-t}$$

c
$$\frac{dx}{dt} = (1 + t)e^{-t} - \left(1 + t + \frac{1}{2}t^2\right)e^{-t}$$
$$= -\frac{1}{2}t^2 e^{-t} \leqslant 0, \text{for all real } t.$$

When $t = 0$, $x = 1$ and x has a negative gradient for all positive t, x is a decreasing function of t. Hence, for $t \geqslant 0$, $x \leqslant 1$, as required.



The graph of x against t, shows the curve crossing the x-axis at $x = 1$ and then decreasing. For all positive t, x is less than 1.

64 a $y = kx \Rightarrow \frac{dy}{dx} = k \Rightarrow \frac{d^2y}{dx^2} = 0$

Substituting into $\frac{d^2y}{dx^2} + y = 3x$

$$0 + kx = 3x$$
$$k = 3$$

b The auxiliary equation is

$m^2 + 1 = 0 \Rightarrow m = \pm i$

The complementary function is given by

$y = A\sin x + B\cos x$

and the general solution is

$y = A\sin x + B\cos x + 3x$

$y = 0,\ x = 0$

$0 = B + 0 \Rightarrow B = 0$

The most general solution is

$y = A\sin x + 3x$

In part **b**, only one condition is given, so only one of the arbitrary constants can be found. The solution is a family of functions, some of which are illustrated in the diagram below.

c At $x = \pi$

$y = A\sin\pi + 3\pi = 3\pi$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** pass through $(\pi, 3\pi)$.

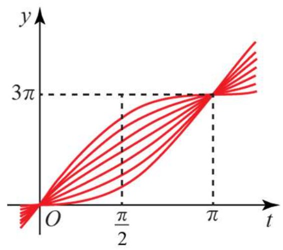
$\frac{dy}{dx} = A\cos x + 3$

At $x = \frac{\pi}{2}$

$\frac{dy}{dx} = A\cos\frac{\pi}{2} + 3 = 3$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** have an equal gradient of 3 at $x = \frac{\pi}{2}$.



As is illustrated by this diagram, the family of curves $y = A\sin x + 3x$ all go through $(0, 0)$ and $(\pi, 3\pi)$. The tangent to the curves at $x = \frac{\pi}{2}$ are all parallel to each other.

d $y = 0,\ x = \frac{\pi}{2}$

Substituting into $y = A\sin x + 3x$

$0 = A\sin\frac{\pi}{2} + \frac{3\pi}{2} = A + \frac{3\pi}{2} \Rightarrow A = -\frac{3\pi}{2}$

The particular solution is

$y = 3x - \frac{3\pi}{2}\sin x$

64 e For a minimum

$$\frac{dy}{dx} = 3 - \frac{3\pi}{2}\cos x = 0$$

$$\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3\pi}{2}\sin x$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}$$

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$

$$\sin x = +\left(\frac{\pi^2 - 4}{\pi^2}\right)^{\frac{1}{2}} = \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$y = 3\arccos\left(\frac{2}{\pi}\right) - \frac{3\pi}{2} \times \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$= 3\arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{\pi^2 - 4}, \text{ as required.}$$

$\cos x = \frac{2}{\pi}$ has an infinite number of solutions. This shows that the solution in the first quadrant gives a minimum as $\sin x$ is positive in that quadrant.

65 a $\int \frac{2}{120-t}\,dt = -2\ln(120-t) = \ln(120-t)^{-2} = \ln\frac{1}{(120-t)^2}$

Using the log law $n\log a = \log a^n$, with $n = -2$.

Hence the integrating factor is

$$e^{\int \frac{2}{120-t}dt} = e^{\ln\frac{1}{(120-t)^2}} = \frac{1}{(120-t)^2}$$

Multiply the equation throughout by $\frac{1}{(120-t)^2}$

$$\frac{1}{(120-t)^2}\frac{ds}{dt} + \frac{2}{(120-t)}S = \frac{1}{4(120-t)^2}$$

$$\frac{d}{dt}\left(\frac{S}{(120-t)^2}\right) = \frac{1}{4}(120-t)^{-2}$$

$\frac{d}{dt}\left(S(120-t)^{-2}\right)$
$= \frac{dS}{dt}(120-t)^{-2} - S\times(-2)(120-t)^{-3}$
$= \frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2}{(120-t)^3}S$

This product enables you to write the differential equation as a complete equation.

