Game theory Mixed exercise

1

	B plays 1	B plays 2	Row min	
A plays 1	4	–2	–2	←
A plays 2	–5	6	–5	
Column max	4	6		
	↑			

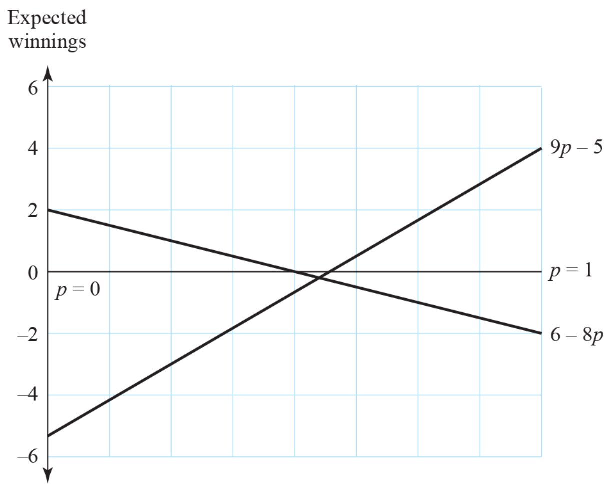
No stable solution since 4 ≠ –2 (column minimax ≠ row maximin)

Let A play 1 with probability p

So A plays 2 with probability $(1 - p)$

If B plays 1 A's expected winnings are $4p - 5(1 - p) = 9p - 5$

If B plays 2 A's expected winnings are $-2p + 6(1 - p) = 6 - 8p$



$$9p - 5 = 6 - 8p$$
$$17p = 11$$
$$p = \frac{11}{17}$$

A should play 1 with probability $\frac{11}{17}$

A should play 2 with probability $\frac{6}{17}$

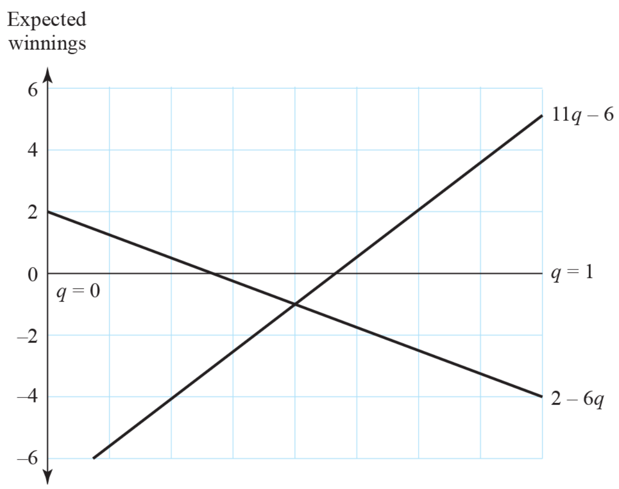
The value of the game to A is $\frac{14}{17}$

Let B play 1 with probability q

Let B play 2 with probability $(1 - q)$

If A plays 1 B's expected winnings are $-[4q - 2(1 - q)] = 2 - 6q$

If A plays 2 B's expected winnings are $-[-5q + 6(1 - q)] = 11q - 6$



$$11q - 6 = 2 - 6q$$
$$17q = 8$$
$$q = \frac{8}{17}$$

B should play 1 with probability $\frac{8}{17}$

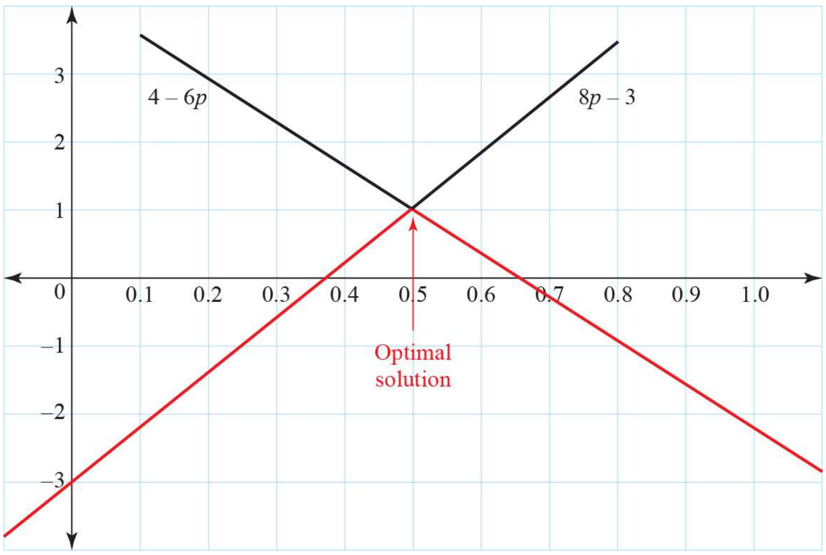
B should play 2 with probability $\frac{9}{17}$

