Review exercise 1

1 a Applying the north-west corner method gives:

	W_1	W_2	W_3	Supply
F_1	2	2		4
F_2		3		3
F_3		4	4	8
Demand	2	9	4	15

$$\text{Cost} = 2\times7+2\times8+3\times2+4\times6+4\times3=£72$$

b Set $S(F_1) = 0$ and so solve the remaining equations.

$D(W_1) = 7 - S(F_1) = 7 - 0 = 7$ $\quad$ $D(W_2) = 8 - S(F_1) = 8 - 0 = 8$

$S(F_2) = 2 - D(W_2) = 2 - 8 = -6$ $\quad$ $S(F_3) = 6 - D(W_2) = 6 - 8 = -2$

$D(W_3) = 3 - S(F_3) = 3 - (-2) = 5$

So, in table form, the shadow costs are:

		7	8	5
		W_1	W_2	W_3
0	F_1	7	8	
–6	F_2		2	
–2	F_3		6	3

The improvement indices are:

$I_{F1W3} = C(F_1W_3) - S(F_1) - D(W_3) = 6 - 0 - 5 = 1$

$I_{F2W1} = C(F_2W_1) - S(F_2) - D(W_1) = 9 - (-6) - 7 = 9 + 6 - 7 = 8$

$I_{F2W3} = C(F_2W_3) - S(F_2) - D(W_3) = 4 - (-6) - 5 = 4 + 6 - 5 = 5$

$I_{F3W1} = C(F_3W_1) - S(F_3) - D(W_1) = 5 - (-2) - 7 = 5 + 2 - 7 = 0$

The table below shows the improvement indices (the routes already in use are marked with a cross):

		7	8	5
		W_1	W_2	W_3
0	F_1	×	×	1
–6	F_2	8	×	5
–2	F_3	0	×	×

c As there are no negative improvement indices, the solution found in part **a** is optimal and gives the minimum cost.

2 a The total supply (9 + 13 + 12 = 34) is greater than the total demand (9 + 11 = 20). The problem is unbalanced. A dummy demand point with demand for 14 units is required to absorb the excess supply and balance the problem.

b There are 3 rows and 3 columns, so a non-degenerate solution would use 3 + 3 – 1 = 5 cells. If there is no zero in cell AK, the solution uses less than 5 cells and is degenerate. The transportation algorithm requires that solutions are non-degenerate, so a zero must be placed in the cell.

c Finding the shadow costs of the solution found after three iterations gives:

		10	15	0
		J	***K***	***L***
0	***A***		15	0
0	***B***			0
–6	***C***	4	9	

The improvement indices are:

$I_{AJ} = 12 - 0 - 10 = 2$ $\quad$ $I_{BK} = 17 - 0 - 15 = 2$

$I_{BJ} = 8 - 0 - 10 = -2$ $\quad$ $I_{CL} = 0 - (-6) - 0 = 6$

The new entering cell is BJ. Entering θ into cell BJ and applying the stepping-stone method gives:

	J	***K***	***L***	**Supply**
A		$8 - \theta$	$1 + \theta$	9
B	θ		$13 - \theta$	13
C	$9 - \theta$	$3 + \theta$		12
Demand	9	11	14	34

The largest possible value of θ is 8, making AK the exiting cell, and giving this improved solution.

	J	***K***	***L***	**Supply**
A			9	9
B	8		5	13
C	1	11		12
Demand	9	11	14	34

Finding the shadow costs of the new solution gives:

		8	13	0
		J	***K***	***L***
0	***A***			0
0	***B***	8		0
–4	***C***	4	9	

2 c (continued)

The improvement indices are:

$I_{AJ} = 12 - 0 - 8 = 4$ $\quad$ $I_{BK} = 17 - 0 - 13 = 4$

$I_{AK} = 15 - 0 - 13 = 2$ $\quad$ $I_{CL} = 0 - (-4) - 0 = 4$

There are no negative improvement indices, so the solution is optimal.

3 a The problem is balanced, so applying the north-west corner method gives:

	D	***E***	***F***	**Supply**
A	20	4		24
B		26	6	32
C			14	14
Demand	20	30	20	80

b Set S(A) = 0 and so solve the remaining equations.

D(D) = 21 − S(A) = 21 − 0 = 21 $\quad$ D(E) = 24 − S(A) = 24 − 0 = 24
S(B) = 23 − D(E) = 23 − 24 = −1 $\quad$ D(F) = 17 − S(B) = 17 − (−1) = 18
S(C) = 25 − D(F) = 25 − 18 = 7

So, in table form, the shadow costs are:

		21	24	18
		D	***E***	***F***
0	***A***	21	24	
−1	***B***		23	17
7	***C***			25

The improvement indices for each unused route are:

$I_{AF} = 16 - 0 - 18 = -2$ $\quad$ $I_{CD} = 15 - 7 - 21 = -13$

$I_{BD} = 18 - (-1) - 21 = -2$ $\quad$ $I_{CE} = 19 - 7 - 24 = -12$

3 c The entering cell is CD. Entering θ into cell CD and applying the stepping-stone method gives:

	D	E	F	**Supply**
A	$20 - \theta$	$4 + \theta$		24
B		$26 - \theta$	$6 + \theta$	32
C	θ		$14 - \theta$	14
Demand	20	30	20	34

The largest possible value of θ is 14, making CF the exiting cell, and giving this improved solution.

	D	E	F	**Supply**
A	6	18		24
B		12	20	32
C	14			14
Demand	20	30	20	80

Cost = $6\times21+14\times15+18\times24+12\times23+20\times17=£1384$

4 a The problem is balanced, so applying the north-west corner method gives:

	D	E	F	**Cars available**
A	6			6
B	0	5		5
C		4	4	8
Cars required	6	9	4	19

Cost = $6\times20+5\times30+4\times20+4\times30=£470$

Note that one cell must have a zero entry to make the solution non-degenerate. As there are 3 rows and 3 columns, the solution should use 3 + 3 – 1 = 5 cells. If there is no zero in cell BD, the solution uses less than 5 cells and is degenerate. However, the zero could be in any other of the blanks cells; for example, this is also a valid solution:

	D	E	F	**Cars available**
A	6	0		6
B		5		5
C		4	4	8
Cars required	6	9	4	19

4 b Using the first solution shown in part **a**, set S(A) = 0 and so solve the remaining equations.

D(D) = 20 − S(A) = 20 − 0 = 20
S(B) = 20 − D(D) = 20 − 20 = 0
D(E) = 30 − S(B) = 30 − 0 = 30
S(C) = 20 − D(E) = 20 − 30 = –10
D(F) = 30 − S(C) = 30 – (–10) = 40

So, in table form, the shadow costs are:

		20	30	40
		D	***E***	***F***
0	***A***	20		
0	***B***	20	30	
–10	***C***		20	30

The improvement indices for each unused route are:

$I_{AE} = 40 - 0 - 30 = 10$ $\quad I_{BF} = 40 - 0 - 40 = 0$

$I_{AF} = 10 - 0 - 40 = -30$ $\quad I_{CD} = 10 - (-10) - 20 = 0$

c The entering cell is AF. Entering θ into cell AF and applying the stepping-stone method gives:

	D	***E***	***F***	**Cars available**
A	$6-\theta$		θ	6
B	θ	$5-\theta$		5
C		$4+\theta$	$4-\theta$	8
Cars required	6	9	4	19

The largest possible value of θ is 4, making CF the exiting cell, and giving this improved solution.

	D	***E***	***F***	**Cars available**
A	2		4	6
B	4	1		5
C		8		8
Cars required	6	9	4	19

The new shadow costs are:

		20	30	10
		D	***E***	***F***
0	***A***	20		10
0	***B***	20	30	
–10	***C***		20	

4 c (continued)

The improvement indices are:

$I_{AE} = 40 - 0 - 30 = 10$ $\quad$ $I_{CD} = 10 - (-10) - 20 = 0$

$I_{BF} = 40 - 0 - 10 = 30$ $\quad$ $I_{CF} = 30 - (-10) - 10 = 30$

There are no negative improvement indices, so the solution is optimal.
Cost = $2 \times 20 + 4 \times 20 + 1 \times 30 + 8 \times 20 + 4 \times 10 = £350$

Note that a different optimal solution can be obtained if the second solution in part **a** is used as an initial solution. This is summarised on the next page.

Choosing the second solution in part **a** as the initial solution:

	D	***E***	***F***	**Cars available**
A	6	0		6
B		5		5
C		4	4	8
Cars required	6	9	4	19

The shadow costs and improvement indices are:

		20	40	50
		D	***E***	***F***
0	***A***	×	×	–40
–10	***B***	10	×	0
–20	***C***	10	×	×

The entering cell is AF. Entering θ into cell AF and applying the stepping-stone method gives:

	D	***E***	***F***	**Cars available**
A	6	$0 - \theta$	θ	6
B		5		5
C		$4 + \theta$	$4 - \theta$	8
Cars required	6	9	4	19

The largest possible value of θ is 0, making AE the exiting cell, and giving this improved solution.

	D	***E***	***F***	**Cars available**
A	6		0	6
B		5		5
C		4	4	8
Cars required	6	9	4	19

4 c (continued)

The new shadow costs and improvement indices are:

		20	0	10
		D	***E***	***F***
0	***A***	×	40	×
30	***B***	–30	×	0
20	***C***	–30	×	×

Cells *BD* and *CD* both have an improvement index of –30.
Choosing *CD* as the entering cell and applying the stepping-stone method gives:

	D	***E***	***F***	**Cars available**
A	$6-\theta$		$0+\theta$	6
B		5		5
C	θ	4	$4-\theta$	8
Cars required	6	9	4	19

The largest possible value of θ is 4, making *CF* the exiting cell, and giving this improved solution.

	D	***E***	***F***	**Cars available**
A	2		4	6
B		5		5
C	4	4		8
Cars required	6	9	4	19

The new shadow costs and improvement indices are:

		20	30	10
		D	***E***	***F***
0	***A***	×	10	×
0	***B***	0	×	30
–10	***C***	×	×	30

There are no negative improvement indices, so this is also an optimal solution.
Cost = $2\times20+4\times10+5\times30+4\times10+4\times20=£350$

4 c (continued)

Note that if *BD* had been chosen as the entering cell (top of this page), then this optimal solution would be obtained:

	D	*E*	*F*	Cars available
A	2		4	6
B	4	1		5
C		8		8
Cars required	6	9	4	19

This is the solution obtained by choosing the first initial solution presented for part **a**.

5 a The problem is balanced as the total supply (35 + 25 + 15 = 75 units) equals the total demand (20 + 25 + 30 = 75 units). Therefore, there is no need to add any dummy supply or demand points. Using the north-west corner method to obtain an initial solution gives:

	B_1	B_2	B_3	Supply
F_1	20	15		35
F_2		10	15	25
F_3			15	15
Demand	20	25	30	

All the stock has been used and all the demand met.
Number of occupied cells (5) = number of supply points (3) + number of demand points (3) – 1
So the solution is non-degenerate.
Total cost of this solution = $20\times10+15\times4+10\times5+15\times8+15\times7=£535$

b Setting $S(F_1)=0$ and solving the remaining equations to find the shadow costs gives:

$D(B_1)=10-S(F_1)=10-0=10$ $\quad$ $D(B_2)=4-S(F_1)=4-0=4$
$S(F_2)=5-D(B_2)=5-4=1$ $\quad$ $D(B_3)=8-S(F_2)=8-1=7$
$S(F_3)=7-D(B_3)=7-7=0$

So, in table form, the shadow costs are:

		10	4	7
		B_1	B_2	B_3
0	F_1	10	4	
1	F_2		5	8
0	F_3			7

The improvement indices are:
$I_{F1B3}=C(F_1B_3)-S(F_1)-D(B_3)=11-0-7=4$
$I_{F2B1}=C(F_2B_1)-S(F_2)-D(B_1)=12-1-10=1$
$I_{F3B1}=C(F_3B_1)-S(F_3)-D(B_1)=9-0-10=-1$
$I_{F3B2}=C(F_3B_2)-S(F_3)-D(B_2)=6-0-4=2$
As I_{F3B1} is negative, the solution is not optimal.

5 b (continued)

The improvement indices can be presented in table form, with the routes already in use marked with a cross.