Integrating both sides with respect to t

$$\frac{S}{(120-t)^2} = \frac{1}{4}\int(120-t)^{-2}dt = -\frac{1}{4}\frac{(120-t)^{-1}}{-1} + C$$

$$\frac{S}{(120-t)^2} = \frac{1}{4(120-t)} + C$$

Multiply this equation by $(120-t)^2$.

$$S = \frac{120-t}{4} + C(120-t)^2$$

Remember to multiply the C by $(120-t)^2$. It is a common error to obtain C instead of $C(120-t)^2$ at this stage.

$S = 6$ when $t = 0$

$$6 = 30 + C\times 120^2 \Rightarrow C = -\frac{24}{120^2} = -\frac{1}{600}$$

$$S = \frac{120-t}{4} - \frac{(120-t)^2}{600}$$

b For a maximum value

$$\frac{dS}{dt} = -\frac{1}{4} + \frac{2(120-t)}{600} = 0$$

$$240 - 2t = 150 \Rightarrow t = \frac{240-150}{2} = 45$$

$$\frac{d^2S}{dt^2} = -\frac{1}{300} < 0 \Rightarrow \text{maximum}$$

Maximum values is given by

$$S = \frac{120-45}{4} - \frac{(120-45)^2}{600} = \frac{75}{4} - \frac{75}{8} = \frac{75}{8} = 9\frac{3}{8}$$

The maximum mass of salt predicted is $9\frac{3}{8}$ kg.

66 a Three quarters of the nutrient is $\frac{3}{4}\times 100m_0 = 75m_0$

At time t, the nutrient consumed is $5(m-m_0)$

Hence

$$5(m-m_0)=75m_0$$
$$5m-5m_0=75m_0 \Rightarrow 5m=80m_0$$
$$m=\frac{80m_0}{5}=16m_0\text{, as required.}$$

b Rate of increase of mass $=\mu\times\text{mass}\times\text{nutrient remaining}$

$$\frac{dm}{dt}=\mu\times m\times[100m_0-5(m-m_0)]$$
$$\frac{dm}{dt}=\mu m(100m_0-5m+5m_0)$$
$$=\mu m(105m_0-5m)$$
$$=5\mu m(21m_0-m)\text{, as required.}$$

The nutrient remaining is the nutrient consumed, $5(m-m_0)$, subtracted from the original nutrient $100m_0$.

c $$\frac{dm}{dt}=5\mu m(21m_0-m)$$

This is a separable equation.

$$\int 5\mu\,dt=\int\frac{1}{m(21m_0-m)}\,dm$$

Separating the variables.

Let $\frac{1}{m(21m_0-m)}=\frac{A}{m}+\frac{B}{21m_0-m}$

To integrate the right hand side of this equation, you must break the expression up into partial fractions using one of the methods you learnt in C4.

Multiplying throughout by $m(21m_0-m)$

$$1=A(21m_0-m)+Bm$$

Let $m=0$

$$1=A\times 21m_0 \Rightarrow A=\frac{1}{21m_0}$$

Let $m=21m_0$

$$1=B\times 21m_0 \Rightarrow B=\frac{1}{21m_0}$$

Hence

$$5\mu t=\frac{1}{21m_0}\int\left(\frac{1}{m}+\frac{1}{21m_0-m}\right)dm$$
$$105\mu m_0 t=\int\left(\frac{1}{m}+\frac{1}{21m_0-m}\right)dm=\ln m-\ln(21m_0-m)+C$$

66 c When $t = 0$, $m = m_0$

$0 = \ln m_0 - \ln 20m_0 + C$

$C = \ln 20m_0 - \ln m_0 = \ln \frac{20m_0}{m_0} = \ln 20$

$105\mu m_0 t = \ln m - \ln(21m_0 - m) + \ln 20 = \ln\left(\frac{20m}{21m_0 - m}\right)$

From part **a**, when $t = T$, $m = 16m_0$

$105\mu m_0 T = \ln\left(\frac{20 \times 16m_0}{21m_0 - 16m_0}\right) = \ln\left(\frac{320m_0}{5m_0}\right)$

$= \ln 64$, as required.

