Game theory 6C

1 a To find the expected pay-offs for each of the strategies we need to multiply the probability of each outcome by the outcome for each of the scenarios:

Strategy *V*:

If *B* plays 1:

Expected pay-off = $0.5\cdot(-2)+0.5\cdot0=-1$

If *B* plays 2:

Expected pay-off = $0.5\cdot4+0.5\cdot5=4.5$

If *B* plays 3:

Expected pay-off = $0.5\cdot2+0.5\cdot(-1)=0.5$

Strategy *W*:

If *B* plays 1:

Expected pay-off = $0.4\cdot(-2)+0.6\cdot0=-0.8$

If *B* plays 2:

Expected pay-off = $0.4\cdot4+0.6\cdot5=4.6$

If *B* plays 3:

Expected pay-off = $0.4\cdot2+0.6\cdot(-1)=0.2$

Strategy *X*:

If *B* plays 1:

Expected pay-off = $0.3\cdot(-2)+0.7\cdot0=-0.6$

If *B* plays 2:

Expected pay-off = $0.3\cdot4+0.7\cdot5=4.7$

If *B* plays 3:

Expected pay-off = $0.3\cdot2+0.7\cdot(-1)=-0.1$

Strategy *Y*:

If *B* plays 1:

Expected pay-off = $0.2\cdot(-2)+0.8\cdot0=-0.4$

If *B* plays 2:

Expected pay-off = $0.2\cdot4+0.8\cdot5=4.8$

If *B* plays 3:

Expected pay-off = $0.2\cdot2+0.8\cdot(-1)=-0.4$

Strategy *Z*:

If *B* plays 1:

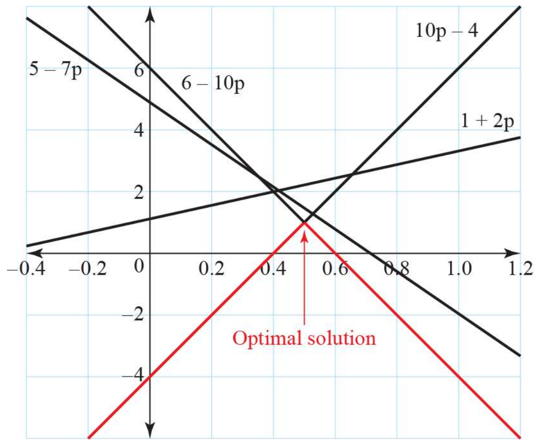
Expected pay-off = $0.1\cdot(-2)+0.9\cdot0=-0.2$

If *B* plays 2:

Expected pay-off = $0.1\cdot4+0.9\cdot5=4.9$

If *B* plays 3:

Expected pay-off = $0.1\cdot2+0.9\cdot(-1)=-0.7$



b The optimal strategy for player *A* maximises the minimal pay-off. In this case, strategy ***Y*** is the one *A* should choose. The minimum pay-off for this strategy is –0.4.

2 a i

	B plays 1	B plays 2	Row min	
A plays 1	2	–4	–4	
A plays 2	–1	3	–1	←
Column max	2	3		
	↑			

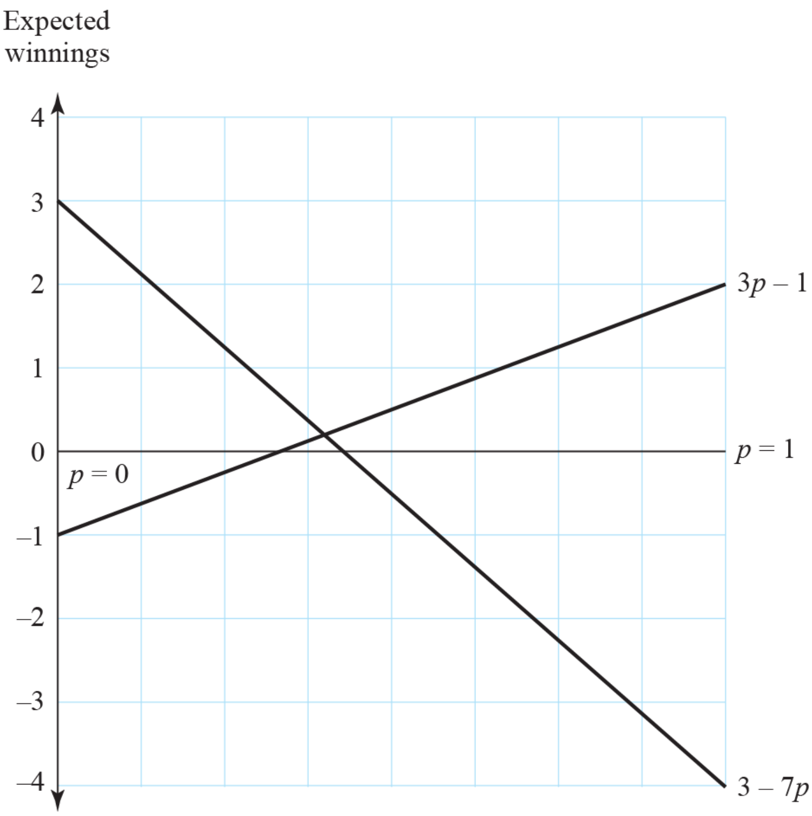
Since 2 ≠ –1 (column minimax ≠ row maximin) the game is not stable

ii Let A play 1 with probability p

So A plays 2 with probability $(1 - p)$

If B plays 1 A's expected winning are $2p - 1(1 - p) = 3p - 1$

If B plays 1 A's expected winnings are $4p + 3(1 - p) = 3 - 7p$



$$3p - 1 = 3 - 7p$$
$$10p = 4$$
$$p = \frac{2}{5}$$

A should play 1 with probability $\frac{2}{5}$

A should play 2 with probability $\frac{3}{5}$

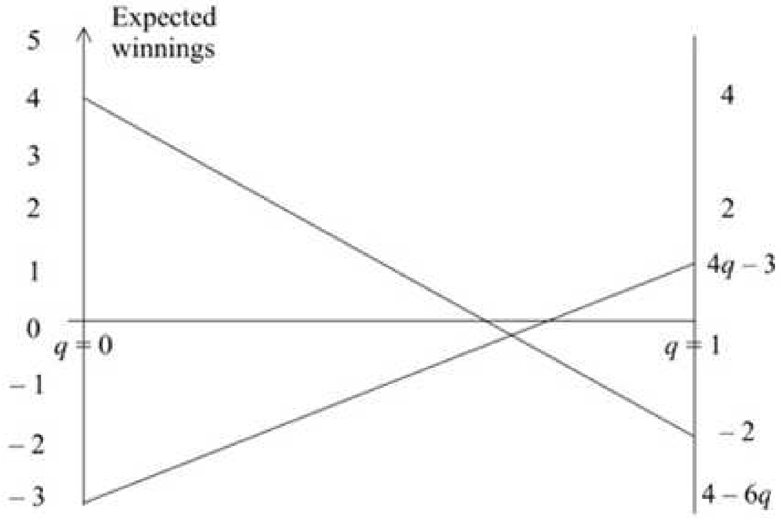
The value of the game to A is $3\left(\frac{2}{5}\right) - 1 = \frac{1}{5}$