		10	4	7
		B_1	B_2	B_3
0	F_1	×	×	4
1	F_2	1	×	×
0	F_3	−1	2	×

c The entering cell is F_3B_1. Entering θ into cell F_3B_1 and applying the stepping-stone method gives:

	B_1	B_2	B_3	Supply
F_1	$20-\theta$	$15+\theta$		35
F_2		$10-\theta$	$15+\theta$	25
F_3	θ		$15-\theta$	15
Demand	20	25	30	75

The largest possible value of θ is 10, making F_2B_2 the exiting cell, and giving this new solution.

	B_1	B_2	B_3	Supply
F_1	10	25		35
F_2			25	25
F_3	10		5	15
Demand	20	25	30	

d The new shadow costs are:

		10	4	8
		B_1	B_2	B_3
0	F_1	10	4	
0	F_2			8
−1	F_3	9		7

The improvement indices are:

$I_{F1B3} = C(F_1B_3) - S(F_1) - D(B_3) = 11 - 0 - 8 = 3$
$I_{F2B1} = C(F_2B_1) - S(F_2) - D(B_1) = 12 - 0 - 10 = 2$
$I_{F2B2} = C(F_2B_2) - S(F_2) - D(B_2) = 5 - 0 - 4 = 1$
$I_{F3B2} = C(F_3B_2) - S(F_3) - D(B_2) = 6 - (-1) - 4 = 3$

There are no negative improvement indices, so this solution is optimal.
Total cost of this solution $= 10\times10 + 25\times4 + 25\times8 + 10\times9 + 5\times7 = 525$ units

6 a The transportation algorithm could be used to solve a problem involving many supply and demand points, where there are many units to be moved, the costs are variable and dependent upon the supply and demand points, and the aim is to minimise costs.

b Supply = 45 + 35 + 40 = 120 Demand = 50 + 60 = 110

So total supply ≠ total demand and the problem is not balanced. It is therefore necessary to add a dummy demand point, *F*, to absorb the excess supply. The problem becomes:

	D	***E***	***F***	**Supply**
A	5	3	0	45
B	4	6	0	35
C	2	4	0	40
Demand	50	60	10	120

c Using the north-west corner method to obtain an initial solution gives:

	D	***E***	***F***	**Supply**
A	45			45
B	5	30		35
C		30	10	40
Demand	50	60	10	120

Total cost of this solution = $45\times5+5\times4+30\times6+30\times4=545$

d The shadow costs are:

		5	7	3
		D	***E***	***F***
0	***A***	5		
–1	***B***	4	6	
–3	***C***		4	0

The improvement indices are:

I_{AE} = C(*AE*) − S(*A* − D(*E*) = 3 − 0 − 7 = –4

I_{AF} = C(*AF*) − S(*A*) − D(*F*) = 0 − 0 − 3 = –3

I_{BF} = C(*BF*) − S(*B*) − D(*F*) = 0 − (–1) − 3 = –2

I_{CD} = C(*CD*) − S(C_3) − D(*D*) = 2 − (–3) − 5 = 0

6 e The entering cell is *AE*. Entering θ into cell *AE* and applying the stepping-stone method gives:

	D	***E***	***F***	**Supply**
A	$45-\theta$	θ		45
B	$5+\theta$	$30-\theta$		35
C		30	10	40
Demand	50	60	10	120

The largest possible value of θ is 30, making *BE* the exiting cell, and giving this new solution.

	D	***E***	***F***	**Supply**
A	15	30		45
B	35			35
C		30	10	40
Demand	50	60	10	120

Total cost of this solution = $15\times5+30\times3+35\times4+30\times4=425$

7 a The problem is balanced as the total supply (6 + 7 + 5 + 8 = 26 units) equals the total demand (7 + 5 + 6 + 8 = 26 units). Therefore, there is no need to add any dummy supply or demand points.

Using north-west corner method to obtain an initial solution gives:

	P	***Q***	***R***	***S***	**Supply**
A	6				6
B	1	5	1		7
C			5	0	5
D				8	8
Demand	7	5	6	8	26

Total cost = $6\times5+1\times9+5\times6+1\times7+5\times7+8\times8=£175$

Note that one cell must have a zero entry to make the solution non-degenerate. As there are 4 rows and 4 columns, the solution should use 4 + 4 – 1 = 7 cells. If there is no zero in cell *CS*, the solution uses less than 7 cells and is degenerate. However, the zero could be in any other of the blanks cells; for example, this is also a valid solution:

	P	***Q***	***R***	***S***	**Supply**
A	6				6
B	1	5	1		7
C			5		5
D			0	8	8
Demand	7	5	6	8	26

7 a (continued)

The worked answer for part **b** uses the first initial solution above, with the zero in cell CS. It may be possible to find other optimal solutions working from a different valid initial solution, such as the second table shown above, with a zero in cell DR.

b Setting S(A) = 0 and solving the remaining equations to get the shadow costs gives:

$D(P) = 5 - S(A) = 5 - 0 = 5$
$S(B) = 9 - D(P) = 9 - 5 = 4$
$D(Q) = 6 - S(B) = 6 - 4 = 2$
$D(R) = 7 - S(B) = 7 - 4 = 3$
$S(C) = 7 - D(R) = 7 - 3 = 4$
$D(S) = 9 - S(C) = 9 - 4 = 5$
$S(D) = 8 - D(S) = 8 - 5 = 3$

In table form, the shadow costs are:

		5	2	3	5
		P	***Q***	***R***	***S***
0	***A***	5			
4	***B***	9	6	7	
4	***C***			7	9
3	***D***				8

The improvement indices are:

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 2 = 4$
$I_{AR} = C(AR) - S(A) - D(R) = 8 - 0 - 3 = 5$
$I_{AS} = C(AS) - S(A) - D(S) = 10 - 0 - 5 = 5$
$I_{BS} = C(BS) - S(B) - D(S) = 11 - 4 - 5 = 2$
$I_{CP} = C(CP) - S(C) - D(P) = 8 - 4 - 5 = -1$
$I_{CQ} = C(CQ) - S(C) - D(Q) = 9 - 4 - 2 = 3$
$I_{DP} = C(DP) - S(D) - D(P) = 10 - 3 - 5 = 2$
$I_{DQ} = C(DQ) - S(D) - D(Q) = 11 - 3 - 2 = 6$
$I_{DR} = C(DR) - S(D) - D(R) = 8 - 3 - 3 = 2$

As I_{CP} is negative, the solution is not optimal.
The entering cell is CP. Entering θ into cell CP and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A	6				6
B	$1 - \theta$	5	$1 + \theta$		7
C	θ		$5 - \theta$	0	5
D				8	8
Demand	7	5	6	8	26

The largest possible value of θ is 1, making BP the exiting cell, and giving this new solution.

	P	***Q***	***R***	***S***	**Supply**
A	6				6
B		5	2		7
C	1		4	0	5
D				8	8
Demand	7	5	6	8	26

7 b (continued)

The new shadow costs are:

		5	3	4	6
		P	***Q***	***R***	***S***
0	***A***	5			
3	***B***		6	7	
3	***C***	8		7	9
2	***D***				8

The improvement indices are:

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 3 = 3$

$I_{AS} = C(AS) - S(A) - D(S) = 10 - 0 - 6 = 4$

$I_{BS} = C(BS) - S(B) - D(S) = 11 - 3 - 6 = 2$

$I_{DP} = C(DP) - S(D) - D(P) = 10 - 2 - 5 = 3$

$I_{DR} = C(DR) - S(D) - D(R) = 8 - 2 - 4 = 2$

$I_{AR} = C(AR) - S(A) - D(R) = 8 - 0 - 4 = 4$

$I_{BP} = C(BP) - S(B) - D(P) = 9 - 3 - 5 = 1$

$I_{CQ} = C(CQ) - S(C) - D(Q) = 9 - 3 - 3 = 3$

$I_{DQ} = C(DQ) - S(D) - D(Q) = 11 - 3 - 2 = 6$

There are no negative improvement indices, so this solution is optimal.
Total cost = $6\times5+1\times8+5\times6+2\times7+4\times7+8\times8=$ £174

8 a x_{AD} number of coaches moved from A to D
x_{AE} number of coaches moved from A to E
x_{AF} number of coaches moved from A to F
x_{BD} number of coaches moved from B to D
x_{BE} number of coaches moved from B to E
x_{BF} number of coaches moved from B to F
x_{CD} number of coaches moved from C to D
x_{CE} number of coaches moved from C to E
x_{CF} number of coaches moved from C to F

This can be expressed more simply as x_{ij} is the number of coaches moved from i to j where

$i \in \{A, B, C\}$
$j \in \{D, E, F\}$
$x_{ij} \geq 0$

b Minimise distance $S = 40x_{AD} + 70x_{AE} + 25x_{AF} + 20x_{BD} + 40x_{BE} + 10x_{BF} + 35x_{CD} + 85x_{CE} + 15x_{CF}$

8 c Constraints

$x_{AD} + x_{AE} + x_{AF} = 8$ (number of coaches at depot A)

$x_{BD} + x_{BE} + x_{BF} = 5$ (number of coaches at depot B)

$x_{CD} + x_{CE} + x_{CF} = 7$ (number of coaches at depot C)

$x_{AD} + x_{BD} + x_{CD} = 4$ (number required at depot D)

$x_{AE} + x_{BE} + x_{CE} = 10$ (number required at depot E)

$x_{AF} + x_{BF} + x_{CF} = 6$ (number required at depot F)

The number of coaches at A, B and C = number of coaches at D, E and F.

9 The problem is balanced as the total stock (34 + 57 + 25 = 116 units) equals the total demand (20 + 56 + 40 = 116 units). Therefore, there is no need to add any dummy supply or demand points.

Let x_{ij} be the number of televisions transported from i to j where

$i \in \{W, X, Y\}$, $j \in \{J, K, L\}$ and x_{ij} Ö 0

Minimise $C = 3x_{WJ} + 6x_{WK} + 3x_{WL} + 5x_{XJ} + 8x_{XK} + 4x_{XL} + 2x_{YJ} + 5x_{YK} + 7x_{YL}$

Subject to:

$x_{WJ} + x_{WK} + x_{WL}$ Ñ 34 $\quad$ $x_{WJ} + x_{XJ} + x_{YJ}$ Ö 20

$x_{XJ} + x_{XK} + x_{XL}$ Ñ 57 $\quad$ $x_{WK} + x_{XK} + x_{YK}$ Ö 56

$x_{YJ} + x_{YK} + x_{YL}$ Ñ 25 $\quad$ $x_{WL} + x_{XL} + x_{YL}$ Ö 40

10 The problem is not balanced, supply exceeds demand so introduce a dummy demand point, V, to absorb the excess supply.

Let x_{ij} be the amount of petrol (in thousands of litres) transported from i to j where

$i \in \{F, G, H\}$, $j \in \{S, T, U, V\}$ and x_{ij} Ö 0 and V represents a dummy point.

Minimise $C = 23x_{FS} + 31x_{FT} + 46x_{FU} + 35x_{GS} + 38x_{GT} + 51x_{GU} + 41x_{HS} + 50x_{HT} + 63x_{HU}$

Subject to:

$x_{FS} + x_{FT} + x_{FU} + x_{FV}$ Ñ 540 $\quad$ $x_{FS} + x_{GS} + x_{HS}$ Ö 257

$x_{GS} + x_{GT} + x_{GU} + x_{GV}$ Ñ 789 $\quad$ $x_{FT} + x_{GT} + x_{HT}$ Ö 348

$x_{HS} + x_{HT} + x_{HU} + x_{HV}$ Ñ 673 $\quad$ $x_{FU} + x_{GU} + x_{HU}$ Ö 410

$x_{FV} + x_{GV} + x_{HV}$ Ö 987

11 To maximise, subtract all entries from 278, the largest value in the original matrix, to obtain:

	H	***I***	***P***	***S***
A	11	6	2	17
B	14	7	0	15
C	11	5	3	15
D	17	9	4	12

Reducing rows gives:

	H	***I***	***P***	***S***
A	9	4	0	15
B	14	7	0	15
C	8	2	0	12
D	13	5	0	17

Reducing columns:

	H	***I***	***P***	***S***
A	1	2	0	3
B	6	5	0	3
C	0	0	0	0
D	5	3	0	5

The zeros in the reduced cost matrix are covered by two lines. The smallest uncovered element is 1. So using the short cut to the Hungarian algorithm, add 1 to the element covered by two lines and subtract 1 from the uncovered elements to obtain:

	H	***I***	***P***	***S***
A	0	1	0	2
B	5	4	0	2
C	0	0	1	0
D	4	2	0	4

The zeros can now be covered by three lines. (This can be done in more than one way, only one approach is illustrated here.) The smallest uncovered element is 2. So using the short cut to the Hungarian algorithm, add 2 to the elements covered by two lines and subtract 2 from the uncovered elements to obtain:

	H	***I***	***P***	***S***
A	0	1	2	2
B	3	1	0	0
C	0	0	3	0
D	2	0	0	2

11 (continued)

This solution is optimal as it requires four lines to cover all the zeros.