Initially, the mass of the embryo is m_0. This enables you to find the particular solution of the differential equation. The initial conditions are often known in scientific applications of mathematics.

Combining the logarithms at this stage simplifies the next stage of the calculation. The form of the simplification is

$\ln a - \ln b + \ln c = \ln \frac{ac}{b}$

67 a $t\frac{\mathrm{d}v}{\mathrm{d}t} - v = t$

Divide throughout by t

$\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{v}{t} = -1$ **(1)**

The integrating factor is

$e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t}$

Multiply (**1**) throughout by $\frac{1}{t}$

$\frac{1}{t}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{v}{t^2} = \frac{1}{t}$

$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{t}\right) = \frac{1}{t}$

$\frac{v}{t} = \int \frac{1}{t}\mathrm{d}t = \ln t + c$

$v = t(\ln t + c)$, as required.

The product rule for differentiating, in this case $\frac{\mathrm{d}}{\mathrm{d}t}(v \times t^{-1}) = \frac{\mathrm{d}v}{\mathrm{d}t} \times t^{-1} + v \times (-1)t^{-2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to t.

b $v = 3$ when $t = 2$

$3 = 2(\ln 2 + c) = 2\ln 2 + 2c \Rightarrow c = 1.5 - \ln 2$

$v = t(\ln t + 1.5 - \ln 2)$

When $t = 4$

$v = 4(\ln 4 + 1.5 - \ln 2)$

≈ 8.77

Use your calculator to evaluate this expression.

The speed of the particle when $t = 4$ is $8.77\,\mathrm{m\,s^{-1}}$ (3 s.f.)

68 $a = \frac{dv}{dt} = e^{2t}$

$v = \int e^{2t}\,dt = \frac{1}{2}e^{2t} + A$

To find the acceleration, you integrate the velocity with respect to time. Remember to include a constant of integration.

When $t = 0, v = 0$

$0 = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2}$

Hence $v = \frac{1}{2}(e^{2t} - 1)$, as required.

69 a $a = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = -3e^{-\frac{1}{6}t} + A$

Using $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + A$, then

$\int \frac{1}{2}e^{-\frac{1}{2}}\,dt = \frac{1}{2 \times (1 - \frac{1}{6})}e^{-\frac{1}{6}} + A$

$= -3e^{-\frac{1}{6}} + A.$

When $t = 0,\ v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180\ldots$

The speed of P when $t = 3$ is $11.2\,\text{m s}^{-1}$ (3 s.f.).

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if $t = 120$, then $e^{-\frac{1}{6}t} \approx 2.06 \times 10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

70 a $a = \frac{dv}{dt} = -4e^{2t}$

$v = \int 4e^{-2t}\,dt = 2^{-2t} + A$

At $t = 0,\ v = 1$

$1 = 2 + A \Rightarrow A = -1$

Hence $v = 2e^{-2t} - 1$

70 b P is instantaneously at rest when $v = 0$.

$0 = 2e^{-2t} - 1$

$e^{-2t} = \frac{1}{2} \Rightarrow e^{2t} = 2$

$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$

$x = \int v \, dt = \int (2e^{-2t})\,dt$

$= -e^{-2t} - t + B$

When $t = 0$, $x = 0$

$0 = -1 - 0 + B \Rightarrow B = 1$

Hence $x = 1 - e^{-2t} - t$

When $t = \frac{1}{2}\ln 2$

$x = 1 - e^{-2\left(\frac{1}{2}\ln 2\right)} - \frac{1}{2}\ln 2 = 1 - e^{-\ln 2} - \frac{1}{2}\ln 2$

$= 1 - e^{\ln\frac{1}{2}} - \frac{1}{2}\ln 2 = 1 - \frac{1}{2} - \frac{1}{2}\ln 2$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

The distance of P from O when P comes to instantaneous rest is $\frac{1}{2}(1 - \ln 2)$m.

> To find the speed when P is instantaneously at rest, you will need to know the value of t when $v = 0$.