The value of the game to B is $\frac{-14}{17}$

2 a No, Nigel is not correct. We can imagine a game where one person always wins 1 and the other person always loses 1. This is a zero sum game, but the value of this game is not 0.

b By the stable solutions theorem, we know that a game has a stable solution if the row maximin = column minimax. In this example, the row minima are –4, –3 and –5. So the row maximin is –3. The column maxima are 2, –3, 0 and 4. So the column minimax is –3, which is the same as the row maximin. Thus the game has a stable solution.

c The play-safe strategy minimises the potential loss. So for Mariette the play-safe strategy means picking the row which has the maximin value, i.e. Mariette should play 2. When choosing the play-safe strategy for Nigel, we need to remember to look at the negatives of the values in the table! Or, in other words, pick the column with the maximin value, i.e. Nigel should play 2. The value of the game to Mariette is equal to the minimum win when both players play safe, so in this case it is –3.



3 a The play-safe strategy minimises the potential loss. In other words, it looks at the worst possible outcome for each strategy, and then chooses the strategy with the least bad option.

b To find the play safe strategy for player A, we need to find the row with the maximin value. The row minima are –3, –4, –3 and –1. So the maximin value is –1 and A should play 4. To find the play-safe strategy for B, we look for the column minimax. The column maxima are 4, –2 and 3, so the minimax is –2. Thus B should play 2.

c The row maximin (–1) is equal to the column minimax (–1) so there is a stable solution to the game.

d A saddle point in a pay-off matrix is a value which is the smallest in its row and the largest in its column. Based on part **c** we see that the saddle point is –1.

e The value of the game for player A is equal to the minimum win when both players play safe, so in this case it is –1.

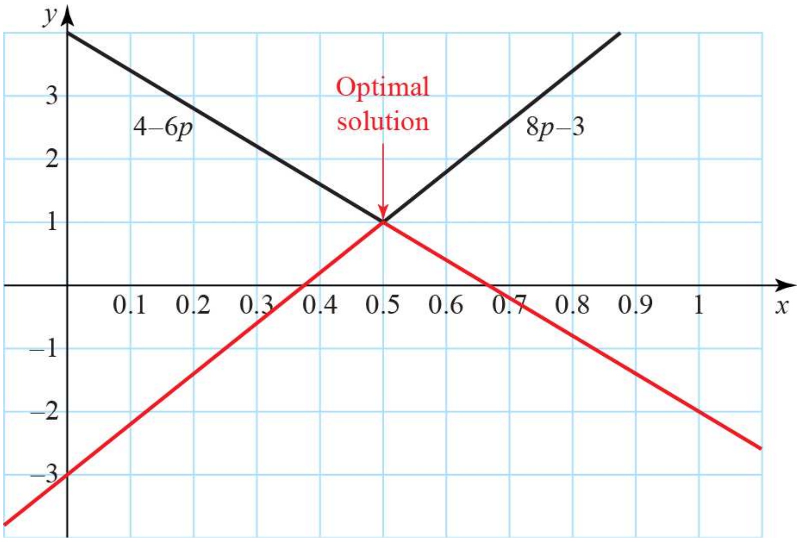
4 a By the stable solution theorem we know that a game has a stable solution if the row maximin = column minimax. In this game, the row minima are –2 and –3, so the row maximin is –2. The column maxima are 5 and 4, so the column minimax is 4. Hence row maximin (–2) ≠ column minimax (4), and so there is no stable solution.

