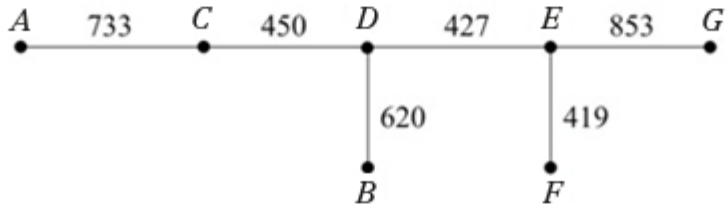
The travelling salesman problem Mixed exercise

1 a Either Kruskal: *EF*, *DE*, *CD*, *BD*, *AC*, *EG* or Prim (e.g.): *AC*, *CD*, *DE*, *EF*, *BD*, *EG*



b $2 \times 3502 = 7004$

c For example use *AB* and *DG*
Route *A C D E F E G D B A* length 6005

2 a

	A	*B*	*C*	*D*	*E*
A	–	7	13	4	3
B	7	–	17	7	10
C	13	17	–	10	13
D	4	7	10	–	5
E	3	10	13	5	–

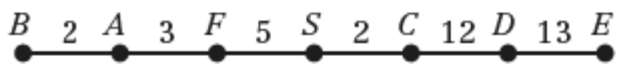
b $A_3E_5D_7B_{17}C_{13}A = 45$

c *A E D B D C A* (*BC* is not on the original network.)

d In our network there are 5 vertices. As the algorithm is of cubic order, we can estimate the running time as $0.85 \times \left(\frac{12}{5}\right)^3 = 11.75$ seconds (2 d.p.)

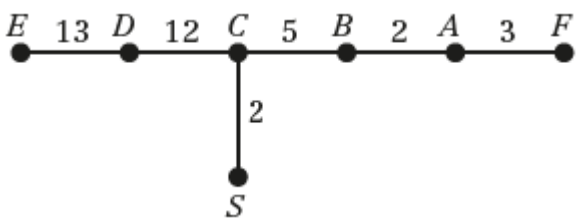
e The time taken is not exactly proportional to the cube of number of vertices, so our calculation was only approximate.

3 a *SC SF FA AB CD DE* – tree 1



and

SC CB BA AF CD DE – tree 2



b Weight of each tree is 37
So initial upper bound is $2 \times 37 = 74$

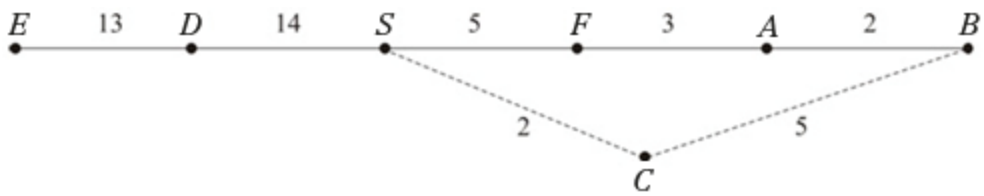
3 c From tree 1
Use BE as a shortcut
(Route is $S\ C\ D\ E\ B\ A\ F\ S$) length 56
From tree 2
Use EF as a shortcut
(Route is $S\ C\ B\ A\ F\ E\ D\ C\ S$) length 53

d $C_2S_5F_3A_2B_{17}D_{13}E_{21}C = 63$
$D_{12}C_2S_5F_3A_2B_{19}E_{13}D = 56$

e The better upper bound is 53 since it is smaller.

f The route is $S\ C\ B\ A\ F\ E\ D\ C\ S$

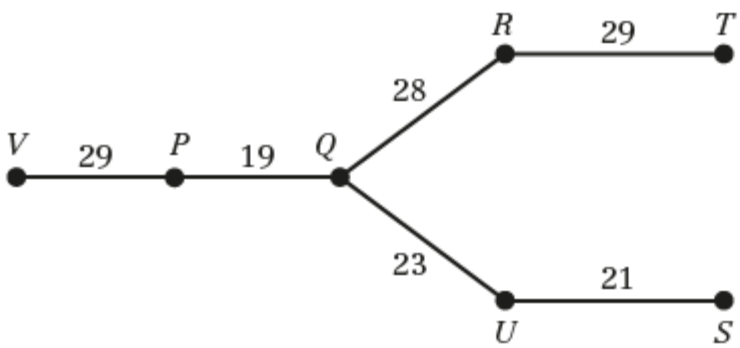
g



Weight of residual minimum spanning tree = 37
Two least arcs from C are CS and CB
Lower bound $= 37 + 2 + 5$
$= 44$

4 a In the classical problem each vertex must be visited exactly once before returning to the start.
In the practical problem each vertex must be visited at least once before returning to the start.