2 a iii Let B play 1 with probability q

so B plays 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[2q-4(1-q)] = 4-6q$

If A plays 2 B's expected winnings are $-[-q+3(1-q)] = 4q-3$



$$4-6q = 4q-3$$
$$10q = 7$$
$$q = \frac{7}{10}$$

B should play 1 with probability $\frac{7}{10}$

B should play 2 with probability $\frac{3}{10}$

The value of the game to B is $4\left(\frac{3}{10}\right)-3 = \frac{-1}{5}$

b i

	B plays 1	B plays 2	Row min	
A plays 1	–3	5	–3	←
B plays 2	2	–4	–4	
Column max	2	5		
	↑			

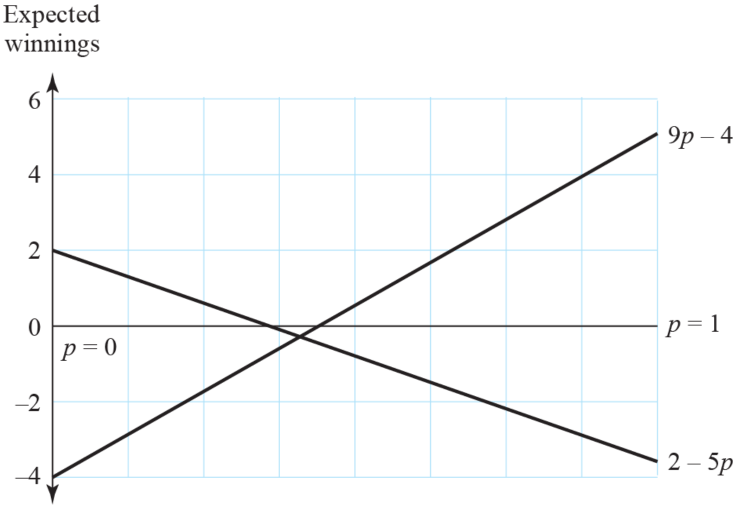
Since 2 ≠ –3 (column minimax ≠ row maximin) the game is not stable

2 b ii Let A play row 1 with probability p

So A plays row 2 with probability $(1-p)$

If B plays 1 A's expected winnings are $-3p-2(1-p)=2-5p$

If B plays 2 A's expected winnings are $5p-4(1-p)=9p-4$



$$2-5p=9p-4$$
$$14p=6$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with probability $\frac{4}{7}$

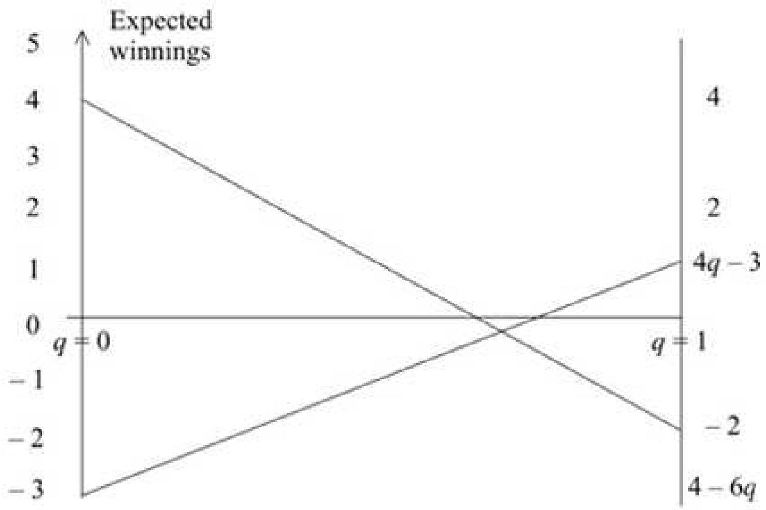
The value of the game to A is $2-5\left(\frac{3}{7}\right)=\frac{-1}{7}$

iii Let B play column 1 with probability q

So B plays column 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[-3q+5(1-q)]=8q-5$

If A plays 2 B's expected winnings are $-[2q-4(1-q)]=4-6q$



$$4-6q=8q-5$$
$$14q=9$$
$$q=\frac{9}{14}$$

B should play 1 with probability $\frac{9}{14}$

B should play 2 with probability $\frac{5}{14}$

The value of the game to B is $8\left(\frac{9}{14}\right)-5=\frac{1}{7}$

2 c i

	B plays 1	B plays 2	Row min	
A plays 1	5	–1	–1	←
A plays 2	–2	1	–2	
Column max	5	1		
		↑		

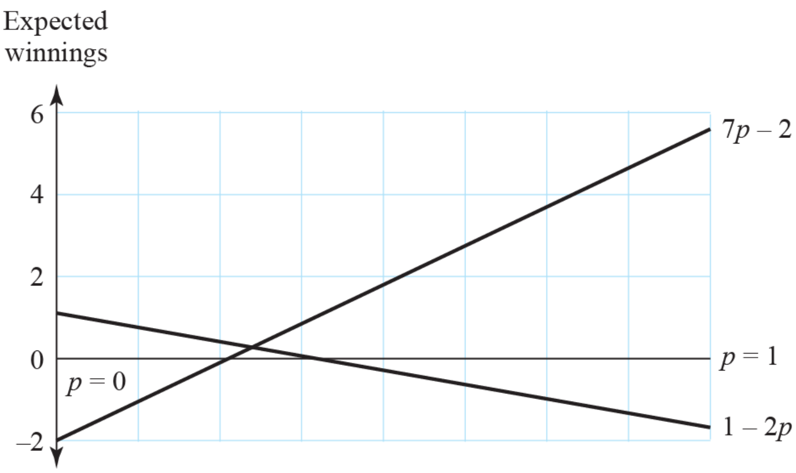
Since $-1 \neq 1$ (column minimax $\neq$ row maximin) the game is not stable

ii Let A play row 1 with probability p

So A plays rows 2 with probability $(1-p)$

If B plays 1 A's expected winnings are $5p-2(1-p)=7p-2$

If B plays 2 A's expected winnings are $-p+1(1-p)=1-2p$



$$7p-2=1-2p$$
$$9p=3$$
$$p=\frac{1}{3}$$

A should play 1 with probability $\frac{1}{3}$

A should play 2 with probability $\frac{2}{3}$

The value of the game to A is $7\left(\frac{1}{3}\right)-2=\frac{1}{3}$

iii Let B play column 1 with probability q

so B plays column 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[5q-1(1-q)]=1-6q$