4 b To find the best strategy for Tadashi, assume he plays 1 with probability p and 2 with probability 1 – p. Then his expected winnings are:

If Molly plays 1 = $5p-3(1-p)=8p-3$

If Molly plays 2 = $-2p+4(1-p)=4-6p$

This can be illustrated on a diagram:



The optimal solution is the probability which maximises the minimal pay-off. Thus we need to solve:

$4-6p=8p-3$

$14p=7$

$p=0.5$

Hence Tadashi should play 1 with probability 0.5 and 2 with probability 0.5. The expected winnings when $p=0.5$ are expected winnings = $4-6\cdot 0.5=1$

5 a By the stable solution theorem we know that a game has a stable solution if the row maximin = column minimax. In this game, the row minima are –6 and –1, so the row maximin is –1. The column maxima are 7, 7 and 1, so the column minimax is 1. Hence row maximin (–1) ≠ column minimax (1), and so there is no stable solution.

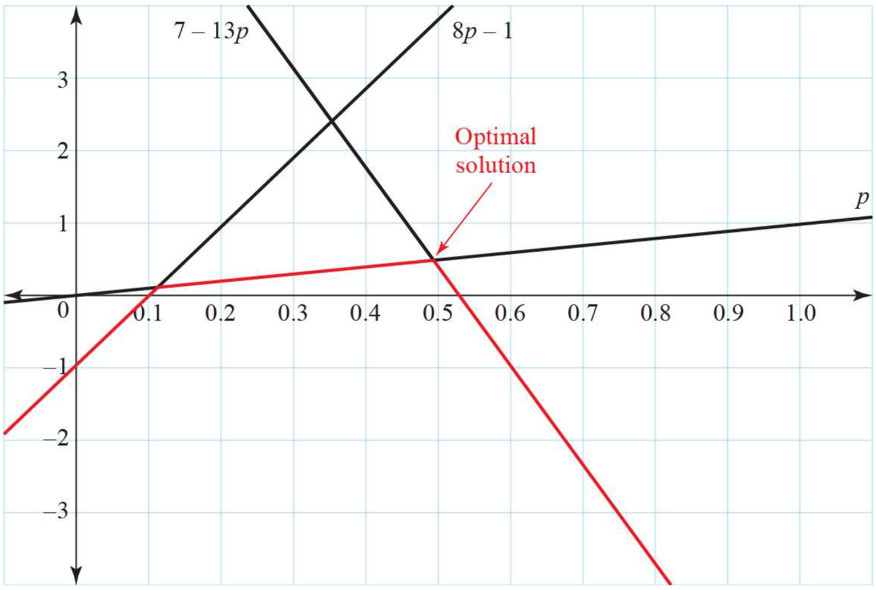
5 b Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

If B plays 1 = $7p-(1-p)=8p-1$

If B plays 2 = $-6p+7(1-p)=7-13p$

If B plays 3 = $p+0=p$.

This can be illustrated on a diagram:



The optimal solution is the probability which maximises the minimum pay-off. So we need to solve:

$p=7-13p$

$14p=7$

$p=0.5$

So A should play 1 with probability 0.5 and 2 with probability 0.5

c The value of the game for player A is equal to the expected winnings when $p=0.5$, i.e. value $= 7-13\cdot 0.5=0.5$

6 a A pure strategy is always the same, i.e. the player's choice never changes. In a mixed strategy, each choice is assigned a probability with which the player picks it.