> Take natural logarithms of both sides of this equation and use the property that, for any x, $\ln(e^x) = x$

> Using $e^0 = 1$. It is a common error to obtain $B = 0$ by carelessly writing $e^0 = 0$.

> Using the law of logarithms, $\ln a^n = n\ln a$ with $n = -1$,
> $-\ln 2 = (-1)\ln 2 = \ln 2^{-1} = \ln\frac{1}{2}$. Then as for any x, $e^{\ln x} = x$, $e^{\ln\frac{1}{2}} = \frac{1}{2}$.

71 a Rewrite the equation as $\frac{dv}{dt} + \frac{3v}{t+3} = 9.8$. This equation has integrating factor

$e^{\int \frac{3}{t+3}dt} = e^{3\ln(t+3)} = (t+3)^3$. Multiplying both factors by this yields:

$(t+3)^3 \frac{dv}{dt} + (t+3)^2 3v = 9.8(t+3)^3$

$\frac{d}{dt}\left(v(t+3)^3\right) = 9.8(t+3)^3$

$v(t+3)^3 = 9.8\frac{(t+3)^4}{4} + C$

$v = 9.8\frac{t+3}{4} + \frac{C}{(t+3)^3}$

$v = \frac{9.8(t+3)^4 + 4C}{4(t+3)^3}$

$v = \frac{49(t+3)^4 + 20C}{20(t+3)^3}$

71 a Since the droplet starts falling from rest, the equation must satisfy the condition $v(0)=0$, so $49\cdot 3^4+20C=0$. Then $C=-\frac{3969}{20}$, and the velocity can be expressed as $v=\frac{49(t+3)^4-3969}{20(t+3)^3}$.

b After six seconds, the velocity of the droplets is given by $\frac{49(9)^4-3969}{20(9)^3}=21.78\text{ms}^{-1}$.

c According to this model the velocity of the droplets is always increasing (it is clear that the limit for $t\to\infty$ is infinity, so for every possible value of velocity there is a time t such that that value is reached). However, it is known from fluid dynamics that a particle falling through air must at some point reach the so-called terminal velocity and stop accelerating.

72 a The amount of liquid inside the bottle at the minute t is given by $400-30t+40t=400+10t$. Of this amount we know that x is acid, so the concentration of acid in the bottle is given simply by $\frac{x}{400+10t}$. Now the liquid that leaks out of this bottle has this concentration of water, so at each minute the rate of acid leaking out is $30\frac{x}{400+10t}=\frac{3x}{40+t}$; the quantity of acid is 10% of 40ml, so 4ml. Then the equation that describes the rate of change in the quantity of acid at minute t is exactly $\frac{dx}{dt}=4-\frac{3x}{40+t}$.

b Solve the equation, by finding the integrating factor $e^{\int\frac{3}{40+t}dt}=(40+t)^3$. Then the equation is equivalent to $\frac{d}{dt}x(40+t)^3=4(40+t)^3$, which is solved by $x=(40+t)^4+\frac{C}{(40+t)^3}$. By forcing the condition $x(0)=0$ we find that the value of the constant C is $-2\,560\,000$, and we easily compute $x(7)=22.3\text{ ml}$.

c Of course, acid cannot disperse immediately on entry, so a correct model would also take into account the velocity at which this process takes place.

73 a From the differential equation relating position and acceleration it is clear that it is a simple harmonic motion.

b The auxiliary equation for this equation is $m^2+49=0$, so the general solution is $x=A\cos 7t+B\sin 7t$. By considering the initial condition for position and velocity we have that $A=0.3$ and $7B=0$, so the equation of the motion is $x=0.3\cos 7t$.

c The period is $\frac{2\pi}{7}$, as is known by a simple analysis of the cosine function. Since the maximum of the cosine function is 1, the maximum of x is 0.3. Differentiating we find that the maximum speed is 7(0.3) = 2.1 m per second.