If A plays 2 B's expected winning are $-[-2q+1(1-q)]=3q-1$

d i

	B plays 1	B plays 2	Row min	
A plays 1	–1	3	–1	←
A plays 2	1	–2	–2	
Column max	1	3		
	↑			

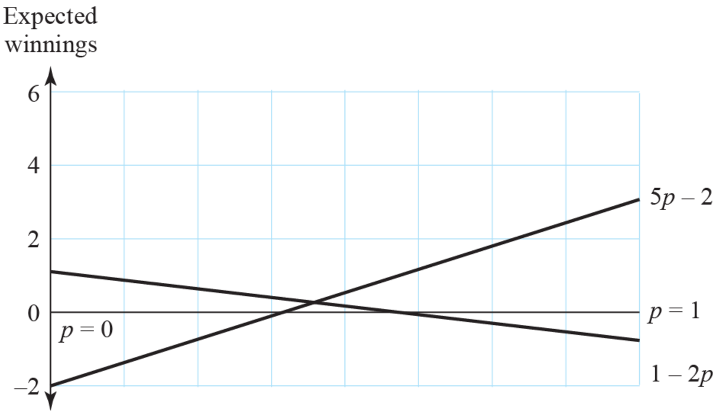
Since $-2 \neq 1$ (column minimax $\neq$ row maximin) the game is not stable.

2 d ii Let A play with probability p

So A plays with probability $(1-p)$

If B plays 1 A's expected winnings are $-p+(1-p)=1-2p$

If B plays 2 A's expected winnings are $3p-2(1-p)=5p-2$



$$5p-2=1-2p$$
$$7\wp=3$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with probability $\frac{4}{7}$

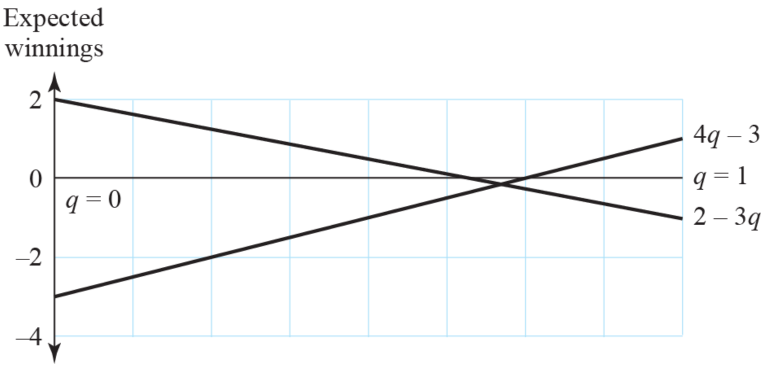
The value of the game to A is $1-2\left(\frac{3}{7}\right)=\frac{1}{7}$

iii Let B play 1 with probability q

So B plays 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[-q+3(1-q)]=4q-3$

If A plays 1 B's expected winnings are $-[q-2(1-q)]=2-3q$



$$4q-3=2-3q$$
$$7q=5$$
$$q=\frac{5}{7}$$

B should play 1 with probability $\frac{5}{7}$

B should play 2 with probability $\frac{2}{7}$

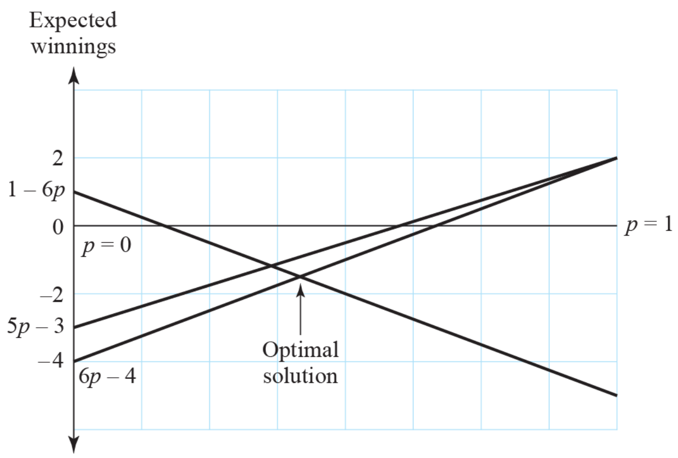
The value of the game to B is $4\left(\frac{5}{7}\right)-3=-\frac{1}{7}$

3 a i

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	–5	2	2	–5	
A plays 2	1	–3	–4	–4	←
Column max	1	2	2		
	↑				

Since 1 ≠ –4 (column minimax ≠ row maximin) the game is not stable

ii Let A play 1 with probability p
So A plays 2 with probability $(1 – p)$
If B plays 1 A's expected winnings are $-5p+1(1-p)=1-6p$
If B plays 2 A's expected winnings are $2p-3(1-p)=5p-3$
If B plays 3 A's expected winnings are $2p-4(1-p)=6p-4$



$$6p-4=1-6p$$
$$12p=5$$
$$p=\frac{5}{12}$$

A should play 1 with probability $\frac{5}{12}$

A should play 2 with probability $\frac{7}{12}$

The value of the game to A is

$$1-6\left(\frac{5}{12}\right)=-\frac{3}{2}$$

b i

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	6	–2	–2	←
A plays 2	–1	–4	3	–4	
Column max	2	6	3		
	↑				

Since 2 ≠ –2 (column minimax ≠ row maximin) the game is not stable

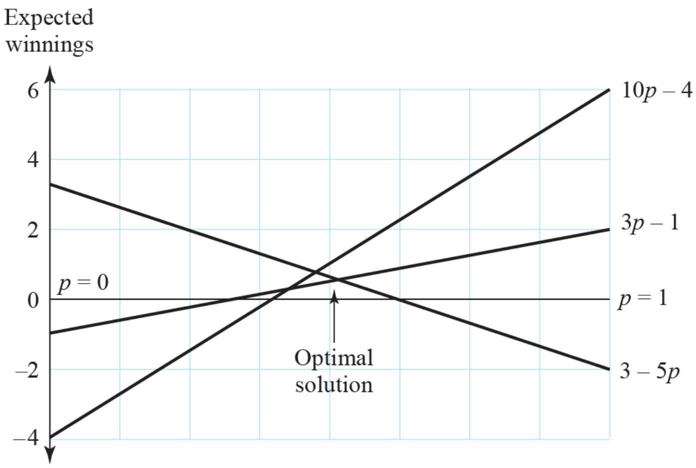
3 b ii Let A play 1 with probability p

So A plays 2 with probability $(1 - p)$

If B plays 1 A's expected winnings are $2p-(1-p)=3p-1$

If B plays 2 A's expected winnings are $6p-4(1-p)=10p-4$

If B plays 3 A's expected winnings are $-2p+3(1-p)=3-5p$



$$3p-1=3-5p$$
$$8p=4$$
$$p=\frac{1}{2}$$

A should play 1 with probability $\frac{1}{2}$

A should play 2 with probability $\frac{1}{2}$

The value of the game to A is

$$3\left(\frac{1}{2}\right)-1=\frac{1}{2}$$

c i By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –4 and the column minimax = 3, so there is no stable solution.

ii Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

If B plays 1 = $-2\cdot p+5\cdot(1-p)=5-7p$

If B plays 2 = $3\cdot p+1\cdot(1-p)=1+2p$

If B plays 3 = $6\cdot p-4\cdot(1-p)=10p-4$

If B plays 4 = $4\cdot p+6\cdot(1-p)=6-2p$

This can be illustrated on a diagram (remember p is a probability, so it takes values between 0 and 1 only!)