6 b Assume Olivia plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings can be calculated as follows:

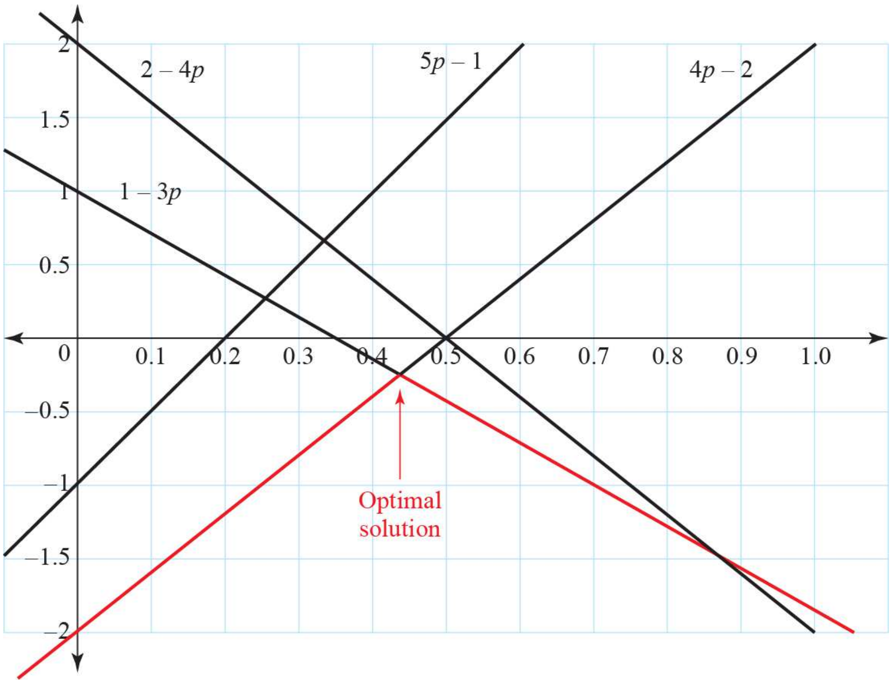
If Jacob plays 1 = $2p-2(1-p)=4p-2$

If Jacob plays 2 = $4p-(1-p)=5p-1$

If Jacob plays 3 = $-2p+2(1-p)=2-4p$

If Jacob plays 4 = $-2p+(1-p)=1-3p$

This can be illustrated on a diagram:



The optimal solution is the probability which maximises the minimum pay-off. So we need to solve:

$4p-2=1-3p$

$7p=3$

$p=\frac{3}{7}$

Thus Olivia should play 1 with probability $\frac{3}{7}$ and 2 with probability $\frac{4}{7}$

c The value of the game for Olivia is equal to her expected pay-off when $p = \frac{3}{7}$

For instance, value = $4 \cdot \frac{3}{7} - 2 = -\frac{2}{7}$

7 a Column 3 dominates column 2 (since 3 < 4 and –4 < –1)

b

	A play 1	A play 2	Row min	
B plays 1	–5	3	–5	
B plays 2	1	–4	–4	←
Col max	1	3		
	↑			

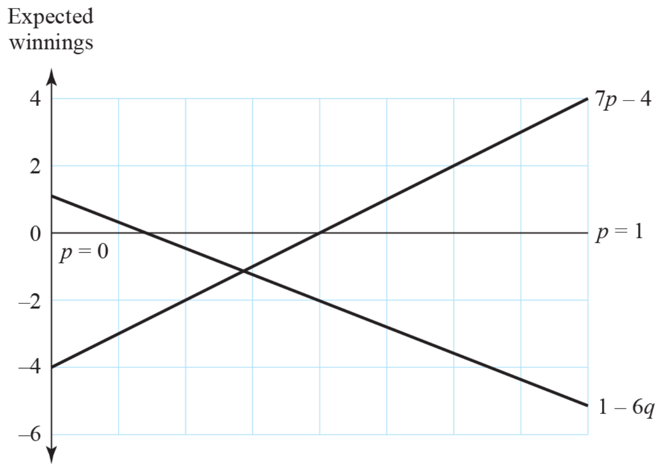
Since 1 ≠ –4 (column minimax ≠ row maximin) game is not stable