There are two optimal solutions:

Site A – H
Site B – P
Site C – S
Site D – I
Total income = 267 + 278 + 263 + 269 = £1077

or

Site A – H
Site B – S
Site C – I
Site D – P
Total income = 267 + 263 + 273 + 274 = £1077

12 Reducing rows gives:

	Job 1	Job 2	Job 3	Job 4
Machine 1	9	0	3	2
Machine 2	0	10	4	3
Machine 3	4	5	0	6
Machine 4	0	2	4	8

Reducing columns:

	Job 1	Job 2	Job 3	Job 4
Machine 1	9	0	3	0
Machine 2	0	10	4	1
Machine 3	4	5	0	4
Machine 4	0	2	4	6

The zeros can now be covered by three lines. (This can be done in more than one way, only one approach is illustrated here.) The smallest uncovered element is 1. So using the short cut to the Hungarian algorithm, add 1 to the elements covered by two lines and subtract 1 from the uncovered elements to obtain:

	Job 1	Job 2	Job 3	Job 4
Machine 1	10	0	3	0
Machine 2	0	9	3	0
Machine 3	5	5	0	4
Machine 4	0	1	3	5

12 (continued)

Four lines are now needed to cover the zeros, so the solution is optimal. This solution is:

Machine 1 – Job 2
Machine 2 – Job 4
Machine 3 – Job 3
Machine 4 – Job 1

Total time required = 5 + 5 + 3 + 2 = 15 hours

13 a i Reducing rows gives:

	A	*B*	*C*	*D*
I	0	7	2	1
II	0	11	4	3
III	0	8	5	2
IV	0	12	6	2

Reducing columns:

	A	*B*	*C*	*D*
I	0	0	0	0
II	0	4	2	2
III	0	1	3	1
IV	0	5	4	1

The zeros can now be covered by two lines. The smallest uncovered element is 1. So add 1 to the elements covered by two lines and subtract 1 from the uncovered elements to obtain:

	A	*B*	*C*	*D*
I	1	0	0	0
II	0	3	1	1
III	0	0	2	0
IV	0	4	3	0

Four lines are needed to cover all the zeros, so this table can be used to make the assignment.

ii The solution is:

Team *C* – Task I (only zero in column 3)
Team *B* – Task III (task I already allocated, so only available zero for task III)
Team *D* – Task IV (task I and III already allocated, so only leaving task IV)
Team *A* – Task II

In summary: $\text{I} - C, \text{II} - A, \text{III} - B, \text{IV} - D$

b Minimum time for room to be prepared = 19 + 12 + 24 + 14 = 69 minutes

14 a The smallest numbers in rows 1, 2, 3, 4, 5 and 6 are 11, 10, 12, 10, 11 and 14 respectively. So reducing rows by subtracting these numbers from each element in the row gives:

	P	*Q*	*R*	*S*	*T*	*U*
A	5	7	3	10	0	12
B	9	10	7	9	0	12
C	6	9	4	8	0	12
D	5	9	7	10	0	11
E	7	9	4	11	0	14
F	3	10	2	9	0	14

The smallest numbers in columns 1, 2, 3, 4, 5 and 6 are 3, 7, 2, 8, 0 and 11 respectively. Reducing columns by subtracting these numbers from each element in the column gives:

	P	*Q*	*R*	*S*	*T*	*U*
A	2	0	1	2	0	1
B	6	3	5	1	0	1
C	3	2	2	0	0	1
D	2	2	5	2	0	0
E	4	2	2	3	0	3
F	0	3	0	1	0	3

The zeros can now be covered by five lines. The smallest uncovered element is 2. So using the short cut to the Hungarian algorithm, add 2 to the elements covered by two lines and subtract 2 from the uncovered elements to obtain:

	P	*Q*	*R*	*S*	*T*	*U*
A	2	0	1	4	2	3
B	4	1	3	1	0	1
C	1	0	0	0	0	1
D	0	0	3	2	0	0
E	2	0	0	3	0	3
F	0	3	0	3	0	5

Six lines are now needed to cover all the zeros, so this table can be used to make the assignment. The optimal solution is:

$A – Q$
$B – T$
$C – S$
$D – U$
$E – R$
$F – P$

b Minimum cost = 18 + 10 + 20 + 21 + 15 + 17 = £101

15 a As the table is not an $n \times n$ matrix, an additional (dummy) column is needed to make the problem balanced. This ensures that the number of tasks matches the number of workers, which is a requirement of the Hungarian algorithm.

b Adding an extra task, 4, to the table gives this $n \times n$ matrix:

	1	**2**	**3**	**4**
Beth	15	18	24	0
Josh	16	15	23	0
Louise	15	17	22	0
Oliver	14	19	23	0

As there is already a zero in each row, the rows cannot be reduced further. The smallest numbers in columns 1, 2, 3 and 4 are 14, 15, 22 and 0 respectively. Reducing the columns by subtracting these numbers from each element in the column gives this reduced cost matrix:

	1	**2**	**3**	**4**
Beth	1	3	2	0
Josh	2	0	1	0
Louise	1	2	0	0
Oliver	0	4	1	0

Four lines are needed to cover all the zeros, so this table can be used to find the optimal solution:

Beth – no task
Josh – Task 2
Louise – Task 3
Oliver – Task 1

c Minimum time = 0 + 15 + 22 + 14 = 51 minutes

16 a As the table is not an $n \times n$ matrix, an additional (dummy) row is needed to make the problem balanced. By adding an extra mechanic to the table this $n \times n$ matrix is obtained:

	A	*B*	*C*	*D*
Bill	570	375	440	520
Steve	510	420	480	470
Jo	550	395	410	490
Dummy	0	0	0	0

The smallest numbers in rows 1, 2, 3 and 4 are 375, 420, 395 and 0. Reducing rows gives:

	A	*B*	*C*	*D*
Bill	195	0	65	145
Steve	90	0	60	50
Jo	155	0	15	95
Dummy	0	0	0	0

There is a zero already in each column, so the reduced cost matrix has been obtained. Covering the row and column that contain all zeros, the smallest uncovered element is 15. So adding 15 to the elements covered by two lines and subtracting 15 gives:

	A	*B*	*C*	*D*
Bill	180	0	50	130
Steve	75	0	45	35
Jo	140	0	0	80
Dummy	0	15	0	0

The zeros can be covered by three lines. The smallest uncovered element is 35. So adding 35 to the elements covered by two lines and subtracting 35 from the uncovered elements to obtain:

	A	*B*	*C*	*D*
Bill	145	0	15	95
Steve	40	0	10	0
Jo	140	35	0	80
Dummy	0	50	0	0

Four lines are needed to cover all the zeros, so this table can be used to find the optimal solution:

Bill – *B*
Steve – *D*
Jo – *C*
No mechanic is allocated to centre *A*

b Minimum cost = 375 + 470 + 410 + 0 = £1255

17 To maximise, subtract all entries from 35, the largest value in the original matrix, to obtain:

	Art	Literature	Music	Science
Donna	4	11	3	0
Kerwin	16	21	15	14
Hannah	19	25	16	13
Thomas	17	20	14	12

Reducing rows gives:

	Art	Literature	Music	Science
Donna	4	11	3	0
Kerwin	2	7	1	0
Hannah	6	12	3	0
Thomas	5	8	2	0

Reducing rows gives:

	Art	Literature	Music	Science
Donna	2	4	2	0
Kerwin	0	0	0	0
Hannah	4	5	2	0
Thomas	3	1	1	0

The zeros can be covered by two lines. The smallest uncovered element is 1. So adding 1to the elements covered by two lines and subtracting 1 from the uncovered elements to obtain:

	Art	Literature	Music	Science
Donna	1	3	1	0
Kerwin	0	0	0	1
Hannah	3	4	1	0
Thomas	2	0	0	0

The zeros can be covered by three lines. The smallest uncovered element is 1. So adding 1to the elements covered by two lines and subtracting 1 from the uncovered elements to obtain:

	Art	Literature	Music	Science
Donna	0	2	0	0
Kerwin	0	0	0	2
Hannah	2	3	0	0
Thomas	2	0	0	1

Four lines are needed to cover all the zeros, so this table can be used to find the optimal solution.

17 (continued)

There are four solutions:

Donna – *S*	Donna – *A*	Donna – *A*	Donna – *M*
Kerwin – *A*	Kerwin – *M*	Kerwin – *L*	Kerwin – *A*
Hannah – *M*	Hannah – *S*	Hannah – *S*	Hannah – *S*
Thomas – *L*	Thomas – *L*	Thomas – *M*	Thomas – *L*

The respective scores are:

35 + 19 + 19 + 15 = 88 points
31 + 20 + 22 + 15 = 88 points
31 + 14 + 22 + 21 = 88 points
32 + 19 + 22 + 15 = 88 points

18 a i Reducing rows gives:

	I	**II**	**III**	**IV**
Clive	0	22	16	4
Julie	1	20	24	0
Nicky	1	18	18	0
Steve	1	23	26	0

Reducing columns:

	I	**II**	**III**	**IV**
Clive	0	4	0	4
Julie	1	2	8	0
Nicky	1	0	2	0
Steve	1	5	10	0

The zeros can now be covered by three lines. (This can be done in more than one way, only one approach is illustrated here.) The smallest uncovered element is 1. So adding 1 to the elements covered by two lines and subtracting 1 from the uncovered elements to obtain:

	I	**II**	**III**	**IV**
Clive	0	4	0	5
Julie	0	1	7	0
Nicky	1	0	2	1
Steve	0	4	9	0

ii Four lines are needed to cover all the zeros, so this table can be used to find the solution.

Clive – III		Clive – III
Julie – I	or	Julie – IV
Nicky – II		Nicky – II
Steve – IV		Steve – I

Both times are 83 minutes (28 + 13 + 32 + 10 = 83 and 28 + 12 + 32 + 11 = 83) so the earliest the meeting could end is 11:23 am.

18 b To find the latest time the meeting could end, maximise the time of the meeting. To maximise, subtract all entries from 36, the largest value in the original matrix, to obtain:

	I	**II**	**III**	**IV**
Clive	24	2	8	20
Julie	23	4	0	24
Nicky	21	4	4	22
Steve	25	3	0	10

And then solve using the Hungarian algorithm.

19 a To maximise, subtract all entries 30, the largest value in the original matrix, to obtain:

	1	**2**	**3**	**4**
Ann	4	0	0	0
Brenda	0	7	4	1
Connor	0	5	3	6
Dave	0	3	5	9

This is already a reduced cost matrix. The zeros can be covered by two lines. The smallest uncovered element is 1. So adding 1 to the elements covered by two lines and subtracting 1 from the uncovered elements to obtain:

	1	**2**	**3**	**4**
Ann	5	0	0	0
Brenda	0	6	3	0
Connor	0	4	2	5
Dave	0	2	4	8

The zeros can be covered by three lines. (This can be done in more than one way, only one approach is illustrated here.) The smallest uncovered element is 2. So adding 2 to the elements covered by two lines and subtracting 1 from the uncovered elements to obtain:

	1	**2**	**3**	**4**
Ann	7	0	0	0
Brenda	2	6	3	0
Connor	0	2	0	3
Dave	0	0	2	6

Four lines are needed to cover all the zeros, so this table can be used to find the solution.