We choose the intersection which gives maximal minimum pay-off. Thus, we need to solve:

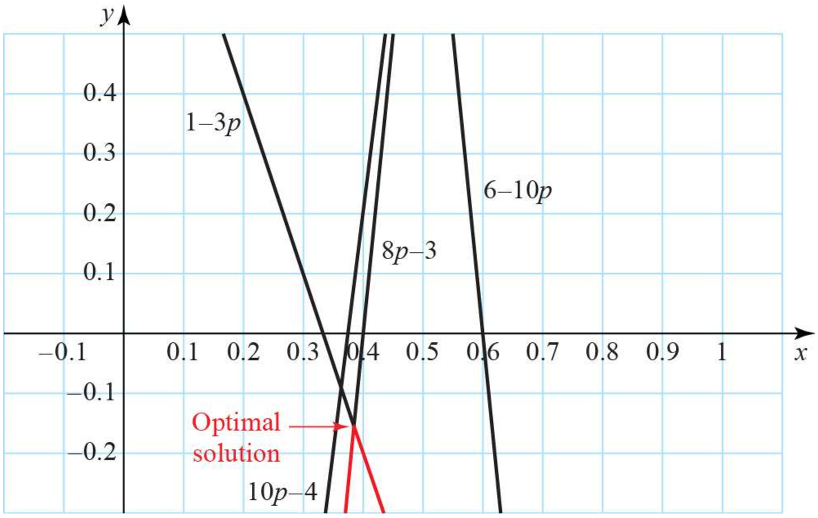
$$5-7p=10p-4$$
$$9=17p$$
$$p=\frac{9}{17}$$

So A should play 1 with probability $\frac{9}{17}$ and 2 with probability $\frac{8}{17}$. The value of the game is then calculated as the expected pay-off at the optimal solution, i.e.:

value = $5-7\cdot\frac{9}{17}=\frac{22}{17}$

3 d i As in part **c** we need to check whether row maximin = column minimax.
We have row maximin = –4 and the column minimax = 1, so there is no stable solution.

ii Assume A plays 1 with probability p and 2 with probability $1 - p$.
Then the expected winnings are:
If B plays 1 = $5 \cdot p - 3 \cdot (1 - p) = 8p - 3$
If B plays 2 = $-2 \cdot p + 1 \cdot (1 - p) = 1 - 3p$
If B plays 3 = $-4 \cdot p + 6 \cdot (1 - p) = 6 - 10p$
If B plays 4 = $6 \cdot p - 4 \cdot (1 - p) = 10p - 4$
This can be shown on a diagram:



We choose the intersection which maximises the minimal pay-off. Hence we need to solve:
$8p - 3 = 1 - 3p$
$11p = 4$
$p = \frac{4}{11}$
So A should play 1 with probability $\frac{4}{11}$ and play 2 with probability $\frac{7}{11}$. The value of the game is then equal to the expected pay-off at the optimal solution, i.e.:
value = $8 \cdot \frac{4}{11} - 3 = -\frac{1}{11}$.

4 a i

	B plays 1	B plays 2	Row min	
A plays 1	–1	1	–1	←
A plays 2	3	–4	–4	
A plays 3	–2	2	–2	
Column max	3	2		
		↑		

Since 2 ≠ – 1 (column minimax ≠ row maximin) the game is not stable

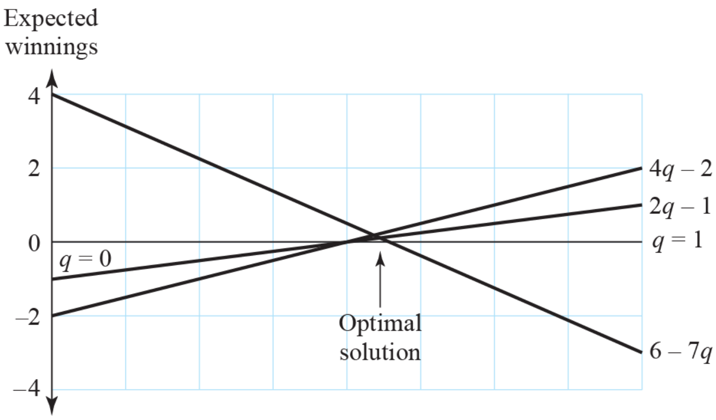
4 a ii Let B play 1 with probability q

So B plays 2 with probability $(1 - q)$

If A plays 1 B's expected winnings are $-[-q+1(1-q)] = 2q-1$

If A plays 2 B's expected winnings are $-[3q-4(1-q)] = 4-7q$

If A plays 3 B's expected winnings are $-[-2q+2(1-q)] = 4q-2$



$$2q-1 = 4-7q$$
$$9q = 5$$
$$q = \frac{5}{9}$$

B should play 1 with probability $\frac{5}{9}$

B should play 2 with probability $\frac{4}{9}$

The value of the game to B is

$$2\left(\frac{5}{9}\right)-1 = \frac{1}{9}$$

b i

	B plays 1	B plays 2	Row min	
A plays 1	–5	4	–5	
A plays 2	3	–3	–3	
A plays 3	1	–2	–2	←
Column max	3	4		
	↑			