Let A play 1 with probability p

So A plays 2 with probability $(1 - p)$

If B plays 1 A's expected winnings are $-5p + 1(1 - p) = 1 - 6p$

If B plays 2 A's expected winnings are $3p - 4(1 - p) = 7p - 4$



$$7p - 4 = 1 - 6p$$
$$13p = 5$$
$$p = \frac{5}{13}$$

A should play 1 with probability $\frac{5}{13}$

A should play 2 with probability $\frac{8}{13}$

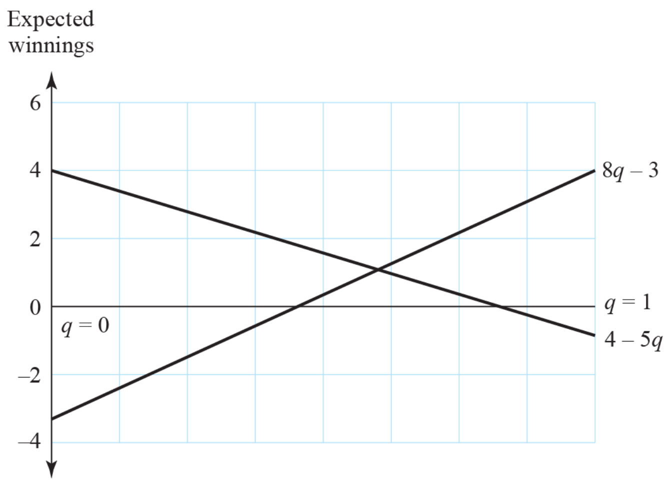
The value of the game is $\frac{-17}{13}$

Let B play 1 with probability q

Let B play 2 with probability $(1 - q)$

If A plays 1 B's expected winnings are $-[-5q + 3(1 - q)] = 8q - 3$

If A plays 2 B's expected winnings are $-[q - 4(1 - q)] = 4 - 5q$



$$8q - 3 = 4 - 5q$$
$$13q = 7$$
$$q = \frac{7}{13}$$

B should play 1 with probability $\frac{7}{13}$

B should play 2 with probability $\frac{6}{13}$

The value of the game is $\frac{17}{13}$

8

	G plays 1	G plays 2	G plays 3	Row min	
C plays 1	–5	2	3	–5	
C plays 2	1	–3	–4	–4	←
C plays 3	–7	0	1	–7	
Column max	1	2	3		
	↑				

a Play safe: Cait plays 2 Georgi plays 1

b 1 ≠ –4 (column minimax ≠ row maximin) so no stable solution

c Row 1 dominates row 3 (since –5 > –7   2 > 0   3 > 1)

	G plays 1	G plays 2	G plays 3
C plays 1	–5	2	3
C plays 2	1	–3	–4

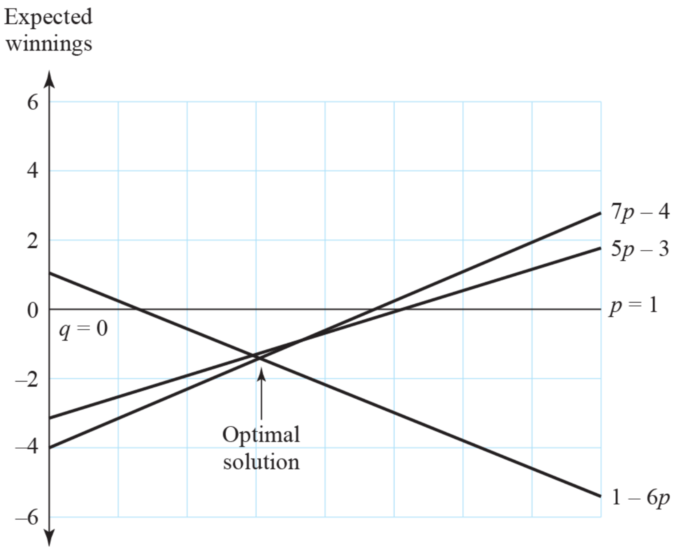
d Let C play 1 with probability p

So C plays 2 with probability $(1 – p)$

If G plays 1 C's expected winnings are $-5p + 1(1-p) = 1-6p$

If G plays 2 C's expected winnings are $2p - 3(1-p) = 5p - 3$

If G plays 3 C's expected winnings are $3p - 4(1-p) = 7p - 4$



$$7p-4=1-6p$$
$$13p=5$$
$$p=\frac{5}{13}$$

Cait should play 1 with probability $\frac{5}{13}$

Cait should play 2 with probability $\frac{8}{13}$

Cait should play 3 never

The value of the game is $\frac{-17}{13}$

e The value of the game to Georgi is $\frac{17}{13}$

9 a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	–1	–3	–3	
A plays 2	–2	1	4	–2	←
A plays 3	–3	1	–3	–3	
A plays 4	–1	2	–2	–2	←
Column max	2	2	4		
	↑	↑			