74 a The auxiliary equation is $m^2 + 1.6 = 0$, so the general solution of the differential equation is:
$x = A\cos\left(\sqrt{1.6}t\right) + B\sin\left(\sqrt{1.6}t\right)$. Assuming the initial conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 1$ we see that the two coefficients have to satisfy the conditions $A = 0$ and $\sqrt{1.6}B = 1$, so the correct solution is $x = \frac{\sqrt{1.6}}{1.6}\sin\left(\sqrt{1.6}t\right)$. Calculating $\frac{\sqrt{1.6}}{1.6}$ yields the maximum displacement: it is 0.791 m to (3 s. f.).

b The boat is at its maximum displacement when $\sqrt{1.6}t = \frac{\pi}{2}$, and at its minimum when $\sqrt{1.6}t = \frac{3\pi}{2}$.
Then the time elapsing between these two moments is given by:

$$\frac{3\pi}{2\sqrt{1.6}} - \frac{\pi}{2\sqrt{1.6}}$$
$$= \frac{\pi}{\sqrt{1.6}} = 2.48$$

So the time is 2.48 minutes.

c It is very unlikely that for large values of t the boat keeps moving with such regularity, surely other factors should be taken into account.

75 a $\frac{d^2x}{dt^2}+2\frac{dx}{dx}+2x=12\cos 2t-6\sin 2t$

Auxiliary equation: $m^2+2m+2=0$

An expression for x is needed. Refer to book FP2 Chapter 5 for the method of solving these equations.

$m=\frac{-2\pm\sqrt{(4-8)}}{2}=-1\pm i$

Complementary function is

$x=e^{-t}(A\cos t+B\sin t)$

Let $x=p\cos 2t+q\sin 2t$

Try this for a particular integrate.

$\dot{x}=-2p\sin 2t+2q\cos 2t$

$\ddot{x}=-4p\cos 2t-4q\sin 2t$

$\therefore -4p\cos 2t-4q\sin 2t$

$+2(-2p\sin 2t+2q\cos 2t)$

$+2(p\cos 2t+q\sin 2t)$

$=12\cos 2t-6\sin 2t$

Substitute the expressions for x, $\dot{x}$ and $\ddot{x}$ into the differential equation.

$\cos 2t(-4p+4q+2p)$

$+\sin 2t(-4q-4p+2q)$

$=12\cos 2t-6\sin 2t$

$-4p+4q+2p=12$

$-2p+4q=12$ **(1)**

Equate coefficients of cos $2t$...

$-4q-4p+2q=-6$

$-4p-2q=-6$

... and of sin $2t$.

$-2p-q=-3$ **(2)**

$5q=15$

Solve **(1)** and **(2)**.

$q=3, p=0$

$\therefore x=e^{-t}(A\cos t+B\sin t)+3\sin 2t$

$t=0,\ x=0 \therefore 0=A$

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-e^{-t}B\sin t+e^{-t}B\cos t+6\cos 2t$

$t=0,\ \dot{x}=0 \therefore 0=B+6$

$B=-6$

$\therefore x=3\sin 2t-6e^{-t}\sin t$

b $\dot{x}=6e^{-t}\sin t-6e^{-t}\cos t+6\cos 2t$

$t=\frac{\pi}{4}, \dot{x}=6\left[e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}-e^{-\frac{\pi}{4}}\cos\frac{\pi}{4}+\cos\frac{\pi}{2}\right]$

$=0$

$\sin\frac{\pi}{4}=\cos\frac{\pi}{4}\quad\cos\frac{\pi}{2}=0$

$\therefore P$ comes to instantaneous test when $t=\frac{\pi}{4}$

75 c $x = 3\sin 2t - 6\mathrm{e}^{-t}\sin t$

$t = \frac{\pi}{4}$ $x = 3\sin\frac{\pi}{2} - 6\mathrm{e}^{-\frac{\pi}{4}}\sin\frac{\pi}{4}$

$= 3 - 6\mathrm{e}^{-\frac{\pi}{4}} \times \frac{1}{\sqrt{2}}$

$= 1.07 (3\text{s.f.})$

d $t \to \infty$

Large values of t needed, so let $t \to \infty$

$x \approx 3\sin 2t$

$\mathrm{e}^{-t} \to 0 \text{ as } t \to \infty$

$\therefore$ approximate period is π

76 a The value of the constants can be found by forcing the conditions $x(0) = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = U$: these give us that $A = 0$ and $-\omega A + B\omega = U$, so the correct equation for this motion is $x = e^{-\omega t}\left(\frac{U}{\omega}\sin \omega t\right)$.