Ann – 3		Ann – 2
Brenda – 4	or	Brenda – 4
Connor – 1		Connor – 3
Dave – 2		Dave – 1

b Value of sales = 30 + 29 + 30 + 27 = 116, or value of sales = 30 + 29 + 27 + 30 = 116
So maximum value of sales = 116 × 10000 = £1 160000

19 c Two solutions can be found from the table found in part **a**. Note that if the zeros in the second table shown in the solution to part **a** are covered with two vertical lines (in columns 1 and 4) and one horizontal line (in row 1), applying the Hungarian algorithm gives this table:

	1	**2**	**3**	**4**
Ann	7	0	0	2
Brenda	0	4	1	0
Connor	0	2	0	5
Dave	0	0	2	8

This table yields the same two solutions.

20 a Replacing the '–' forbidden allocation with the value of 50 makes the assignment 'unattractive'. The matrix becomes:

	1	**2**	**3**	**4**
Amy	21	19	23	20
Bob	20	18	21	19
Charles	22	20	50	21
Davina	20	19	22	20

Reducing rows gives:

	1	**2**	**3**	**4**
Amy	2	0	4	1
Bob	2	0	3	1
Charles	2	0	30	1
Davina	1	0	3	1

Reducing columns gives:

	1	**2**	**3**	**4**
Amy	1	0	1	0
Bob	1	0	0	0
Charles	1	0	27	0
Davina	0	0	0	0

Four lines are needed to cover all the zeros, so this table can be used to find the solution.

Amy – 2		Amy – 4
Bob – 3	or	Bob – 3
Charles– 4		Charles – 2
Davina – 1		Davina – 1

b Both solutions give a minimum time of 81 minutes
Minimum time = 19 + 21 + 21 + 20 = 81 minutes
Minimum time = 20 + 21 + 20 + 20 = 81 minutes

21 The total number of deluxe and majestic seats should be most half of the number of standards seats:

$$y+z \leqslant \frac{1}{2}x \Rightarrow 2(y+z) \leqslant x$$

The number of deluxe seats should be at least 10% of the total number of seats:

$$y \geqslant \frac{10}{100}(x+y+z) \Rightarrow x+z \leqslant 9y$$

The number of deluxe seats should be at most 20% of the total number of seats:

$$y \geqslant \frac{20}{100}(x-y+z) \Rightarrow x+z \geqslant 4y$$

The number of majestic seats should be at least half of the number of deluxe seats:

$$z \geqslant \frac{1}{2}y \Rightarrow 2x \geqslant y$$

The total number of seats should be at least 250:

$$x+y+z \geqslant 250$$

The problem becomes let x, y and z be the number of standard, majestic and deluxe seats to be bought

Minimise $C = 20x + 26y + 36z$

Subject to:

$$x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$$
$$2(y+z) \leqslant x$$
$$x+z \leqslant 9y$$
$$x+z \geqslant 4y$$
$$2x \geqslant y$$
$$x+y+z \geqslant 250$$

22 a $C_1 = 7+14+0+14 = 35$

$C_2 = 7+14+5 = 26$

$C_3 = 8+9+6+8 = 31$

b The capacity of cut C_2 is equal to the flow of 26, so by the maximum flow–minimum cut theorem, the flow of 26 is maximal.

Alternatively, cut C_2 only passes through saturated arcs (*FI*, *EH* and *EG*) so it is the minimum cut. By the maximum flow–minimum cut theorem, the maximum flow = the capacity of C_2 = 26.

c Adding arc *E* to *J* (capacity 5) will increase the flow by 1 along *SEJL* since only one more unit can leave *E*. The maximum inflow into *E* is 22 (the capacity of *SE* plus *FE*). So maximum flow will increase to 27.

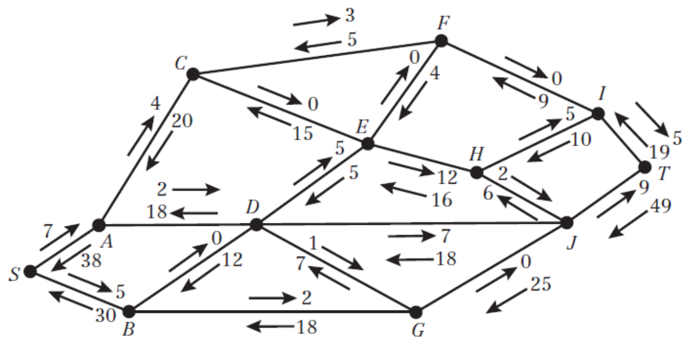
Adding arc *F* to *H* (capacity 3) will increase the flow by 2 along *SFHJL*. Arcs SF and HJ will become saturated. A cut through *IK*, *HJ* and *EG* will only pass through saturated arcs; this cut will have capacity of 12 + 9 + 7 = 28, which will become the new maximal flow. So the government should choose this option.

23 a Using the conservation condition as follows.
Flow into E = flow out of E, so $15+5=4+x \Rightarrow x=16$
Flow into D = flow out of D, so $18+12=5+18+y \Rightarrow y=7$

b Flowing into cut C_1 are arcs CF, CE, DE, DJ, DG and BG.
So the value of cut C_1 = 8 + 15 + 10 + 25 + 8 + 20 = 86

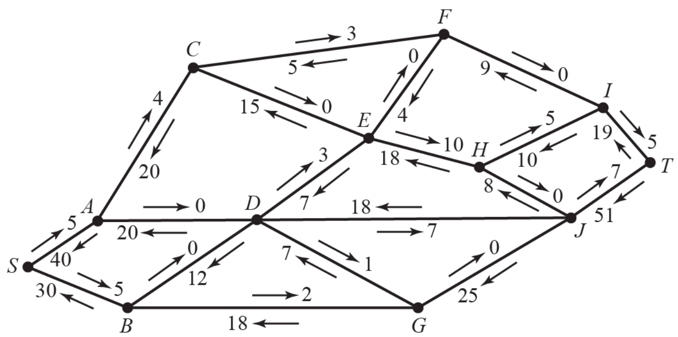
Flowing into cut C_2 are arcs FI, CE, AD, BD and GJ. (Arcs EF and DG flow out of the cut.)
So the value of cut C_2 = 9 + 15 + 20 + 12 + 25 = 81.

c Applying the labelling procedure to the initial feasible flow of 68 gives:

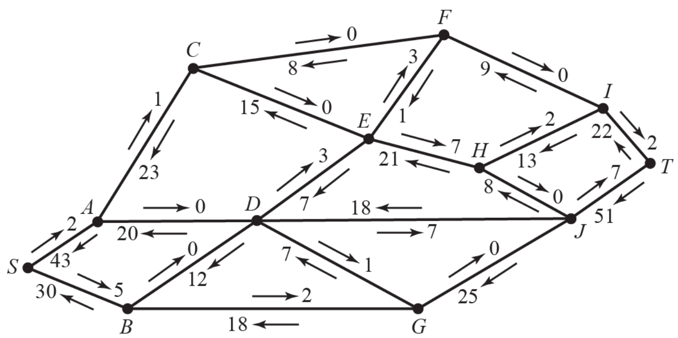


There are several flow augmenting routes that could be chosen; the worked answer for parts **c** and **d** shows one set of choices.

First, choose the flow-augmenting route *SADEHJT*. An additional flow of 2 can be sent along this route because arcs *AD* and *HJ* have a maximum spare capacity of 2. The updated diagram is:

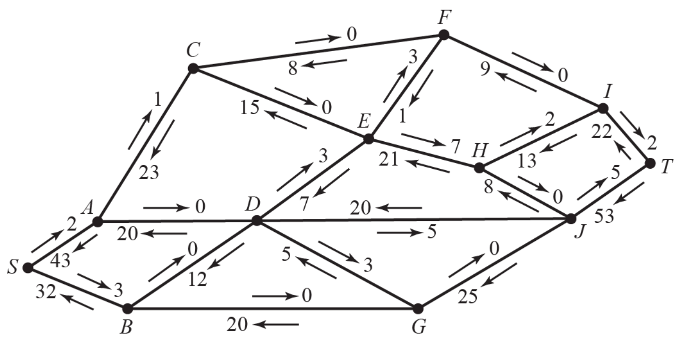


Now, choose the flow-augmenting route *SACFEHIT*. After *C* the route is forced, since arcs *CE*, *FI* and *HJ* are saturated. An additional flow of 3 can be sent along this route because *CF* has a maximum spare capacity of 3. The updated diagram is:



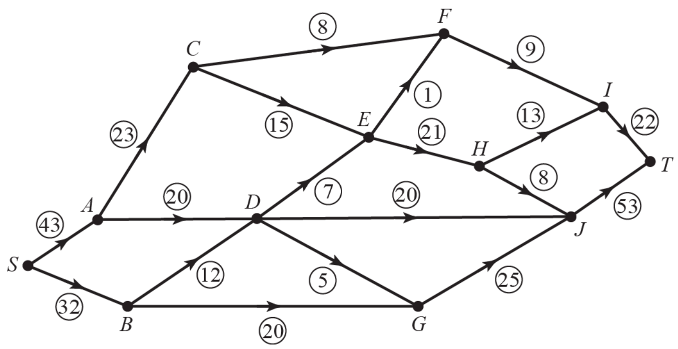
23 c (continued)

There are no further routes starting *SA* because *AD* is saturated and all arcs leaving *C* are also saturated. So choose *SBGDJT*. An additional flow of 2 can be sent along this route because *BG* has a maximum spare capacity of 2. The updated diagram is:



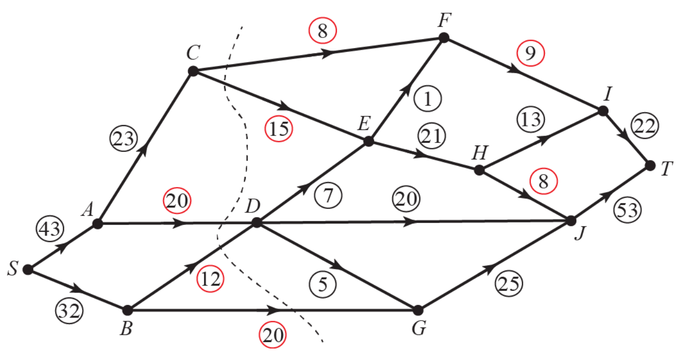
There are no further flow-augmenting routes, as all arcs leaving *B* and *C* are saturated and *AD* is saturated, so the flow is maximal.

d This is the final flow diagram:



The maximal flow is 75 (arcs leaving the source: *SA* and *SB* = 43 + 32 = 75 or arcs entering the sink: *IT* and *JT* = 24 + 51 = 75).

e This diagram shows the saturated arcs (represented by the red circles). The cut (represented by the dashed line) must be a minimum because it only passes through saturated arcs.



Flowing into the minimum cut are arcs *CF*, *CE*, *AD*, *BD* and *BG*. So the value of the minimum cut = 8 + 15 + 20 + 12 + 20 = 75.
As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

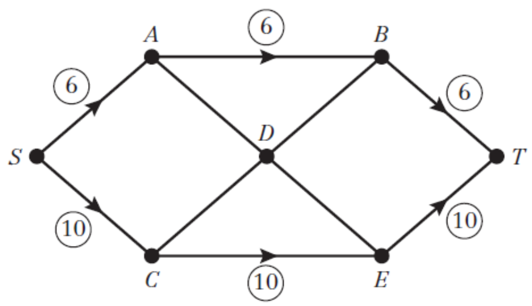
24 a If there is a cut with capacity of less than 30, then by the maximum flow–minimum cut theorem the maximum flow must be less than 30. The cut through arcs *AB*, *AD*, *CD* and *CE* has capacity 6 + 4 + 5 + 10 = 25, so the maximum flow cannot be 30. Similarly the cut through arcs *AB*, *BD* and *ET* has capacity 6 + 6 + 12 = 24, so again the maximum flow cannot be 30.

Alternatively consider the flow into and out of *A*. This must be equal. The maximum flow out of *A* can only be 6 + 4 = 10, so the maximum flow along *SA* can only be 10. Similarly, the flow into and out of *C* must be equal. The maximum flow out of *A* can only be 5 + 10 = 15, so the maximum flow along *SC* can only be 15. Therefore the maximum flow out of *S* can only be 10 + 15 =25. So the maximum flow cannot be 30.