Since 2 ≠ –2 (column minimax ≠ row maximin) there is no stable solution.

b A row x dominates a row y, if, in each column, the element in row x $\geqslant$ the element in row y.

c Row 4 dominates row 3

	B plays 1	B plays 2	B plays 3
A plays 1	2	–1	–3
A plays 2	–2	1	4
A plays 3	–1	2	–2

d Add 4 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	6	3	1
A plays 2	2	5	8
A plays 3	3	6	2

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v so $V = v + 4$

Maximise $P = V$

Subject to:

$$6p_1 + 2p_2 + 3p_3 \geqslant V$$
$$3p_1 + 5p_2 + 6p_2 \geqslant V$$
$$p_1 + 8p_2 + 2p_3 \geqslant V$$
$$p_1 + p_2 + p_3 \leqslant 1$$

10

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	5	–3	1	–3	
A plays 2	–1	–4	4	–4	
A plays 3	3	2	–1	–1	←
Column max	5	2	4		
		↑			

a Play safe (A plays 1, B plays 2)

b Since 2 ≠ –1 (column minimax ≠ row maximin) there is no stable solution

c Column 2 dominates column 1 (–3 < 5, –4 < –1, 2 < 3) B would always choose to minimise A's winnings by playing 2 rather than 1

	B plays 2	B plays 3
A plays 1	–3	1
A plays 2	–4	4
A plays 3	2	–1

d

	A plays 1	A plays 2	A plays 3
B plays 2	3	4	–2
B plays 3	–1	–4	1

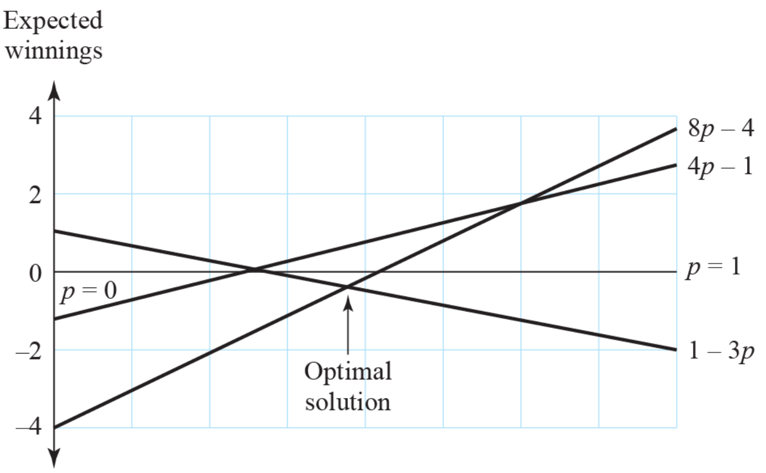
e Let B play 2 with probability p

So B plays 3 with probability $(1 – p)$

If A plays 1 B's expected winnings are $3p - 1(1 - p) = 4p - 1$

If A plays 2 B's expected winnings are $4p - 4(1 - p) = 8p - 4$

If A plays 3 B's expected winnings are $-2p + 1(1 - p) = 1 - 3p$



$$8p - 4 = 1 - 3p$$
$$11p = 5$$
$$p = \frac{5}{11}$$

B should play 2 with probability $\frac{5}{11}$

B should play 3 with probability $\frac{6}{11}$

The value of the game is $\frac{-4}{11}$

11

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	7	–1	–1	
A plays 2	5	0	8	0	←
A plays 3	–2	3	5	–2	
Column max	5	7	8		
	↑				

a Play safe is (A plays 2, B plays 1)

b Since 5 ≠ 0 (column minimax ≠ row maximin) there is no stable solution

c

	A plays 1	A plays 2	A plays 3
B plays 1	–2	–5	2
B plays 2	–7	0	–3
B plays 3	1	–8	–5

d Adding 9 to all elements

	A plays 1	A plays 2	A plays 3
B plays 1	7	4	11
B plays 2	2	9	6
B plays 3	10	1	4

Let B play 1 with probability p_1, play 2 with probability p_2 and play 3 with probability p_3.

