Game theory 6B

1 Row 3 dominates row 1 ($3>1, -3>-5$) so game can be reduced to

Ellie would always choose to play row 3 over row 1

	Freya plays 1	Freya plays 2
Ellie plays 2	–1	6
Ellie plays 3	3	–3

2 a Note that every entry in column 3 is smaller than every respective entry in column 2. Hence column 3 dominates column 2 and so column 2 can be deleted.

	H plays 1	H plays 3
D **plays 1**	–5	–1
D **plays 2**	2	–6

b To write out the matrix from Harry's perspective, we need to multiply each number by -1:

	D plays 1	D plays 2
H **plays 1**	5	–2
H **plays 3**	1	6

3 a First, we notice that every entry in row 1 is greater than each respective entry in row 2. Hence row 1 dominates row 2 and row 2 can be deleted. Next, we inspect the columns and notice that every entry in column 1 is smaller than each respective entry in column 3. So column 1 dominates column 3 and column 3 can be deleted. Reduced matrix:

	N plays 1	N plays 2
D **plays 1**	1	2
D **plays 3**	2	–1

b To write out Nick's pay-off matrix we need to multiply each number by –1:

	D plays 1	D plays 3
N **plays 1**	–1	–2
N **plays 2**	–2	1

4 a We notice that row 1 dominates row 2 and so row 2 can be deleted, as it would never be chosen. Reduced matrix:

	Y plays 1	Y plays 2	Y plays 3
S **plays 1**	4	–6	2
S **plays 3**	–5	7	–8

4 b To determine the play-safe strategy for Sakiya, we look for the row maximin. The minima of the two rows are –6 and –8 respectively, so the maximin is –6 and Sakiya should play 1.
To determine the play-safe strategy for Yin we look for the column minimax. The maxima of the three columns are 4, 7 and 2 respectively. So the column minimax is 2 and so Yin should play 3.

c Based on part **b** and the stable solution theorem, we see that this situation has no stable solution. This is because the row maximin (–6) ≠ column minimax (2).

d If both players play safe, i.e. Sakiya plays 1 and Yin plays 3, Sakiya will win 2.

e Because this is a zero-sum game, part **d** means that Yin will lose 2 (or win –2).

f To write out Yin's pay-off matrix we need to multiply all numbers by –1:

	***S* plays 1**	***S* plays 3**
***Y* plays 1**	–4	5
***Y* plays 2**	6	–7
***Y* plays 3**	–2	8

5 a Each value in column 3 less than or equal to the corresponding value in column 2. This means that when playing column 3 Brian will lose at most as much as when playing column 2.

b i The reduced pay-off matrix looks as follows:

	Brian plays 1	**Brian plays 3**	**Brian plays 4**
Ali plays 1	0	2	12
Ali plays 2	9	8	10
Ali plays 3	10	3	0

ii The row maximin is 8 in row 2. The column minimax is 8 in column 2, so the row maximin = column minimax = 8. Thus the matrix has a saddle point equal to 8. Since the pay-off matrix is written out from Ali's perspective, the value of the game to Brian is −8.

c By inspecting the original pay-off matrix we see that the value 8 appears twice in the row 2. It is still the row maximin. Similarly, it is still the column minimax, except it now appears in both column 2 and 3. So the game has 2 saddle points.

Challenge

To prove an if and only if statement we need to show that the first part of the statement implies the second part **AND ALSO** that the second part implies the first part. We will thus construct our proof in two parts.

1. Assume that G has a stable solution. This means that G has a saddle point, i.e. a point which is the smallest in its row and the largest in its column. Without loss of generality, we will assume that the saddle point is in row r and column s, and we will call that point $x_{r,s}$. Now, suppose we can use the domination argument to reduce G. We will consider two cases here: removing a row and removing a column. We begin with removing a row. Notice that row r cannot be reduced; To remove a row, we need all its values to be smaller than the respective values in another row. But, because $x_{r,s}$ is a saddle point, we know that it is the largest number in column s. Hence, it cannot be smaller than any other number in column s, so there is no row which would have all respective values greater than row r. We can therefore assume that the row we remove is row p. We want to assess whether this influences the previously found saddle point. Removing row p will not affect the fact that $x_{r,s}$ is the smallest value in row r. Now, since $x_{r,s}$ is also the largest value in column s, we have that $x_{r,s} \geqslant x_{p,s}$. So after removing row p, $x_{r,s}$ is still the largest value in column s, so it is still a saddle point. Now, we will consider what happens if we remove a column. Again, column s cannot be removed. To remove a column, we need its values to be larger than the respective values of another column. Since $x_{r,s}$ is a saddle, it is the smallest value in row r. Hence, we cannot find a column which would have a smaller value in the rth row. Thus we can assume that the column we remove is q. We want to assess whether this affects the saddle point. Removing column q does not change the fact that $x_{r,s}$ is the largest value in column s, but we need to check whether it is still the smallest value in row r. Again, because $x_{r,s}$ is a saddle, we have that $x_{r,s} \leqslant x_{r,q}$. Thus, after removing column q, $x_{r,s}$ will still be the smallest value in row r and thus will still be a saddle point. We thus showed that reducing G does not change the saddle point. So if G had a saddle point, G' will also have a saddle point (we have in fact also shown that it will be the same one as G!)

Challenge (continued)

2. Assume that G' has a saddle point in row r and column s, call it $x_{r,s}$. We now want to show that the matrix G from which G' originated also has a saddle point. We will consider two cases: the case where G' arises by removing a row from G, or the case when it arises by removing a column from G. We begin with the row case and we want to assess whether adding that row back will affect the saddle point. This is the reverse of the argument we used in part 1. If a row k was removed, its values were smaller than the respective values of another row in G. Again, adding back a row would not affect the fact that $x_{r,s}$ is the smallest in row r, but it might affect it being the largest in column s. For that to happen, we would need $x_{k,s} \geqslant x_{r,s}$. But since $x_{r,s}$ is now a saddle, it is greater than any other value in column s. Hence we would need $x_{k,s}$ to be even larger, which would mean it was the largest value in column s. This, however, means that row k and could not have been removed because there is no row with a larger value in column s! This is a contradiction, so we deduce that $x_{r,s}$ must have been a saddle before the reduction as well. Next, consider adding back a column l, which was removed by the domination argument. This means that all its values were larger than the respective values of another column in G. Adding back this column will not affect $x_{r,s}$ being the largest in column s, but it might affect it being the smallest in row r. For that to happen, we would need $x_{r,l} \leqslant x_{r,s}$. But this would mean that $x_{r,l}$ was the smallest value in row r. This in turn implies that column l could not have been removed, as its values could not have been larger than the respective values of another column – no column has a smaller value in row r! Again, this leads to a contradiction, meaning that adding back a column will not affect the saddle point $x_{r,s}$. We thus showed that if G' has a saddle point, so does G (we also showed that it is the exact same saddle point!).

   Having proven both implications we can now conclude that the theorem is true.