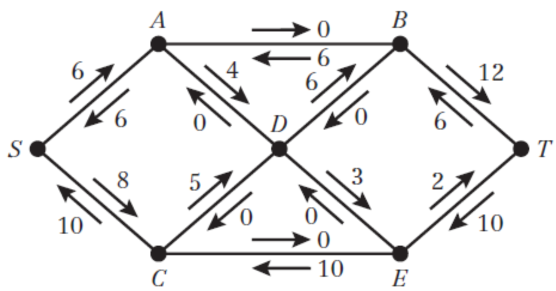
b **i** The maximum flow along *SABT* = 6, the capacity of arc *AB*, which is the arc along the route *SABT* that has the lowest capacity.

ii The maximum flow along *SCET* = 10, the capacity of arc *CE*, which is the arc along the route *SCET* that has the lowest capacity.

c

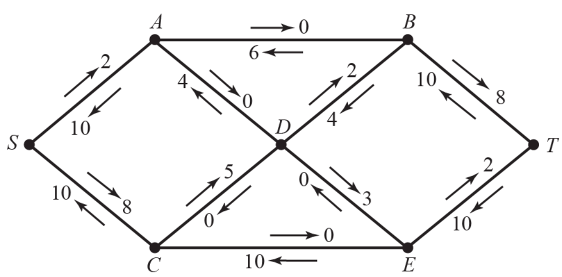


d Applying the labelling procedure to the initial feasible flow of 16 shown in part **c** gives:



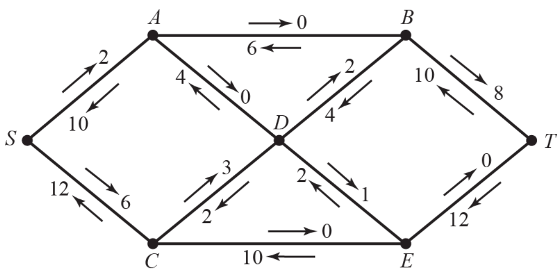
There are several flow augmenting routes that could be chosen; the worked answer for parts **d** and **e** shows one set of choices.

First, choose the flow-augmenting route *SADBT*. An additional flow of 4 can be sent along this route because arc *AD* has a maximum spare capacity of 4. The updated diagram is:

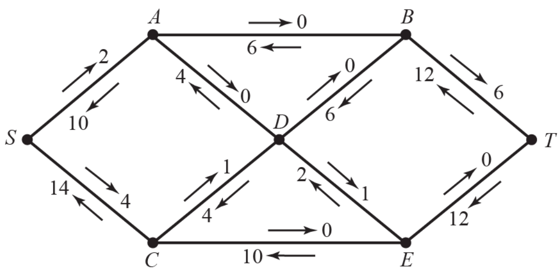


24 d (continued)

There are no further routes starting *SA* because all arcs leaving *A* are now saturated. So, choose the flow-augmenting route *SCDET*. An additional flow of 2 can be sent along this route because *ET* has a maximum spare capacity of 2. The updated diagram is:

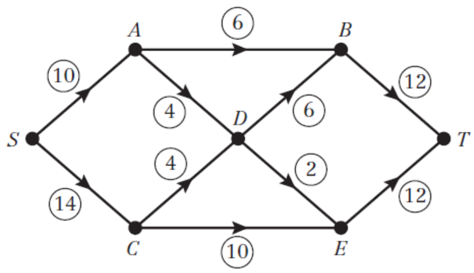


Now choose *SCDBT*. An additional flow of 2 can be sent along this route because *DB* has a maximum spare capacity of 2. The updated diagram is:



There are no further flow-augmenting routes, as arcs *DB* and *ET* are now saturated.

e This is the final flow diagram:



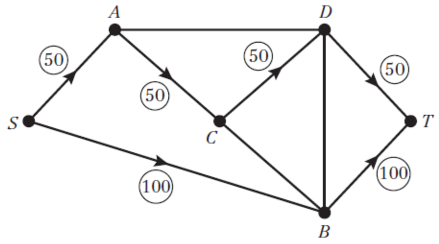
The maximal flow is 24 (arcs leaving the source: *SA* and *SB* = 12 + 14 = 24).

f The cut through *AB*, *DB* and *ET* must be a minimum cut because it only passes through saturated arcs. Its value is 6 + 6 + 12 = 24. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

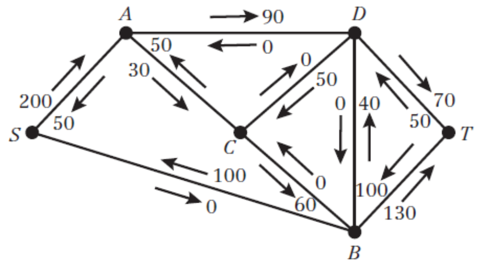
25 a i The maximum flow along *SACDT* = 50, the capacity of arc *CD*, which is the arc along the path that has the lowest capacity.

ii The maximum flow along *SBT* = 100, the capacity of arc *SB*, which is the arc along the path that has the lowest capacity.

25 b



c Applying the labelling procedure to the initial feasible flow of 150 shown in part **d** gives:

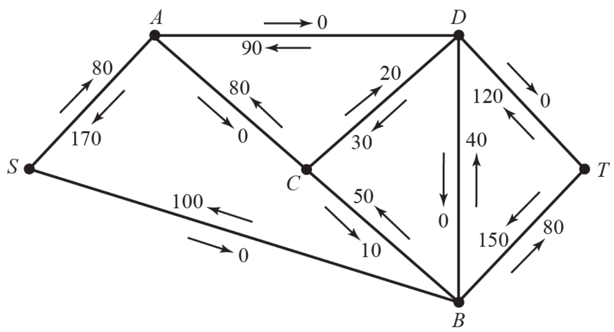


All flow-augmenting routes must start along arc *SA* as arc *SB* is saturated. First choose *SADT*. An additional flow of 70 can be sent along this route because *DT* has a maximum spare capacity of 70.

Now choose *SACBT*. An additional flow of 30 can be sent along this route because *AC* has a maximum spare capacity of 30.

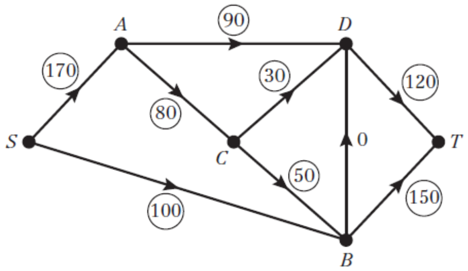
Now choose *SADCBT*. An additional flow of 20 can be sent along this route because *AD* has a maximum spare capacity of 20. (Note that this reduces the flow along arc *CD*.)

There are no further flow-augmenting routes, as all arcs leaving *A* are now saturated and arc *SB* remains saturated, so this flow is maximal The updated diagram is:



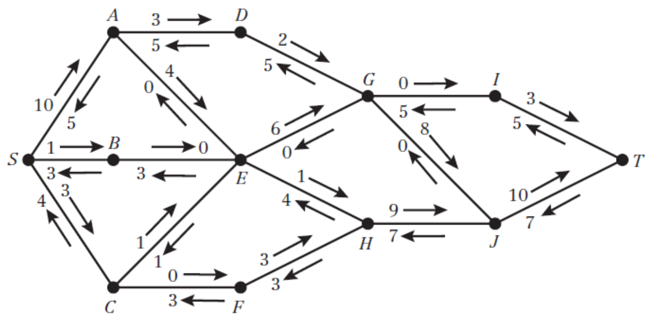
The flow has increased from 150 by 120 (70 + 30 + 20) to 270 the maximum flow.

25 d This is the final flow diagram:



25 e The cut through AD, AC and SB must be a minimum cut because it only passes through saturated arcs. Its value is 90 + 80 + 100 = 270. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

26 a Applying the labelling procedure to the initial feasible flow of 12 students gives:

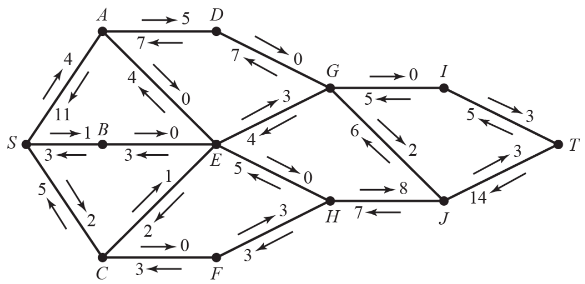


No flow-augmenting routes can go along arc SB as arc BE is saturated. First choose $SADGJT$. An additional flow of 2 can be sent along this route because DG has a maximum spare capacity of 2.

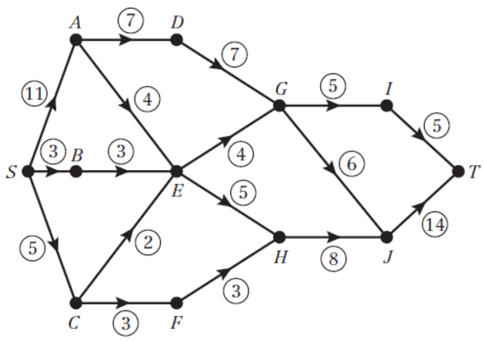
Then $SAEGJT$. An additional flow of 4 can be sent along this route because AE has a maximum spare capacity of 4.

Then $SCEHJT$. An additional flow of 1 can be sent along this route because CE has a maximum spare capacity of 1. (Note that $SCEGJT$ is also an acceptable flow-augmenting route here.)

b There are no further flow-augmenting routes, so this flow is maximal. The updated diagram is:



This is the final flow diagram:



c 19 students (there are 11 + 3 + 5 = 19 leaving Stirling and 5 + 14 = 19 arriving in Truro).

d The cut through DG, AE, BE, CE and CF must be a minimum cut because it only passes through saturated arcs. Its value is 7 + 4 + 3 + 2 + 3 = 19. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

27 a Using the conservation flow condition, the flow into A = flow out of A

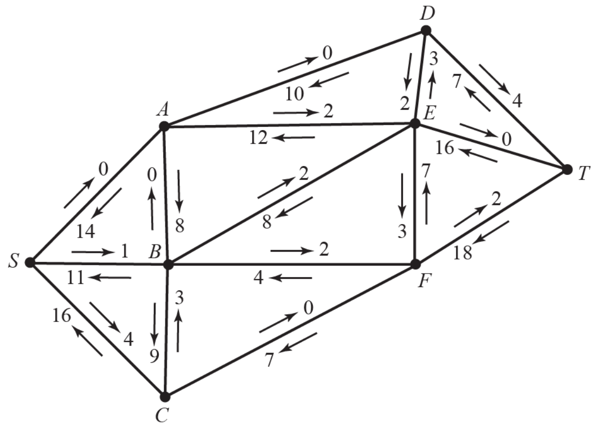
$14+8=10+x \Rightarrow x=12$

Flow into F = flow out of F

$4+7+y=18 \Rightarrow y=7$

b Initial flow = the flow leaving the source S = 14 + 11 + 16 = 41
Initial flow = the flow entering sink T = 7 + 16 + 18 = 41

c The completed labelling procedure diagram for the initial flow is:



d There are several flow augmenting routes that could be chosen. This is one possibility.

No flow-augmenting routes can go along arc SA as arc SA is saturated. First choose $SBEDT$. An additional flow of 1 can be sent along this route because SB has a maximum spare capacity of 1.

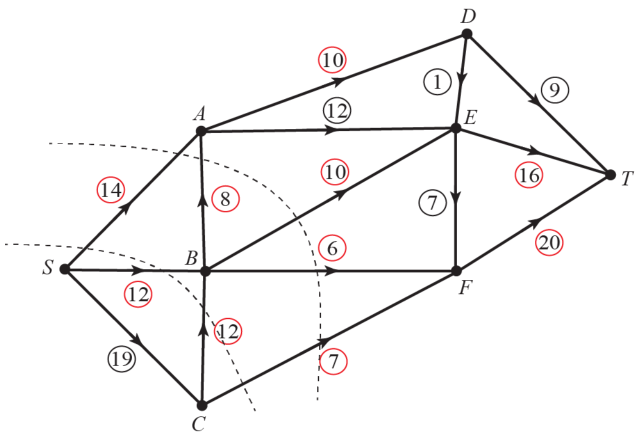
Then $SCBFT$. An additional flow of 2 can be sent along this route because BF has a maximum spare capacity of 2.