Since $3 \neq -2$ (column minimax ≠ row maximin) the game is not stable

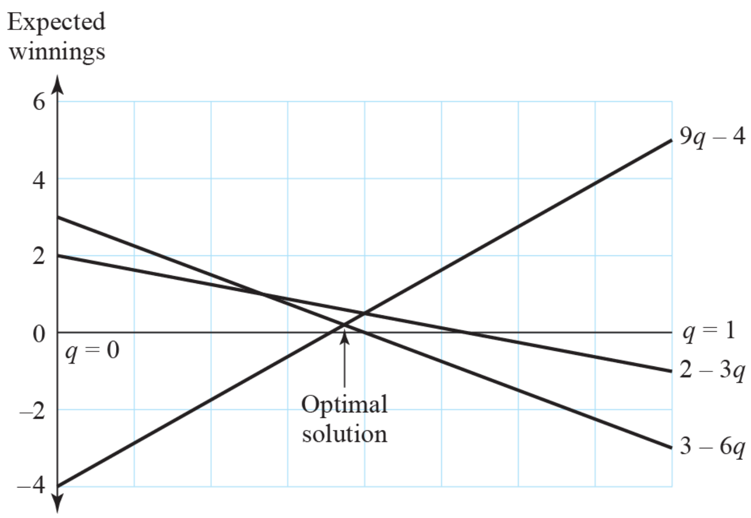
4 b ii Let B play 1 with probability q

So B plays 2 with probability $(1 - q)$

If A plays 1 B's expected winnings are $-[-5q + 4(1-q)] = 9q - 4$

If A plays 2 B's expected winnings are $-[3q - 3(1-q)] = 3 - 6q$

If A plays 3 B's expected winnings are $-[q - 2(1-q)] = 2 - 3q$



$$9q - 4 = 3 - 6q$$
$$15q = 7$$
$$q = \frac{7}{15}$$

B should play 1 with probability $\frac{7}{15}$

B should play 2 with probability $\frac{8}{15}$

The value of the game to B is

$$9\left(\frac{7}{15}\right) - 4 = \frac{3}{15}$$

c i By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –2 and the column minimax = 2, so there is no stable solution.

4 c ii We begin by rewriting the pay-off matrix from *B*'s perspective:

	***A* plays 1**	***A* plays 2**	***A* plays 3**	***A* plays 4**
***B* plays 1**	3	1	–2	–3
***B* plays 2**	–2	2	4	5

Now, assume *B* plays 1 with probability *p* and 2 with probability 1 – *p*.
Then the expected winnings are:

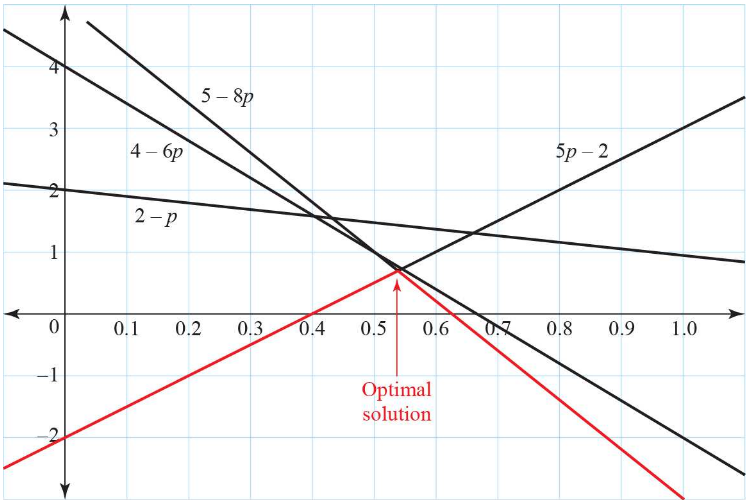
If *A* plays 1 = $3 \cdot p - 2 \cdot (1 - p) = 5p - 2$

If *A* plays 2 = $p + 2 \cdot (1 - p) = 2 - p$

If *A* plays 3 = $-2 \cdot p + 4 \cdot (1 - p) = 4 - 6p$

If *A* plays 4 = $-3 \cdot p + 5 \cdot (1 - p) = 5 - 8p$

This can be illustrated on a diagram:



We choose the solution which maximises the minimal pay-off, so we need to solve:

$5 - 8p = 5p - 2$

$13p = 7$

$p = \frac{7}{13}$

So *B* should play 1 with probability $\frac{7}{13}$ and 2 with probability $\frac{6}{13}$. The value of the game (from *B*'s perspective) is then equal to the expected pay-off at the optimal solution, i.e.:

$5 \cdot \frac{7}{13} - 2 = \frac{9}{13}$

4 d i By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –1 and the column minimax = 2, so there is no stable solution.

ii We begin by rewriting the pay-off matrix from B's perspective:

	***A* plays 1**	***A* plays 2**	***A* plays 3**	***A* plays 4**
***B* plays 1**	–2	2	–1	3
***B* plays 2**	3	–4	1	–5

Now, assume B plays 1 with probability p and 2 with probability 1 – p.
Then the expected winnings are:

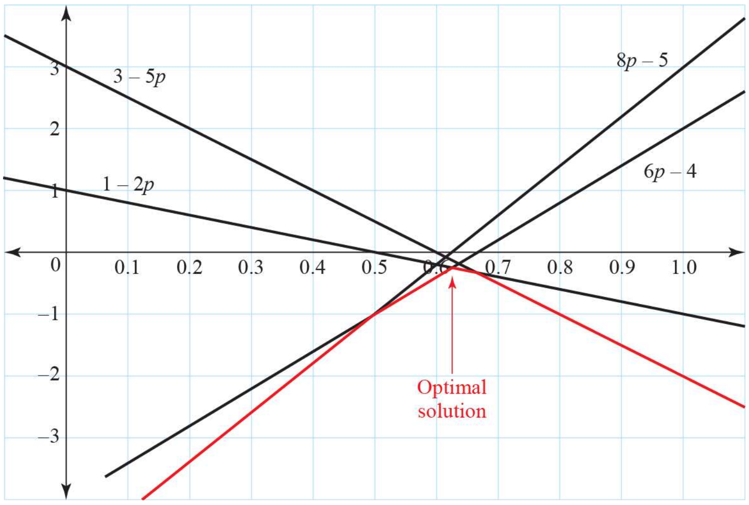
If A plays 1 = $-2\cdot p+3\cdot(1-p)=3-5p$

If A plays 2 = $2\cdot p-4\cdot(1-p)=6p-4$

If A plays 3 = $-p+(1-p)=1-2p$

If A plays 4 = $3\cdot p-5\cdot(1-p)=8p-5$

This can be illustrated on a diagram:



We choose the solution which maximises the minimal pay-off, so we need to solve:

$1-2p=6p-4$

$8p=5$

$p=\frac{5}{8}$

So B should play 1 with probability $\frac{5}{8}$ and play 2 with probability $\frac{3}{8}$. The value of the game for player B equals the expected pay-off at this optimal point, i.e.: $1-2\cdot\frac{5}{8}=-\frac{1}{4}$

5 a A zero-sum game is when one player's winnings are the other player's losses. This means that if player A wins x, then player B has to lose x (or win $-x$). This way the winnings add up to zero, hence the name zero-sum game.

b By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –2 and the column minimax = 0, so there is no stable solution.

c Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

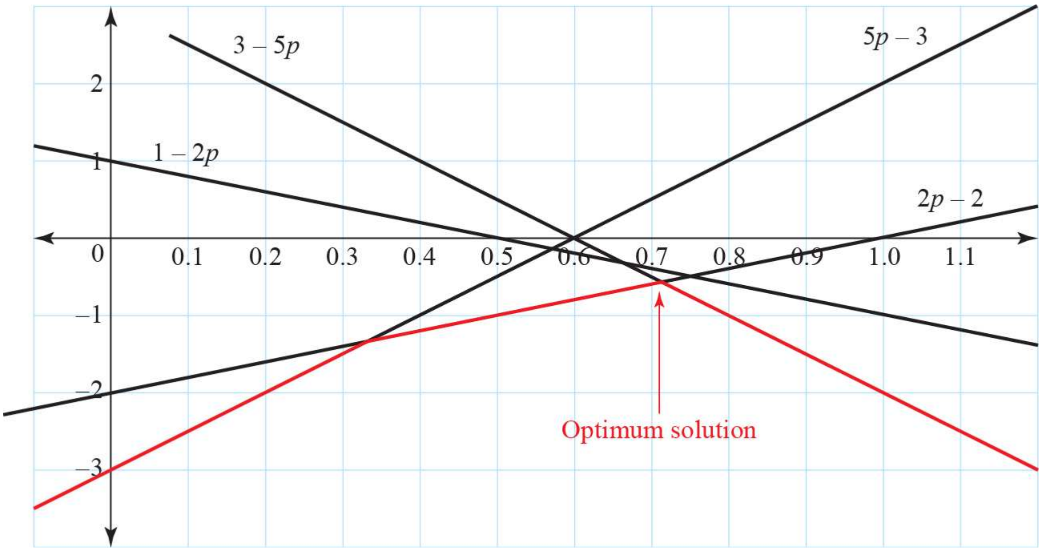
If B plays 1 = $-p+(1-p)=1-2p$

If B plays 2 = $0-2\cdot(1-p)=2p-2$

If B plays 3 = $-2\cdot p+3\cdot(1-p)=3-5p$

If B plays 4 = $2\cdot p-3\cdot(1-p)=5p-3$

This can be illustrated on a diagram (remember p is a probability, so it takes values between 0 and 1 only!)



We choose the intersection which maximises the minimal pay-off. So we need to solve:

$2p-2=3-5p$

$7p=5$

$p=\frac{5}{7}$

So A should play 1 with probability $\frac{5}{7}$ and 2 with probability $\frac{2}{7}$. The value of the game for player A is equal to the expected pay-off at this optimal solution, i.e. $2\cdot\frac{5}{7}-2=-\frac{4}{7}$

6 a By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –2 and the column minimax = 0, so there is no stable solution.

6 b Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

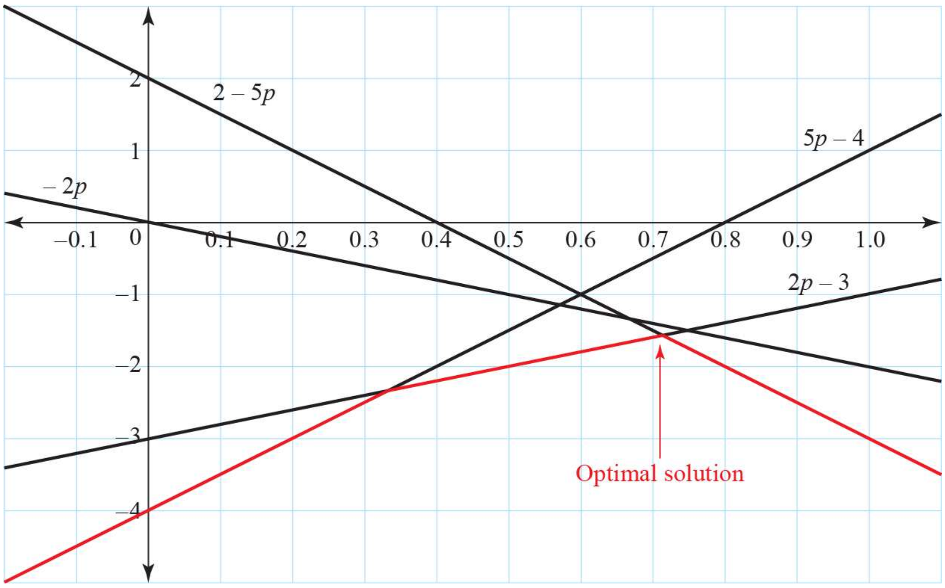
If B plays 1 = $-2 \cdot p + 0 = -2p$

If B plays 2 = $-p - 3 \cdot (1 - p) = 2p - 3$

If B plays 3 = $-3 \cdot p + 2 \cdot (1 - p) = 2 - 5p$

If B plays 4 = $p - 4 \cdot (1 - p) = 5p - 4$

This can be illustrated on a diagram (remember p is a probability, so it takes values between 0 and 1 only!)



We choose the intersection which maximises the minimal pay-off. So we need to solve

$2p - 3 = 2 - 5p$

$7p = 5$

$p = \frac{5}{7}$

So A should play 1 with probability $\frac{5}{7}$ and 2 with probability $\frac{2}{7}$. The value of the game for player A is then equal to the expected pay-off at that optimal point, i.e. $2 \cdot \frac{5}{7} - 3 = -\frac{11}{7}$

c This is a zero-sum game, so, since the value is $-\frac{11}{7}$ for player A, it must be $\frac{11}{7}$ for player B.

7 a From Amy's perspective, the pay-off matrix looks as follows:

	***B* plays 1**	***B* plays 2**
***A* plays 1**	2	–3
***A* plays 2**	–3	4

7 b By the stable solution theorem we know that the row maximin = column minimax. In this case the row maximin = –3 and the column minimax = 2, so there is no stable solution. Now, assume *A* plays 1 with probability *p* and 2 with probability 1 – *p*. Then the expected winnings are:

If *B* plays 1 = $2\cdot p-3\cdot(1-p)=5p-3$

If *B* plays 2 = $-3p+4\cdot(1-p)=4-7p$

This can be illustrated on a diagram (remember *p* is a probability, so it takes values between 0 and 1 only!)

We choose the intersection which maximises the minimal pay-off. So we need to solve:

$5p-3=4-7p$

$12p=7$

$p=\frac{7}{12}$

So *A* should play 1 with probability $\frac{7}{12}$ and 2 with probability $\frac{5}{12}$.

c The value of the game for Amy is equal to the expected pay-off at the optimal solution, i.e. $5\cdot\frac{7}{12}-3=-\frac{1}{12}$. This indeed suggests that the game is unfair and even with optimal strategy, Amy loses on average about £0.08 (8p) each game. This, however, means that with Amy's suggestion the game will be slightly biased in Amy's favour (the value will be about 2p, which means that the value for Barun will be –2p).

