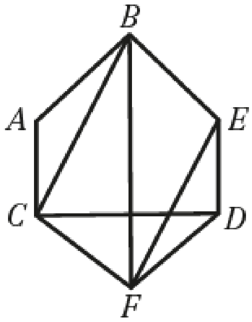
Graphs and networks 2E

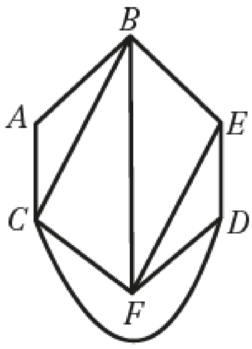
1 a There are only two edges adjacent to A, so if the cycle starts with edge AB it would have to end with CA. Hence, starting with ABC leaves no return path to A to complete the cycle.

b $ABEDFCA$ or $ABEFDCA$

c **Step 1:** Identify Hamiltonian cycle $ABEDFCA$.
Steps 2, 3: Draw a hexagon matching the vertices and add respective edges.



Steps 4, 5: Choose BF to stay inside the hexagon. Take CD outside. Go to **Step 4**.

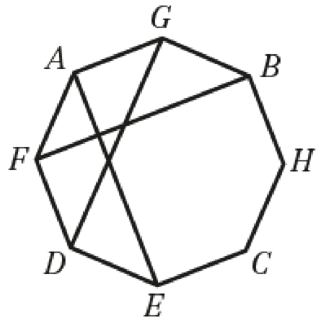


Step 4: There are no crossing edges inside the hexagon. The graph is planar.

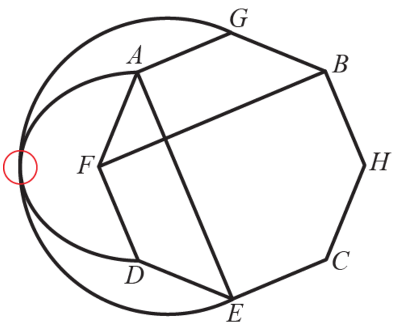
2 a A planar graph is one that can be drawn in a plane such that no two edges meet except at a vertex.

b Note that we need to alternate between the vertices on the left (A, B, C, D) and on the right (E, F, G, H). A possible Hamiltonian cycle is $AGBHCEDFA$

2 c **Step 1:** Use the Hamiltonian cycle identified in part **b.** In our case it is *AGBHCEDFA*
Steps 2, 3: Draw a polygon matching the vertices and add respective edges.



Steps 4, 5: Choose *BF* to stay inside the polygon and take *DG* outside. It is not possible to move outside *AE* and *DG* without crossing another edge, so go to **Step 6**.



Steps 6: Choose *AE* to stay inside the polygon. Go to **Step 5**.

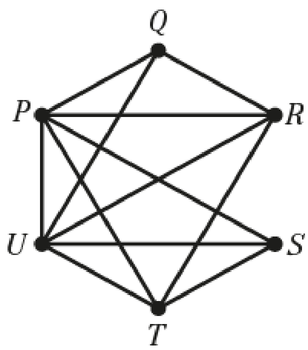
Step 5: It is not possible to move outside *BF* and *DG* . Go to **Step 6**.

Step 6: Choose *DG* to stay inside the polygon. Go to **Step 5**.

Step 5: It is not possible to move outside *AE* and *BF* . Go to **Step 6**.

Step 6: There are no other edges inside the polygon, so we conclude the original graph is not planar.

3 a

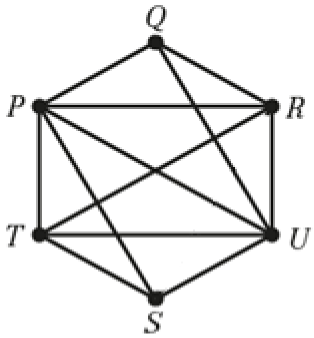


Note that the graph is undirected.

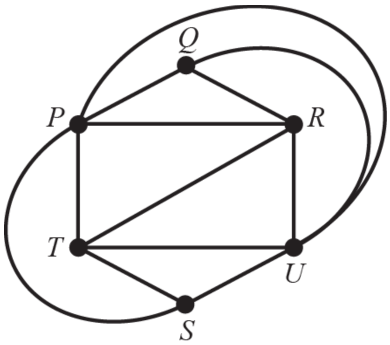
3 b We use the planarity algorithm.

Step 1: Identify Hamiltonian cycle $PQRUSTP$

Step 2, 3: Draw a polygon matching vertices in the cycle and add respective edges.



Steps 4, 5: Choose RT to stay inside the polygon and move QU, PU, PS outside. Go to **Step 4:**



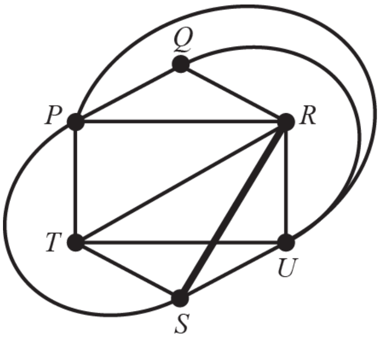
Step 4: There are no crossing edges, so the original graph was planar. Hence, the connections can be made without crossing any wires.

3 c After adding edge *RS*, the graph is not planar any more. To see that we can use the planarity algorithm again. We can repeat the first few steps, i.e.

Step 1: Identify Hamiltonian cycle *PQRUSTP*

Step 2, 3: Draw a polygon matching vertices in the cycle and add respective edges.

Steps 4, 5: Choose RT to stay inside the polygon and move QU, PU, PS outside. Go to **Step 4**.



Now we continue with the updated graph.

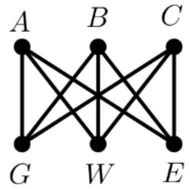
Steps 4, 5: Choose RS to stay inside the polygon. TU cannot be moved outside without crossing another edge. Go to **Step 6**.

Step 6: Choose TU to stay inside the polygon. Go to **Step 5**.

Step 5: RS cannot be moved outside without crossing another edge. Go to **Step 6**.

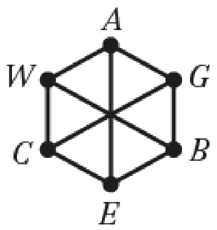
Step 6: There are no other edges inside the polygon, so we conclude the original graph is not planar.

4 We can draw a graph such that the houses correspond to vertices A, B, C and supplies to vertices G, W, E. Each house is connected to each of the supplies.



Use the planarity algorithm.

Step 1: Identify Hamiltonian cycle *AGBECWA*
Step 2, 3: Draw a polygon matching vertices in the cycle and add respective edges.



Step 4, 5: Fix edge *AE* and try to move *CG* and *BW* outside. It is not possible without crossing, so move to **Step 6**.

Step 6: Fix edge *CG* and go to **Step 5**.

Step 5: It is not possible to move outside *AE* and *BW* without crossing. Go to **Step 6**.

Step 6: Fix edge *BW* and go to **Step 5**.

Step 5: It is not possible to move outside *AE* and *CG* without crossing. Go to **Step 6**.

Step 6: All edges inside the polygon have been chosen so we conclude the graph is not planar.

Thus, we have shown that if all houses are connected to all three utilities, two supply lines must cross.

NB. One may skip a few last steps in the algorithm by appealing to symmetry.

Challenge

a The graph does not have a Hamiltonian cycle, so we cannot use the planarity algorithm.

b It suffices to redraw *AC* and *EC* to obtain a graph with no edges crossing.