Then $SCBEDT$. An additional flow of 1 can be sent along this route because arcs CB and BE both have a maximum spare capacity of 1.

There are no further flow-augmenting routes, as all arcs leaving C are now saturated and arcs SA and SB are saturated, so this flow is maximal.

The maximal flow is the initial flow 41 + the augmented flows 1 + 2 + 1 = 4, i.e. 41 + 4 = 45

e This is the final flow diagram, with two cuts C_1 and C_2 shown and with the saturated arcs shown in red:



27 e (continued)

Both C_1 and C_2 are minimum cuts as both only pass across saturated arcs.
Flowing into cut C_1 are arcs SA, SB, CB and CF; the value of cut $C_1 = 14 + 12 + 12 + 7 = 45$
Flowing into cut C_2 are arcs SA, BA, BE, BF and CF; the value of $C_2 = 14 + 8 + 10 + 6 + 7 = 45$

So the value of the minimum cut = 45. As the maximum flow–minimum cut theorem states that the minimum cut value equals the maximum flow value, the flow of 45 is maximal.

28 a Using the conservation flow condition, the flow into F = flow out of F

$2+7+0=x \Rightarrow x=9$

The flow into D = flow out of D

$14+0+2=y \Rightarrow y=16$

b The initial flow = the flow into A + the flow into $E = 20 + 9 + 14 + 6 + 4 = 53$

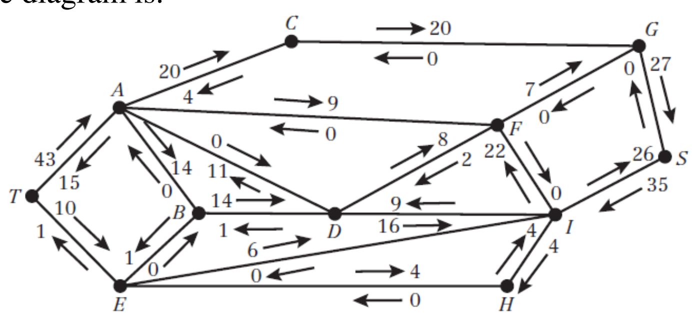
Consider route IDA. An additional flow of 9 can be sent along this route because arc ID has a maximum spare capacity of 9. So the initial flow cannot be maximal.

In general, prove that the flow is not maximal by finding a flow-augmenting route or demonstrating that there not enough saturated arcs for a minimum cut. In this case, the saturated arcs are GC, GF, FA, IE, HE and BA and it is not possible to find a cut that only passes through some (or all) of these arcs.

c Add a super source S and a super sink T to apply the labelling procedure to the initial flow.

Arc SG must have capacity $20 + 7 = 27$ and initial flow $20 + 7 = 27$
Arc SI must have capacity $22 + 25 + 6 + 8 = 61$ and initial flow $0 + 16 + 6 + 4 = 26$
Arc AT must have capacity $24 + 9 + 11 + 14 = 58$ and initial flow $20 + 9 + 0 + 14 = 43$
Arc ET must have capacity $1 + 6 + 4 = 11$ and initial flow $0 + 6 + 4 = 10$

The diagram is:



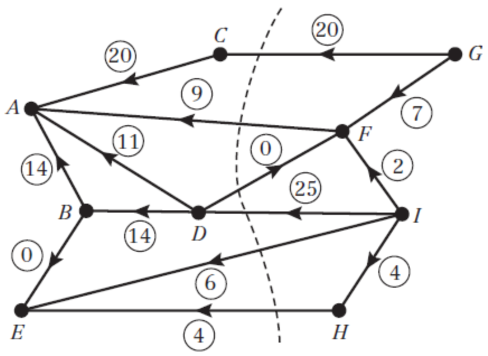
There are several flow augmenting routes that could be chosen. This is one possibility.

First note that no flow-augmenting routes can go along arc SG as arcs GC and GF are saturated. So choose $SIDA$. An additional flow of 9 can be sent along this route because ID has a maximum spare capacity of 1.

Then $SIFDA$. An additional flow of 2 can be sent along this route because of the constraint that the flow along arc DF can only be reduced by 2.

There are no further flow-augmenting routes, so this flow is maximal. The maximal flow is the initial flow 53 + the augmented flows $9 + 2 = 11$, i.e. $53 + 11 = 64$

28 d This shows the maximal flow pattern for the solution found in part **c**:



e The cut through *GC*, *FA*, *DF*, *ID*, *IE* and *HE* must be a minimum cut because it only passes through saturated arcs and one empty arc directed from sink to source (*DF*).
Its value is 20 + 9 + 25 + 6 + 4 = 64. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

(Note: super source or super sink arcs must not be used in looking for the minimum cut.)

29 a Using the conservation flow condition, flow into D = flow out of D

$\text{Flow } SD + 8 + 1 = 12 + 7 \Rightarrow \text{Flow } SD = 10$

Flow into E = flow out of E

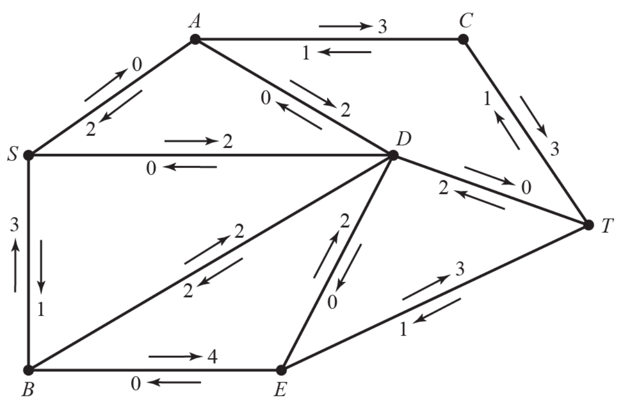
$\text{Flow } BE = 1 + 5 = 6$

Flow into C = flow out of C

$\text{Flow } CT = 19$

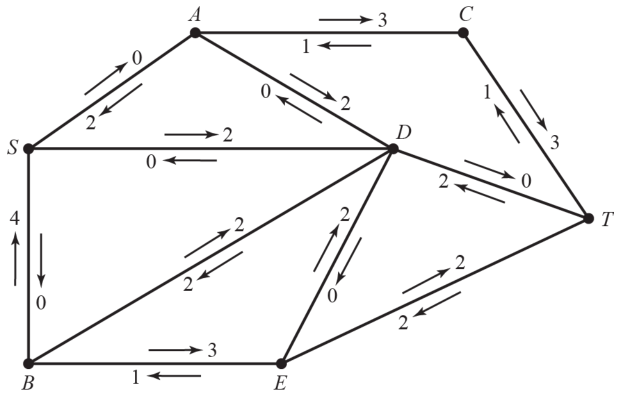
Flow through the network = flow out of the source S = 12 + 10 + 14 = 36
Flow through the network = flow into the sink T = 19 + 12 + 5 = 36

b The labelling procedure has been applied to the network taking into account the upper and lower capacities of each arc (for example, the flow along *SB* can be increased by 1 and can be decreased by 3 to keep within the maximum and minimum capacities). The diagram of the initial flow is:

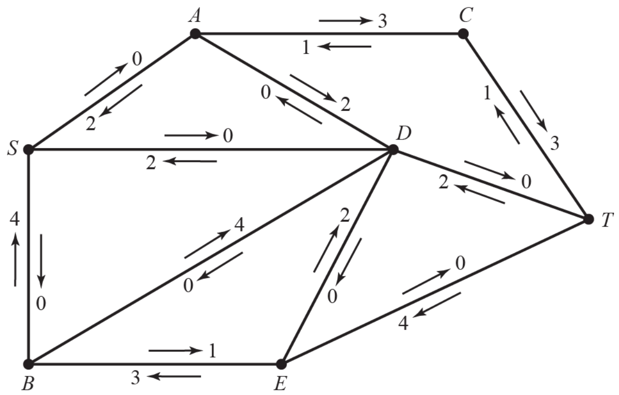


29 b (continued)

The first flow-augmenting route is *SBET* with an additional flow of 1. The updated diagram is:



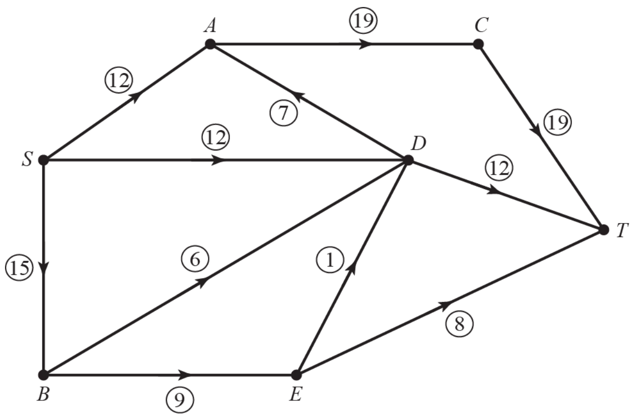
The next flow-augmenting route is *SDBET* with an additional flow of 2. The updated diagram is:



There are no further flow-augmenting routes, as there is no spare capacity on any route from S, so the flow is maximal.

Initial flow through the network = 36. Flow augmentation has increased this by $1 + 2 = 3$, so the maximum flow $= 36 + 3 = 39$

c The final flow diagram is :



The cut through SA, SD and SB must be a minimum cut because it only passes through saturated arcs. Its value is 12 + 12 + 15 = 39. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

30 a Using the conservation flow condition, flow into B = flow out of B

$9+2=8+x \Rightarrow x=3$

Flow into C = flow out of C

$16+y=25+7+10 \Rightarrow y=26$

b Add a super source S and a super sink T

Arc SF_1 must have minimum capacity 19 + 15 + 8 = 42

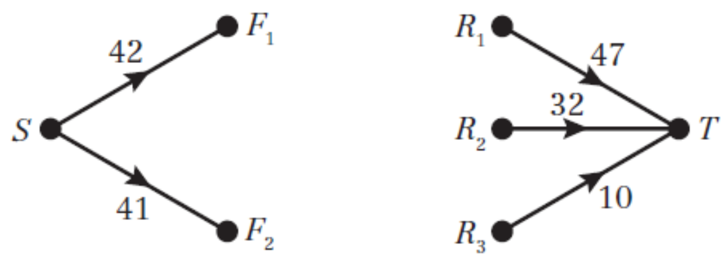
Arc SF_2 must have minimum capacity 10 + 31 = 41

Arc R_1T must have minimum capacity 32 + 15 = 47

Arc R_2T must have minimum capacity 25 + 7 = 32

Arc R_3T must have minimum capacity 10

The relevant additions to the network diagram are shown here:



c Initial flow along arc SF_1 = 12 + 9 + 8 = 29

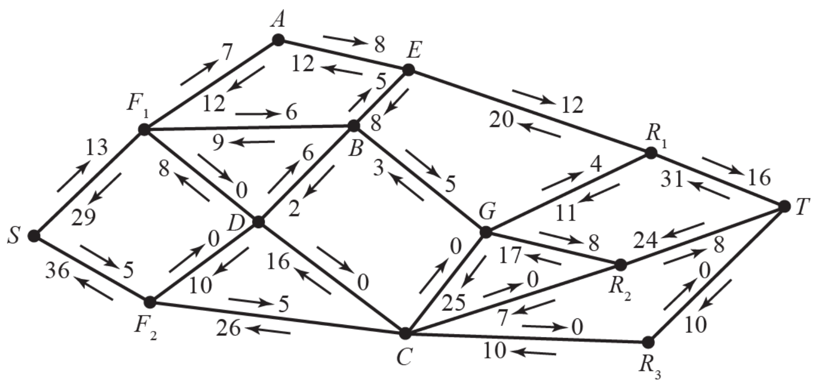
Initial flow along arc SF_2 = 10 + 26 = 36

Initial flow along arc R_1T = 20 + 11 = 31

Initial flow along arc R_2T = 17 + 7 = 24

Initial flow along arc R_3T = 10

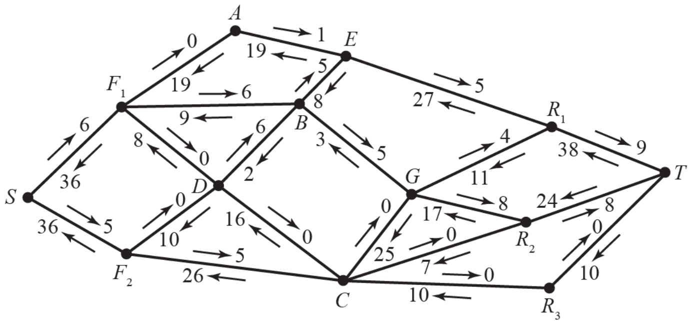
Applying the labelling procedure to the initial flow gives:



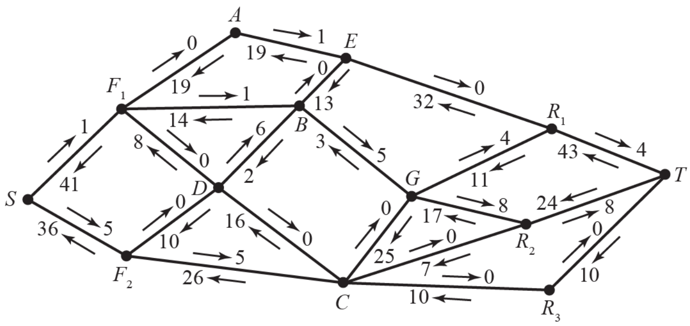
There are several flow augmenting routes that could be chosen. This answer illustrates one set of options.