8 a By the stable solution theorem we know that the stable solution exists if row maximin = column minimax. In this case the row maximin = 1 and the column minimax = 4, so there is no stable solution.

b We notice that row 1 dominates row 3, so row 3 can be deleted. The updated pay-off matrix looks as follows:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	6	–2	2
***A* plays 2**	1	4	5

Further inspection shows that column 2 dominates column 3 (remember that when reducing columns we remove the one with the largest values!). So we remove column 3. Final matrix:

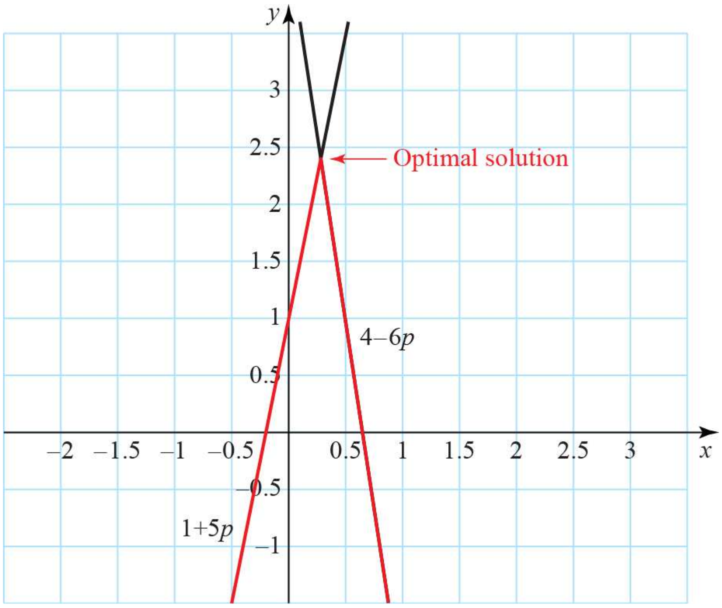
	***B* plays 2**	***B* plays 3**
***A* plays 1**	6	–2
***A* plays 2**	1	4

8 c Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

If B plays 1 = $6p + (1 - p) = 1 + 5p$

If B plays 3 = $-2p + 4(1 - p) = 4 - 6p$

This can be illustrate on a diagram:



We choose the intersection which maximises the minimal pay-off. So we need to solve:

$4 - 6p = 1 + 5p$

$11p = 3$

$p = \frac{3}{11}$

So A should play 1 with probability $\frac{3}{11}$ and 2 with probability $\frac{8}{11}$. The value of the game to A is then equal to the expected pay-off at the optimal solution, i.e. $4 - 6 \cdot \frac{3}{11} = \frac{26}{11} = 2\frac{5}{11}$

9 a A saddle point is the value which is the smallest in its row and largest in its column is called a saddle point.

b Because the pay-off matrix is written out from A's perspective, to find the number of points that B gets we need to multiply all numbers by –1. Thus if A plays 3 and B plays 3, B will get 4 points.

c We can use the domination argument to reduce the pay-off matrix. For columns, it means that we are looking for a column with all values **larger** than the respective values in another column. In our example, we see that all values in column 2 are greater that the respective values in column 1, i.e. 2 is **dominated** by 1. So we can delete column 2. Updated pay-off matrix:

	***B* plays 1**	***B* plays 3**
***A* plays 1**	–4	3
***A* plays 2**	–1	0
***A* plays 3**	1	–4

9 d First, write out the matrix from *B*'s perspective. This helps avoid errors.

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	4	1	–1
***B* plays 3**	–3	0	4

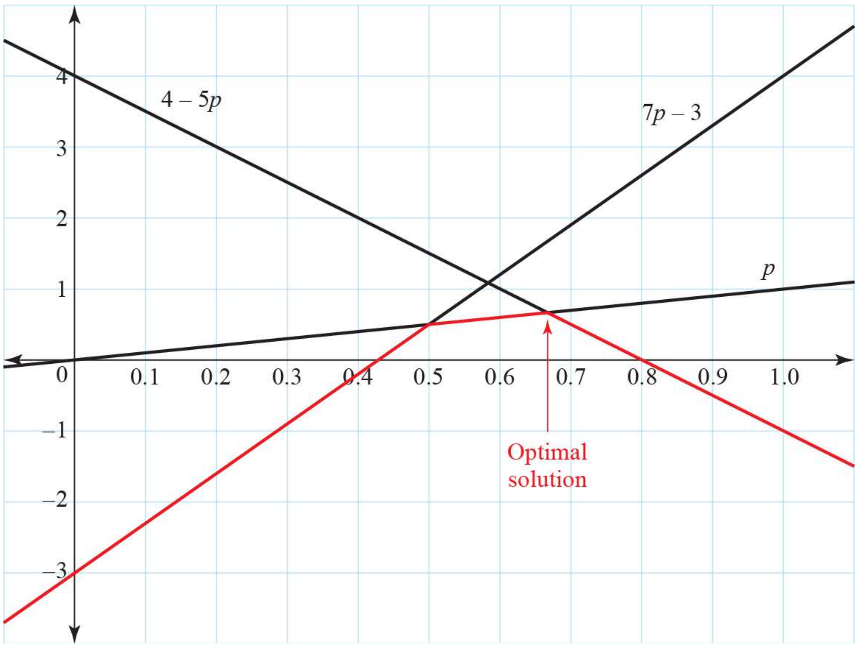
We will look for a mixed strategy. Assume *B* plays 1 with probability *p* and 3 with probability 1 – *p*. Then the expected winnings for *B* are:

If *A* plays 1 = $4\cdot p-3\cdot(1-p)=7p-3$

If *A* plays 2 = $p+0=p$

If *A* plays 3 = $-1\cdot p+4\cdot(1-p)=4-5p$

This can be illustrated on a diagram:



We pick the intersection which maximises the minimal pay-off. So we need to solve:

$p=4-5p$

$6p=4$

$p=\frac{2}{3}$

Thus *B* should play 1 with probability $\frac{2}{3}$ and 3 with probability $\frac{1}{3}$

e The value of the game is then equal to the expected pay-off at this optimal solution, so we substitute $p=\frac{2}{3}$ into the line equation above. So the value of the game is $\frac{2}{3}$

10 To look for the best strategy for player A we begin by reducing the pay-off matrix. We notice that row 1 dominates row 2, so we delete row 2.

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	5	2	3
***A* plays 3**	–1	4	–2

We next notice, that we can also reduce a column – all values in column 3 are smaller than the respective values in column 1. So column 3 **dominates** column 1. Thus we delete column 1.