b

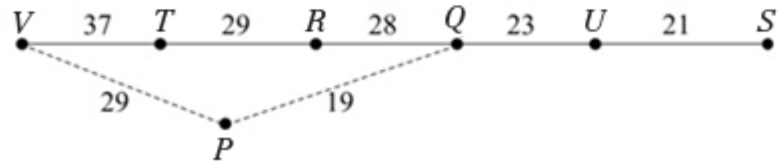


Order of arcs: $PQ, QU, US, QR, \begin{Bmatrix} TR \\ VP \end{Bmatrix}$

c Use VT and QS as shortcuts giving a length of 213
(Route $P\ Q\ U\ S\ Q\ R\ T\ V\ P$)

d $P_{19}Q_{23}U_{21}S_{51}R_{29}T_{37}V_{29}P = 209$

4 e



Weight of residual minimum spanning tree $= 138$

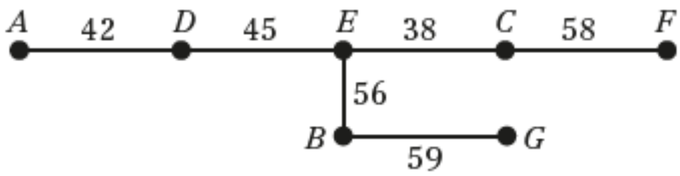
Two least arcs PQ and PV

Lower bound $= 138 + 19 + 29$

$= 186$

f 186 < optimal valueÑ 209

5 a



order of arcs: AD, DE, EC, EB, CF, BG

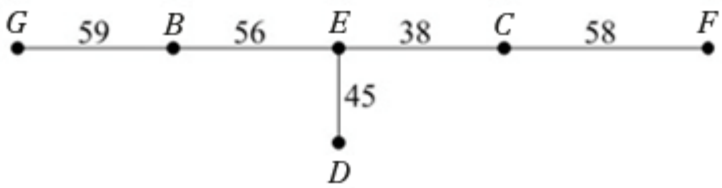
b Initial upper bound $= 2 \times 298$

$= 596$

c The minimum connector has been doubled and each arc in it repeated.

d Use AE and GF as shortcuts – length 427
(route is A D E B G F C E A)

e



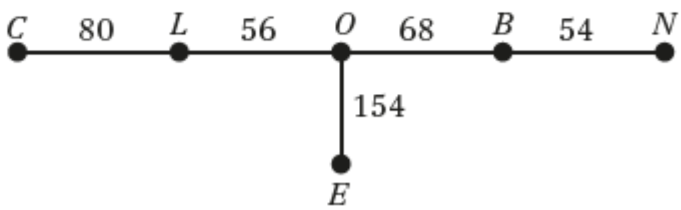
Weight of residual minimum spanning tree $= 256$

Two least arcs from A are AD (42) and AE (54)

Lower bound $= 256 + 42 + 54 = 352$ km

f The lower bound will give the optimal solution if it is a tour.
If the minimum spanning tree has no ‘branches’ – so the two end vertices have valency 1, and all other vertices have valency 2, then if the two least arcs are incident on the 2 vertices of valency 1 an optimal solution cannot be found.

6 a



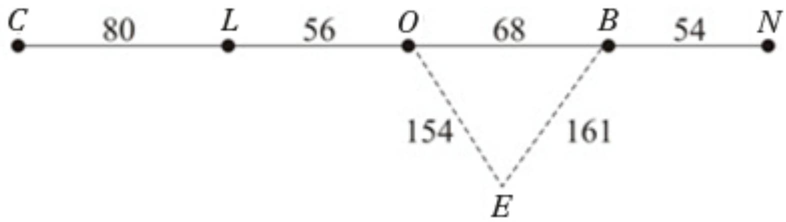
order of selection: LO, OB, BN, LC, OE

6 b i Initial upper bound $= 2 \times 412$

$= 824$ miles

ii Use NC as a shortcut – length is 653
(Route is L O E O B N C L)

c



Weight of residual minimum spanning tree $= 258$

Two least arcs are *EO* and *EB*
Lower bound $= 258 + 154 + 161$

$= 573$

7 a The nearest neighbour route is *AECGBDFA* of length $12 + 22 + 23 + 20 + 18 + x + 15 = 110 + x$
Hence, $140 = 110 + x \Rightarrow x = 30$.

b The nearest neighbour route from *B* is *BAECGFDB* of length
$16 + 12 + 22 + 23 + 30 + x + 18 = 151$ miles.

c By using Prim's algorithm (table below) or otherwise we find the RMST of length
$16 + 21 + 17 + 12 + 15 = 81$

	A	*B*	*C*	*D*	*E*	*F*
A	–	16	21	17	12	15
B	16	–	24	18	30	26
C	21	24	–	31	22	35
D	17	18	31	–	28	*x*
E	12	30	22	28	–	27
F	15	26	35	*x*	27	–

Two shortest edges from *G* to the reduced graph are *GA* and *GB* of lengths 19 and 20, respectively. Hence, we have a lower bound of $81 + 19 + 20 = 120$ miles.

d Using the upper bound of 140 given in the question we have $120 <$ optimal value Ñ 140

Challenge

Consider first finding the nearest neighbour path from a single vertex. For the first step of the algorithm we need to compare n edge weights to select the smallest. Once we move to the next vertex, there are at most $n-1$ edges to compare. For the next one there are only $n-2$ and so on. Recall that finding minimum of n numbers requires $n-1$ operations.

Hence in total we need $(n-1)+(n-2)+\cdots+2+1=\frac{(n-1)n}{2}$ comparisons.

We need to repeat the process for each vertex, which gives $\frac{(n-1)n}{2}\times n=\frac{(n-1)n^2}{2}$ comparisons.

Finally, select the tour of shortest length (n – 1 comparisons) to give $\frac{(n-1)n^2}{2}+n-1=\frac{(n-1)(n^2+2)}{2}$, which is cubic order.