30 c (continued)

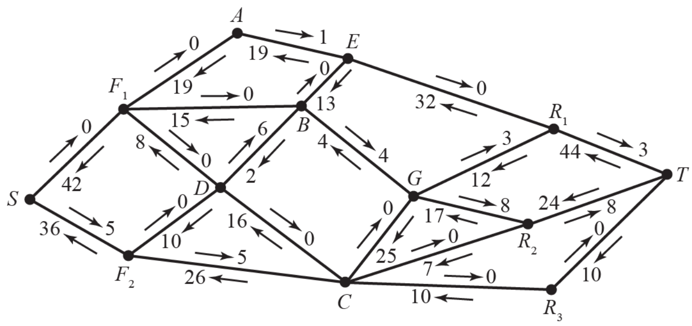
First choose SF_1AER_1T as a flow-augmenting route. An additional flow of 7 can be sent along this route because F_1A has a maximum spare capacity of 7. The updated diagram is:



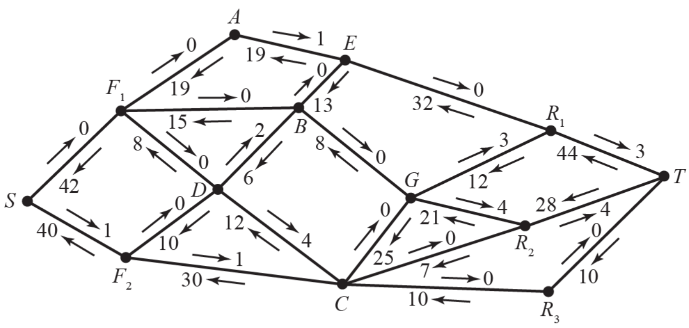
Next choose SF_1BER_1T. An additional flow of 5 can be sent along this route because BE and ER_1 both have a maximum spare capacity of 5. The updated diagram is:



Next choose SF_1BGR_1T. An additional flow of 1 can be sent along this route because SF_1 and F_1B both have a maximum spare capacity of 1. The updated diagram is:

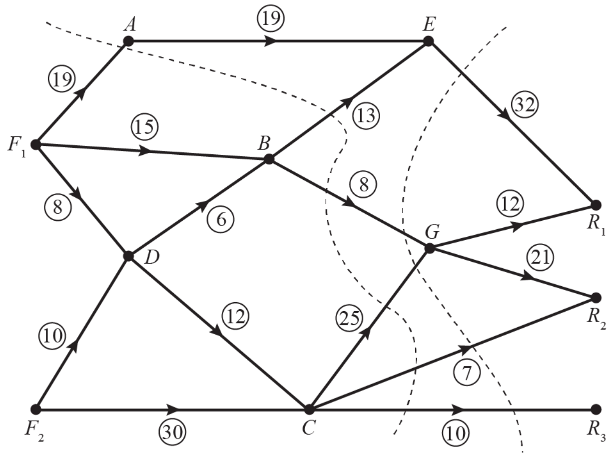


Next choose SF_2CDBGR_2T. An additional flow of 4 can be sent along this route because BG has a maximum spare capacity of 4. The updated diagram is:



There are no further flow-augmenting routes, so the flow is maximal.

30 d The final flow diagram is:



The maximal flow is 82 (flow on arcs leaving F_1 and F_2 = 19 + 15 + 8 + 10 + 30 = 82)

e The dotted lines on the diagram for part **d** show minimum cuts; they both only pass through saturated arcs. The value of the right-hand cut passing through ER_1, BG, CG, CR_2 and CR_3 is 32 + 8 + 25 + 7 + 10 = 82. As the minimum cut value equals the maximum flow value, so by the maximum flow–minimum cut theorem the flow is maximal.

31 a Add a super source S creating arcs SA and SC and a super sink T creating arcs GT and IT.

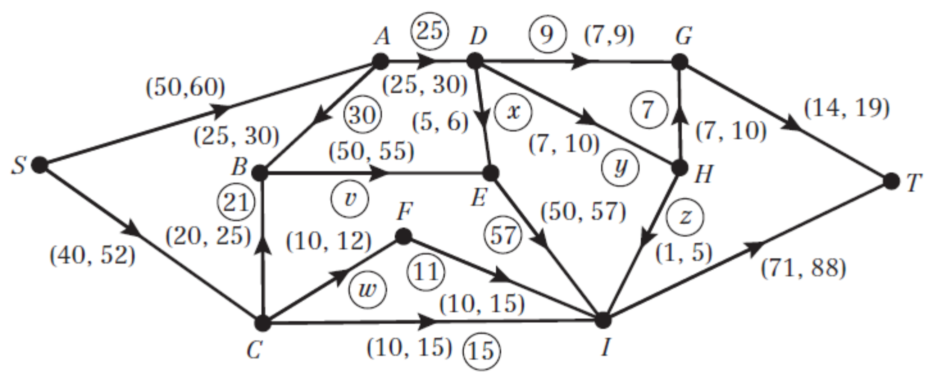
Minimum flow into A = total minimum flow out of A = 25 + 25 = 50
Maximum flow into A = total maximum flow out of A = 30 + 30 = 60

Minimum flow into C = total minimum flow out of C = 20 + 10 + 10 = 40
Maximum flow into C = total maximum flow out of C = 25 + 12 + 15 = 52

Minimum flow out of G = total minimum flow into G = 7 + 7 = 14
Maximum flow out of G = total maximum flow into G = 9 + 10 = 19

Minimum flow out of I = total minimum flow into I = 1 + 50 + 10 + 10 = 71
Maximum flow out of I = total maximum flow into I = 5 + 57 + 11 + 15 = 88

The network diagram becomes:



31 b Using the conservation flow condition, flow into B = flow out of B
$30+21=v \Rightarrow v=51$

Flow into F = flow out of F $\Rightarrow w=11$

Flow into E = flow out of E
$x+v=57 \Rightarrow x=57-v=57-51=6$

Flow into D = flow out of D
$25=9+x+y \Rightarrow y=25-9-x=16-6=10$

Flow into H = flow out of H
$y=7+z \Rightarrow z=y-7=10-7=3$

c Flow through the network = flow leaving the sources = 25 +30 + 21 + w + 15 = 102
Flow through the network = flow entering the sinks = 9 + 7 + z + 57 + 11 + 15 = 102

d The cut through DG, DH, EI, FI and CI must be a minimum cut because it only passes through saturated arcs. Its value is 9 + 10 + 57 + 11 + 15 = 102. As the minimum cut value equals the maximum flow value, by the maximum flow–minimum cut theorem the flow of 102 is maximal.

Note ignore the super source and super sink nodes when determining the cut.

32 a Using the conservation flow condition, flow into A = flow out of A $\Rightarrow w=18+12=30$
Flow into C = flow out of C $\Rightarrow x=21$
Flow into F = flow out of F $\Rightarrow y=18+7+20=45$
Flow into G = flow out of G $\Rightarrow 32+10=20+z \Rightarrow z=22$

Flow through the network = flow leaving the source = 30 + 16 + 21 = 67

The cut through AF, DF, GF and GT must be a minimum cut because it only passes through saturated arcs. Its value is 18 + 7 + 20 + 22 = 67. As the minimum cut value equals the maximum flow value, by the maximum flow–minimum cut theorem the flow of 67 is maximal.

32 b As BE is already saturated the maximum flow entering B after D is closed can be 11 as this is the amount leaving B. Therefore, set the flow along AB to 11 and the flow along SB to 0.

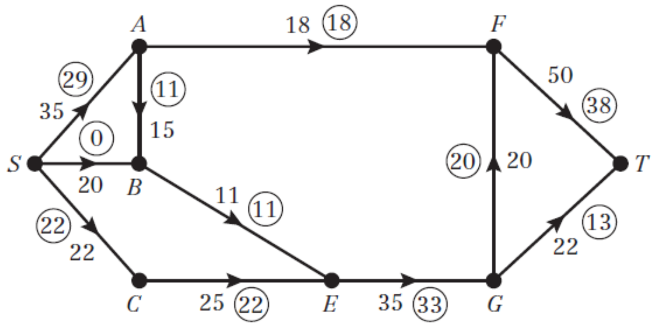
AF is saturated with a flow of 18. The flow leaving A is 11 + 18 = 29, which will equal the flow along SA.

A flow of 22 can be sent along SC, which will also flow along CE. The flow along EG will equal the sum of the flows arriving from BE and CE = 11 + 22 = 33.

A flow of 20 can be sent along GF, which will leave a flow of 33 − 20 = 13 along GT.

The flow along FT will equal the sum of the flows arriving from AF and GF = 18 + 20 = 38.

This represents one possible flow through the network if the junction at D is closed for maintenance. This the diagram representing this network:



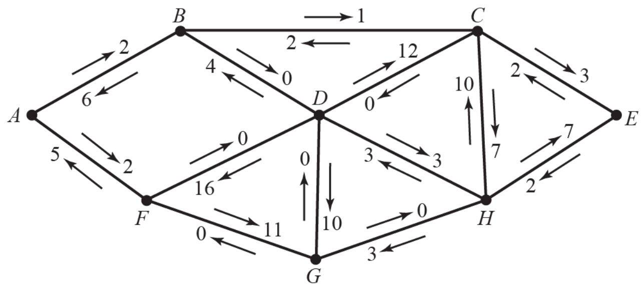
c Flow through the network = flow leaving the source = 29 + 0 + 22 = 51

The cut through AF, BE and SC must be a minimum cut because it only passes through saturated arcs. Its value is 18 + 11 + 22 = 51. As the minimum cut value equals the maximum flow value, by the maximum flow–minimum cut theorem the flow of 51 is maximal.

33 a A, E and G

b Flow through the network = flow leaving the sources = 6 + 5 + 3+ 7 + 11 + 10 + 3 = 45
Flow through the network = flow into sink (D) = 4 + 12 + 3 + 10 + 16 = 45

33 c Applying the labelling procedure to the initial flow gives:



Four of the arcs into sink *D* are saturated (*BD*, *CD*, *GD* and *FD*) and there is only arc *HD* with any spare capacity.

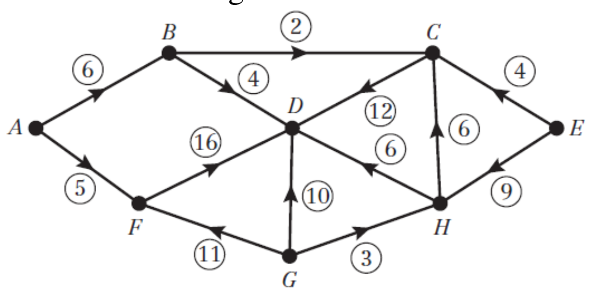
Choose *EHD* as a flow augmenting route. An additional flow of 2 can be sent along this route because *EH* has a maximum spare capacity of 2.