b The velocity of the particle is given by the following expression:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\omega e^{-\omega t}\left(\frac{U}{\omega}\sin\omega t\right) + e^{-\omega t}\left(\frac{U}{\omega}\omega\cos\omega t\right) =$$

$$= -Ue^{-\omega t}\sin\omega t + Ue^{-\omega t}\cos\omega t =$$

$$= Ue^{-\omega t}(\cos\omega t - \sin\omega t)$$

The least value for which this is zero is $\omega t = \frac{\pi}{4}$, so the time is $t = \frac{\pi}{4\omega}$.

77 To find a particular integral for this equation we look at functions of the form $x = \lambda + \mu t$; by deriving it we find that the coefficients must satisfy the conditions $10k^2\mu = 10k^2V$ and $2k\mu + 10k^2\lambda = 0$, so the particular integral will be $x = -\frac{V}{5k} + Vt$. As for the general solution, the auxiliary equation associated with the homogeneous equation is $m^2 + 2km + 10k^2 = 0$, and solutions for this equation are $-k \pm 3ik$, so the general solution for the homogeneous equation is $x = e^{-kt}(A\cos 3kt + B\sin 3kt)$. Then the general solution of the equation is $x = -\frac{V}{5k} + Vt + e^{-kt}(A\cos 3kt + B\sin 3kt)$. To find A and B we force the conditions $x(0) = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$, which yield $-\frac{V}{5k} + A = 0$ and $V - kA + 3Bk = 0$. By solving these simple equations we find that the expression for x is:

$$x = -\frac{V}{5k} + Vt + e^{-kt}\left(\frac{V}{5k}\cos 3kt - \frac{4V}{15k}\sin 3kt\right).$$

78 a A particular integral for this equation is given by the constant function $x = 0.3$. To find the general solution, consider the auxiliary equation $m^2 + 6m + 8 = 0$; this gives us that the general solution of the homogeneous equation is $x = Ae^{-2t} + Be^{-4t}$, so the general solution is $x = 0.3 + Ae^{-2t} + Be^{-4t}$. To find the coefficients we force the conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$, which yield $A + B + 0.3 = 0$ and $-2A - 4B = 0$, so we can easily see that the solution of the equation that describes the motion of P is $x = 0.3 - 0.6e^{-2t} + 0.3e^{-4t}$.

b Deriving the expression for x gives us that at time t the speed of the particle is $1.2\left(e^{-2t} - e^{-4t}\right)$. This is a positive number for every $t > 0$, so the particle never stops moving downward.

79 a Compute the second derivative of x:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d}{dt}(0.1x + 0.1y)$$
$$= 0.1\frac{dx}{dt} + 0.1\frac{dy}{dt}$$
$$= 0.01x + 0.01y - 0.0025x + 0.02y$$
$$= 0.0075x + 0.03y$$

Now we can substitute the derivatives of x in the equation and we get:

$$0.0075x + 0.03y - 0.03x - 0.03y + 0.0225x$$
$$= -0.0225x + 0.0225x = 0$$

b The auxiliary equation is $m^2 - 0.3m + 0.0225 = 0$, which can be rewritten as $(m - 0.15)^2 = 0$, so the general solution is $x = (A + Bt)e^{0.15t}$.

c We must solve $\frac{dy}{dt} = -0.025x + 0.2y$. This is $\frac{dy}{dt} - 0.2y = -0.025x$; the integrating factor for this equation is $e^{-0.2t}$, so we get $\frac{d}{dt}ye^{-0.2t} = -0.025xe^{-0.2t}$. In order to solve this, we must first solve $\int -0.025xe^{-0.2t}\,dt$, as follows:

$$\int -0.025xe^{-0.2t}\,dt =$$
$$= -0.025\int xe^{-0.2t}\,dt =$$
$$= -0.025\int (A + Bt)e^{-0.05t}\,dt =$$
$$= -0.025\left(A\int e^{-0.05t}\,dt + B\int te^{-0.05t}\,dt\right) =$$
$$= -0.025\left(-A\frac{e^{-0.05t}}{0.05} + B\left(-\frac{te^{-0.05t}}{0.05} - \frac{e^{-0.05t}}{0.0025}\right)\right) =$$
$$= 0.5Ae^{-0.05} + 0.5Bte^{-0.05} + 10Be^{-0.05}$$

$$m^2 - m - 6 = 0$$

Multiply this by the integrating factor to get an expression for y:

$$y = 0.5Ae^{0.15t} + 0.5Bte^{0.15t} + 10Be^{0.15t}$$

d Use the given initial conditions to find the value of the constants. In particular, if we fix $t = 0$ at the beginning of 2010, we get that $A = 20$ and $0.5A + 10B = 100$, so the coefficients are 20 and 9. Then after seven months the number of angler fish is given by $(20 + 9 \cdot 7)e^{0,15 \cdot 7} = 237$.

79 e The model does not give any bounds on the growth of the number of fish, so it models a situation in which the number is always increasing – unlikely to be a good model for large values of t.

80 a Compute the second derivative of x:

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=$$
$$=\frac{d}{dt}(2x+y+1)=$$
$$=2\frac{dx}{dt}+\frac{dy}{dt}=$$
$$=4x+2y+2+4x-y+1=$$
$$=8x+y+3$$

Now we can substitute the derivatives of x in the equation and we get:

$$8x+y+3-2x-y-1-6x=$$
$$=6x-6x+3-1=2$$

b First we solve the differential equation that models x; we easily see that a particular integral is given by the constant function $x=-\frac{1}{3}$. Solving the auxiliary equation $m^2-m-6=0$ then tells us that the general solution for the equation is $x=-\frac{1}{3}+Ae^{3t}+Be^{-2t}$. Then we use this to find an expression for y: we need to solve the equation, $\frac{dy}{dt}+y=4x+1$ which has integrating factor e^t and is therefore equivalent to $\frac{d}{dt}ye^t=(4x+1)e^t$. Substitute the value of x to solve this:

$$\int(4x+1)e^t\ dt=$$
$$=\int\left(-\tfrac{1}{3}+4Ae^{3t}+4Be^{-2t}\right)e^t\ dt=$$
$$=\int-\frac{e^t}{3}+4Ae^{4t}+4Be^{-t}\ dt=$$
$$=-\frac{e^t}{3}+Ae^{4t}-4Be^{-t}$$

Therefore:

$$ye^t=-\frac{e^t}{3}+Ae^{4t}-4Be^{-t}$$
$$y=-\tfrac{1}{3}+Ae^{3t}-4Be^{-2t}$$

Now we use the initial conditions to determine the coefficients: we have that $-\frac{1}{3}+A+B=20$ and $-\frac{1}{3}+A-4B=60$, and so $B=-8$ and $A=\frac{85}{3}$.

Hence $x=-\frac{1}{3}+\frac{85}{3}e^{3t}-8e^{-2t}, y=-\frac{1}{3}+\frac{85}{3}e^{3t}+32e^{-2t}$

c Since both expressions feature the term e^{3t}, for large values of t the amount of gas should go to infinity in both tanks, which cannot happen.

Challenge

1 Let $n = 1$

The result $M^n = M$ becomes $M^1 = M$, which is true.

Assume the result is true for $n = k$.

That is

$$M^k = M = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$M^{k+1} = M^k M$$

$$= \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix} \cdot \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^4 x - \cosh^2 x \sinh^2 x & \cosh^4 x - \cosh^2 x \sinh^2 x \\ -\sinh^2 x \cosh^2 x + \sinh^4 x & -\sinh^2 x \cosh^2 + \sinh^4 x \end{pmatrix}$$

You can prove this result using mathematical induction, a method of proof you learnt in the FP1 module. The prerequisites in the FP3 specification state that a knowledge of FP1 is assumed and may be tested.

$$\cosh^4 x - \cosh^2 x \sinh^2 x = \cosh^2 x\left(\cosh^2 x - \sinh^2 x\right)$$
$$= \cosh^2 x$$

You use the identity $\cosh^2 x - \sinh^2 x = 1$ to simplify the terms in the matrix.

$$-\sinh^2 x \cosh^2 x + \sinh^4 x = \sinh^2 x\left(-\cosh^2 x + \sinh^2 x\right)$$
$$= -\sinh^2 x$$

Hence $M^{k+1} = \begin{pmatrix} \cosh^2 x & \cosh^2 x \\ -\sinh^2 x & -\sinh^2 x \end{pmatrix}$

and this is the result for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.
By mathematical induction the result is true for all positive integers n.