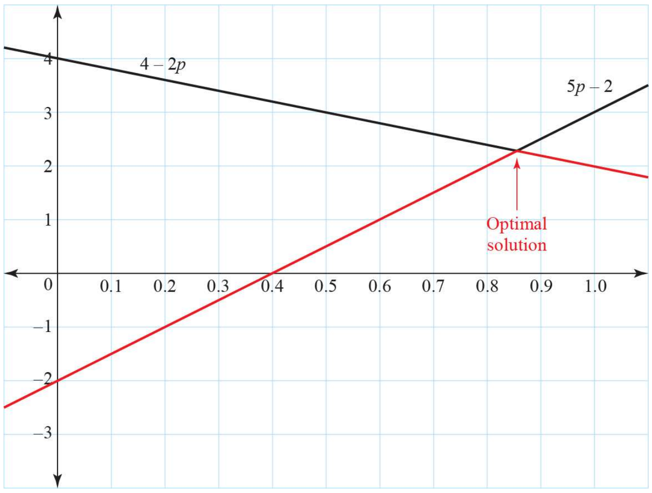
	***B* plays 2**	***B* plays 3**
***A* plays 1**	2	3
***A* plays 3**	4	–2

We are now ready to find the optimal strategy for player A. Assume A plays 1 with probability p and 3 with probability $1-p$. Then the expected winnings are:

If B plays 2 = $2\cdot p+4\cdot(1-p)=4-2p$

If B plays 3 = $3\cdot p-2\cdot(1-p)=5p-2$

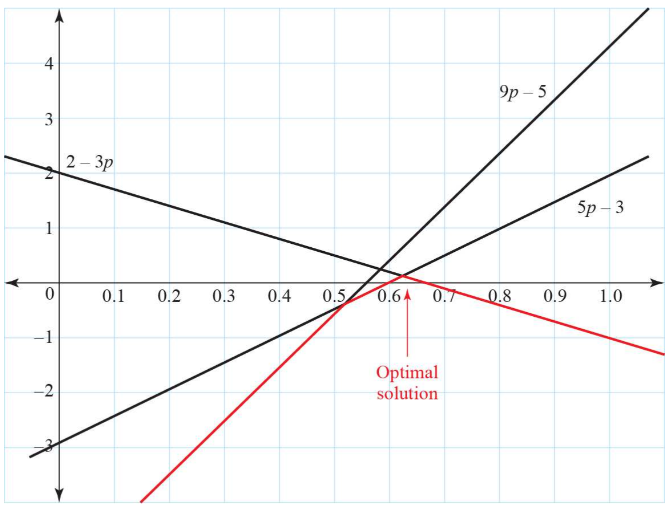
This can be illustrated on a diagram:



So to find the optimal solution we need to solve:

$$4-2p=5p-2$$
$$7p=6$$
$$p=\frac{6}{7}$$

So A should play 1 with probability $\frac{6}{7}$ and 3 with probability $\frac{1}{7}$. The value of the game for A can be computed by substituting p into the expected winnings equation: $4-2\cdot\frac{6}{7}=\frac{16}{7}$



11

	B plays 1	**B plays 2**
A plays 1	–2	3
A plays 2	1	–2
A plays 3	–4	5

We begin by investigating the game further. First, let's reduce the pay-off matrix. Notice that column 2 dominates column 3, so we can remove column 3 as it would never be chosen by Bob.
The matrix cannot be reduced anymore, so we write it out from Bob's perspective. We achieve that by multiplying all numbers by –1.

	A plays 1	**A plays 2**	**A plays 3**
B plays 1	2	–1	4
B plays 2	–3	2	–5

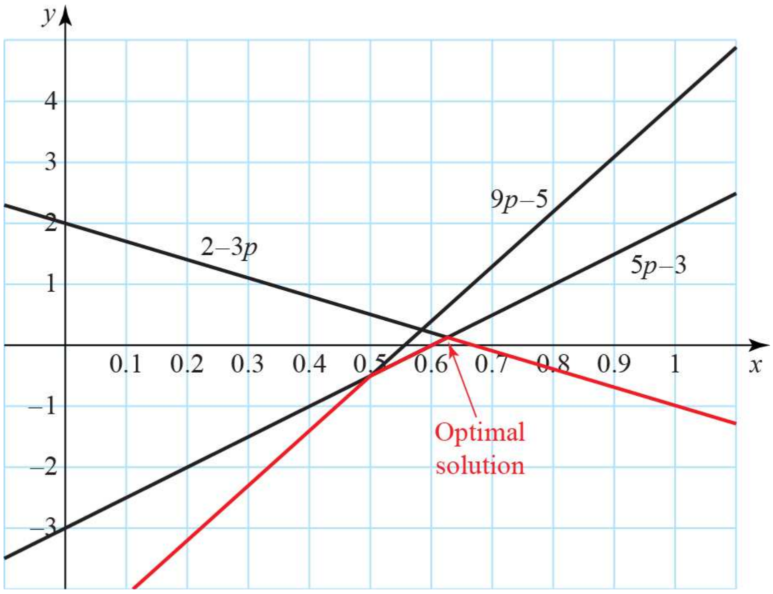
Now we look for the optimal strategy for Bob. Assume he plays 1 with probability p and 2 with probability $1 - p$. Then his expected winnings are:

If Alice plays 1 = $2 \cdot p - 3 \cdot (1 - p) = 5p - 3$

If Alice plays 2 = $-1 \cdot p + 2 \cdot (1 - p) = 2 - 3p$

If Alice plays 3 = $4 \cdot p - 5 \cdot (1 - p) = 9p - 5$

We illustrate this on a diagram:



For the optimal strategy we pick the intersection which maximises Bob's minimal pay-off. Thus we need to solve:

$5p - 3 = 2 - 3p$

$8p = 5$

$p = \frac{5}{8}$

11 (continued)

So Bob should play 1 with probability $\frac{5}{8}$ and 2 with probability $\frac{3}{8}$. This is his optimal strategy. The value of the game for him is then $5 \cdot \frac{5}{8} - 3 = \frac{1}{8}$. Since this is the best strategy Bob can find, he is incorrect about being able to find a strategy with expected winnings of $\frac{1}{2}$

Challenge

To find the value of the game to A, we need to find her optimal strategy. We begin as usual, by assuming that A plays 1 with probability p and 2 with probability $1 - p$. We then proceed to fine the expected pay-offs:

If B plays 1 = $p + 3 \cdot (1 - p) = 3 - 2p$
If B plays 2 = $x \cdot p + 0 = px$

The optimal strategy for A will be the point of intersection of these two lines. Thus we need to solve:

$$3 - 2p = px$$
$$(x + 2) \cdot p = 3$$
$$p = \frac{3}{x + 2}$$

The value of the game to A is then the expected winnings at that point, so value = $p \cdot x = \frac{3x}{x + 2}$, so long as p does not exceed the value of 1, which is true for $1 \leqslant x$

Hence, we need to consider the case when $x < 1$. Revisiting the two player table we see that the minimax = maximin = x so long as $0 \leqslant x \leqslant 1$, and that the minimax = maximin = 0 when $x < 0$

So the value of A is given by:

$$\text{value } = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ \dfrac{3x}{x + 2} & \text{if } x > 1 \end{cases}$$