Let v = value of the game to B and $V = v + 9$
Maximise $P = V$
Subject to:

$$7p_1 + 2p_2 + 10p_3 \geqslant \text{V} \Rightarrow \text{V} - 7p_1 - 2p_2 - 10p_3 + \text{r} = 0$$
$$4p_1 + 9p_2 + p_3 \geqslant V \Rightarrow \text{V} - 4p_1 - 9p_2 - p_3 + \text{s} = 0$$
$$11p_1 + 6p_2 + 4p_3 \geqslant \text{V} \Rightarrow \text{V} - 11p_1 - 6p_2 - 4p_3 + \text{t} = 0$$
$$p_1 + p_2 + p_3 \leqslant 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$
$$\text{where } p_1,\ p_2,\ p_3,\ r,\ s,\ t,\ u \geqslant 0$$

11 e

b.v.	V	P_1	P_2	P_3	r	s	t	u	value
r	1	–7	–2	–10	1	0	0	0	0
s	1	–4	–9	–1	0	1	0	0	0
t	1	–11	–6	–4	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	–1	0	0	0	0	0	0	0	0

Challenge

a Begin with determining the best strategy for player A. Assume A plays 1 with probability p and 2 with probability $1 - p$. Then the expected winnings are:

If B plays 1 = $pa + c(1-p) = c + p(a-c)$

If B plays 2 = $pb + d(1-p) = d + p(b-d)$

The optimal solution is going to be the intersection of these two lines, so we need to solve:

$$c + p(a-c) = d + p(b-d)$$
$$p(a+d-b-c) = d-c$$
$$p = \frac{d-c}{a+d-b-c}$$

So A should play 1 with probability $\frac{d-c}{a+d-b-c}$ and 2 with probability $\frac{a-b}{a+d-b-c}$. For player B we begin be writing out the pay-off matrix from B's perspective.

	***A* plays 1**	***A* plays 2**
***B* plays 1**	$-a$	$-c$
***B* plays 2**	$-b$	$-d$

Now assume B plays 1 with probability q and 2 with probability $1 - q$. Then the expected winnings are:

If A plays 1 = $-qa - b(1-q) = q(b-a) - b$

If A plays 2 = $-qc - d(1-q) = q(d-c) - d$

The optimal solution is the intersection of these two lines, so we need to solve:

$$q(b-a) - b = q(d-c) - d$$
$$q(b+c-a-d) = b-d$$
$$q = \frac{b-d}{b+c-a-d}$$

So B should play 1 with probability $\frac{b-d}{b+c-a-d}$ and 2 with probability $\frac{c-a}{b+c-a-d}$

Challenge

b The value of the game for player A is the expected winnings when $p = \dfrac{d-c}{a+d-b-c}$:

$$\begin{aligned}\text{value}_A &= c + (a-c)\cdot\frac{d-c}{a+d-b-c}\\ &= \frac{ac+cd-bc-c^2+ad-ac-cd-c^2}{a+d-b-c}\\ &= \frac{ad-bc}{a+d-b-c}\end{aligned}$$

The value of the game for player B is the expected winnings when $q = \dfrac{b-d}{b+c-a-d}$:

$$\begin{aligned}\text{value}_B &= \frac{b-d}{b+c-a-d}\cdot(b-a)-b\\ &= \frac{b^2-ab-bd+ad-b^2-bc+ab+bd}{b+c-a-d}\\ &= \frac{ad-bc}{b+c-a-d}\end{aligned}$$

c Note that:

$$\begin{aligned}-\text{value}_B &= -\frac{ad-bc}{b+c-a-d} = \frac{ad-bc}{a+d-b-c}\\ &= \text{value}_A\end{aligned}$$

As required.