2 a Turn one of these into a second order differential equation by noting that $x = \frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$.

Of course this is solved by $x = Ae^t + Be^{-t}$. Now we use this to turn $\frac{dy}{dt} = x$ into $\frac{dy}{dt} = Ae^t + Be^{-t}$, an equation that can be easily solved finding $y = Ae^t - Be^{-t}$. With the given initial conditions we find that x and y must satisfy $A - B = 0$ and $A + B = 1$, so $A = B = \frac{1}{2}$. Substituting this into the expressions for x and y yields $x = \cosh t$ and $y = \sinh t$.

Challenge

2 b Turn one of these into a second order differential equation by noting that q satisfies the equation $\frac{d^2q}{dt^2}-2\frac{dq}{dt}+2q=0$, which is solved by $q=e^t\left(A\cos t+B\sin t\right)$. Now consider the equation $\frac{dp}{dt}=p-q$, which is equivalent to $\frac{d}{dt}pe^{-t}=-qe^{-t}$; with the expression for q that we found this can be easily rewritten as $\frac{d}{dt}pe^{-t}=-\left(A\cos t+B\sin t\right)$, and therefore $pe^{-t}=-\left(A\sin t-B\cos t\right)$.

Then an expression for p is $p=-e^t\left(A\sin t-B\cos t\right)$. Then it is easy to solve the equation for r:

$$\frac{dr}{dt}=p+2q+r$$
$$\frac{dr}{dt}-r=p+2q$$
$$\frac{d}{dt}re^{-t}=\left(p+2q\right)e^{-t}$$
$$re^{-t}=\int 2A\cos t+2B\sin t-A\sin t+B\cos t\ dt$$
$$re^{-t}=A\left(2\sin t+\cos t\right)+B\left(\sin t-2\cos t\right)+C$$
$$r=Ae^t\left(2\sin t+\cos t\right)+Be^t\left(\sin t-2\cos t\right)+Ce^t$$

Now we can use the initial conditions to find the coefficients: from the expression for q we get $A=1$, from the expression for p we get $B=1$ and from the expression for r we get $C=2$. Then the particular solution for r satisfying the given conditions is $r=e^t\left(3\sin t-\cos t+2\right)$.

Challenge

3 The gradient of the line l is clearly the tangent of the angle $\alpha+\theta$. This gradient can be expressed as follows: since the Cartesian coordinates of P are $(r\cos\theta, r\sin\theta)$, the gradient is $\dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$.

Now obviously $\alpha = \alpha+\theta-\theta$, so $\tan\alpha = \tan\left((\alpha+\theta)-\theta\right)$. By a formula, this can be solved as follows:

$$\tan\left((\alpha+\theta)-\theta\right) = \frac{\tan(\alpha+\theta)-\tan\theta}{1+\tan(\alpha+\theta)\tan\theta} =$$

$$= \frac{\dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} - \tan\theta}{1+\dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\tan\theta} =$$

$$= \frac{\dfrac{\frac{dr}{d\theta}\sin\theta\cos\theta + r\cos^2\theta - \frac{dr}{d\theta}\cos\theta\sin\theta + r\sin^2\theta}{\frac{dr}{d\theta}\cos^2\theta - r\sin\theta\cos\theta}}{1+\dfrac{\frac{dr}{d\theta}\sin^2\theta + r\cos\theta\sin\theta}{\frac{dr}{d\theta}\cos^2\theta - r\sin\theta\cos\theta}} =$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\frac{dr}{d\theta}\cos^2\theta + \frac{dr}{d\theta}\sin^2\theta} = \frac{r}{\frac{dr}{d\theta}}$$