Choose *ECHD* as a flow augmenting route. An additional flow of 1 can be sent along this route because *HD* has a maximum spare capacity of 1.

There are no further flow-augmenting routes, as all arcs into sink *D* are now saturated, so the flow is maximal.

(Note: the paths can be reversed, an additional flow of 2 can be sent down *ECHD* and then an additional flow of 1 can be sent down *EHD*.)

d The final flow diagram is:



The maximal flow is 48 (flow into $D = 4 + 12 + 6 + 10 + 16 = 48$)

e The cut through *DB*, *DC*, *DH*, *DG* and *DF* must be a minimum cut because it only passes through saturated arcs. Its value is $4 + 12 + 6 + 10 + 16 = 48$. As the minimum cut value equals the maximum flow value, by the maximum flow–minimum cut theorem the flow of 48 is maximal.

f If the flow through junction *C* is restricted to 8, then the flow along *CD* will reduce from 12 to 8.

All arcs entering the sink *D* other than *HD* are saturated. If the flow along *HD* is increased from 3 to 6 to make it saturated this would give the maximal flow into the sink.

As the flow leaving junction *C* is 8, the flow entering junction *C* must also be 8. *BC* is already saturated at 2, therefore if a flow of 4 is sent along *EC* and 2 along *HC*, there will be a flow of $2 + 4 + 2 = 8$ entering junction *C*.

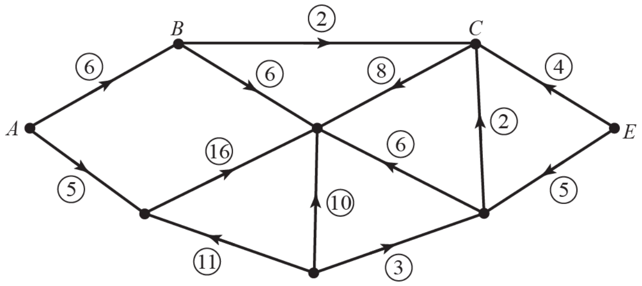
Now ensure that the flow entering junction *H* equals the flow leaving junction *H*. Set the flow along *EH* to 5, then the flow entering junction *H* is $5 + 3 = 8$ and the flow leaving junction *H* is $6 + 2 = 8$. All other arcs remain the same. This ensures that all 5 arcs entering the sink are saturated to give a maximal flow of $4 + 8 + 6 + 10 + 16 = 44$

33 g As the flow along *EC* must be unchanged (at 3), look for an alternative solution to part **f**.

As the flow leaving junction *C* is 8, the flow entering junction *C* must also be 8. *BC* is already saturated at 2, and the flow along *EC* is fixed at 3 so send 3 along *HC*, there will be a flow of $2 + 3 + 3 = 8$ entering junction *C*.

Now to ensure that the flow entering junction *H* equals the flow leaving junction *H*, set the flow along *EH* to 6, then the flow entering junction *H* is $6 + 3 = 9$ and the flow leaving junction *H* is 6 (along *HD*, ensuing *HD* is saturated) + 3 (along *HC* as required) = 9. All other arcs remain the same. This ensures that all 5 arcs entering the sink are saturated to give a maximal flow of $4 + 8 + 6 + 10 + 16 = 44$

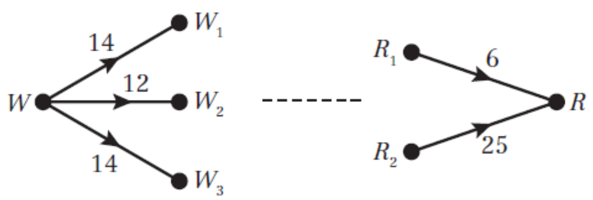
The flow diagram showing this maximum flow is:



34 a Add a super source *W* creating arcs WW_1, WW_2 and WW_3 and a super sink *R* creating arcs R_1R and R_2R.

Minimum capacity of $WW_1 = 8 + 6 = 14$
Minimum capacity of $WW_2 = 12$
Minimum capacity of $WW_3 = 3 + 11 = 14$
Minimum capacity of $R_1R = 6$
Minimum capacity of $R_2R = 9 + 16 = 25$

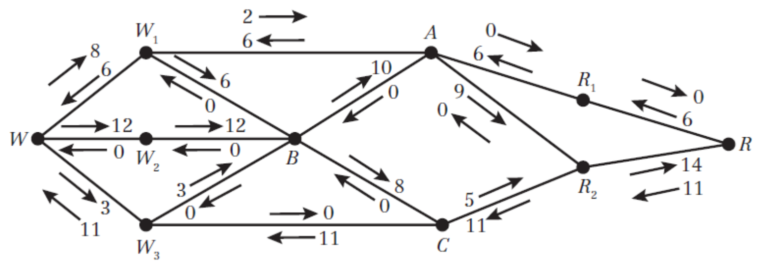
The relevant additions to the network diagram are shown here:



34 b i The maximum flow along WW_1AR_1R is 6, as the arc with the lowest capacity of the route AR_1 has a capacity of 6

ii The maximum flow along WW_3CR_2R is 11, as the arc with the lowest capacity of the route W_1C has a capacity of 11

c Applying the labelling procedure to the initial flow gives:



There are several flow augmenting routes that could be chosen. This answer illustrates one set of options.

Choose WW_1BAR_2R as a flow augmenting route. An additional flow of 6 can be sent along this route because W_1B has a maximum spare capacity of 6

Then use WW_1AR_2R as a flow augmenting route. An additional flow of 2 can be sent along this route because W_1A has a maximum spare capacity of 2

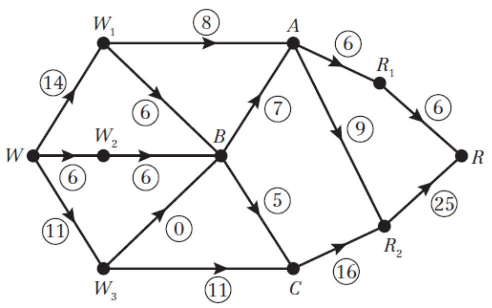
Then use WW_2BCR_2R as a flow augmenting route. An additional flow of 5 can be sent along this route because CR_2 has a maximum spare capacity of 5

Finally use WW_2BAR_2R as a flow augmenting route. An additional flow of 1 can be sent along this route because AR_2 has a maximum spare capacity of 1

There are no further flow-augmenting routes, as all arcs into R_1 and R_2 are now saturated, so the flow is maximal.

This maximal flow is the initial flow of 6 + 11 = 17, together with the augmented flows of 6 + 2 + 5 + 1 = 14, i.e. 17 + 14 = 31

The flow diagram showing this maximum flow is:



d There are 12 van loads passing through B each day for the final flow network shown in part **c**.

(Note: there are other solutions to part **c** that also generate a maximal flow of 31 in which different numbers of vans pass through B.)

34 e The company can make no use of this opportunity. All arcs out of A and C are saturated, so the total flow cannot be increased unless the number of van loads from A or C to R_1 or R_2 is increased.

Challenge

1 a The problem is unbalanced as the total supply (140 + 300 + 200 = 640 boxes) is not equal to the total demand (120 + 160 + 150 + 220 = 650 boxes). As demand is greater than supply, a dummy supply point must be added to balance the problem.

Let D be the dummy supply point. Any boxes supplied from D will be unmet demand and so will incur the penalty payments set out in the question. The balanced problem therefore becomes:

	P	***Q***	***R***	***S***	**Supply**
A	2	6	3	5	140
B	4	4	7	3	300
C	4	8	5	2	200
D	3	1	3	4	10
Demand	120	160	150	220	

b Applying the north-west corner method to obtain an initial solution gives:

	P	***Q***	***R***	***S***	**Supply**
A	120	20			140
B		140	150	10	300
C				200	200
D				10	10
Demand	120	160	150	220	

The number of occupied cells (7) = number of supply points + number of demand points – 1, so the solution is non-degenerate.

The cost of this solution = $120\times2+20\times6+140\times4+150\times7+10\times3+200\times2+10\times4=£2440$

Now find the shadow costs by arbitrarily setting S(A) = 0 and solving the remaining equations:

D(P) = 2 – S(A) = 2 – 0 = 2

D(Q) = 6 – S(A) = 6 – 0 = 6

S(B) = 4 – D(Q) = 4 – 6 = –2

D(R) = 7 – S(B) = 7 – (–2) = 7 + 2 = 9

D(S) = 3 – S(B) = 3 – (–2) = 3 + 2 = 5

S(C) = 2 – D(S) = 2 – 5 = –3

S(D) = 4 – D(S) = 4 – 5 = –1

The shadow costs in table form are:

		2	6	9	5
		P	***Q***	***R***	***S***
0	***A***	2	6		
–2	***B***		4	7	3
–3	***C***				2
–1	***D***				4

Challenge
1 b (continued)

The improvement indices are:

$I_{AR} = C(AR) - S(A) - D(R) = 3 - 0 - 9 = -6$
$I_{AS} = C(AS) - S(A) - D(S) = 5 - 0 - 5 = 0$
$I_{BP} = C(BP) - S(B) - D(P) = 4 - (-2) - 2 = 4$
$I_{CP} = C(CP) - S(C) - D(P) = 4 - (-3) - 2 = 5$
$I_{CQ} = C(CQ) - S(C) - D(Q) = 8 - (-3) - 6 = 5$
$I_{CR} = C(CR) - S(C) - D(R) = 5 - (-3) - 9 = -1$
$I_{DP} = C(DP) - S(D) - D(P) = 3 - (-1) - 2 = 2$
$I_{DQ} = C(DQ) - S(D) - D(Q) = 1 - (-1) - 6 = -4$
$I_{DR} = C(DR) - S(D) - D(R) = 3 - (-1) - 9 = -5$

As there are negative improvement indices, the solution is not optimal.

First iteration
The entering cell is AR as it has the most negative value. Entering θ into cell AR and applying the stepping-stone method:

	P	***Q***	***R***	***S***
A	120	$20 - \theta$	θ	
B		$140 + \theta$	$150 - \theta$	10
C				200
D				10

The maximum value of θ is 20, making AQ the exiting cell. This is the improved solution.

	P	***Q***	***R***	***S***
A	120		20	
B		160	130	10
C				200
D				10

The cost of this solution $= 120 \times 2 + 20 \times 3 + 160 \times 4 + 130 \times 7 + 10 \times 3 + 200 \times 2 + 10 \times 4 = £2320$

The shadow costs of the new solution are:

		2	0	3	–1
		P	***Q***	***R***	***S***
0	***A***	2		3	
4	***B***		4	7	3
3	***C***				2
5	***D***				4

The improvement indices are:

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 0 = 6$
$I_{AS} = C(AS) - S(A) - D(S) = 5 - 0 - (-1) = 6$
$I_{BP} = C(BP) - S(B) - D(P) = 4 - 4 - 2 = -2$
$I_{CP} = C(CP) - S(C) - D(P) = 4 - 3 - 2 = -1$
$I_{CQ} = C(CQ) - S(C) - D(Q) = 8 - 3 - 0 = 5$
$I_{CR} = C(CR) - S(C) - D(R) = 5 - 3 - 3 = -1$
$I_{DP} = C(DP) - S(D) - D(P) = 3 - 5 - 2 = -4$
$I_{DQ} = C(DQ) - S(D) - D(Q) = 1 - 5 - 0 = -4$
$I_{DR} = C(DR) - S(D) - D(R) = 3 - 5 - 3 = -5$

Challenge
1 b (continued)

Second iteration
As there are negative improvement indices, the solution is not optimal. The entering cell is *DR* as it has the most negative value. Entering θ into cell *DR* and applying the stepping-stone method:

	P	***Q***	***R***	***S***
A	120		20	
B		160	$130 - \theta$	$10 + \theta$
C				200
D			θ	$10 - \theta$

The maximum value of θ is 10, making *DS* the exiting cell. This is the improved solution.

	P	***Q***	***R***	***S***
A	120		20	
B		160	120	20
C				200
D			10	

The cost of this solution = $120\times2+20\times3+160\times4+120\times7+20\times3+200\times2+10\times3=£2270$

The shadow costs of the new solution are:

		2	0	3	–1
		P	***Q***	***R***	***S***
0	***A***	2		3	
4	***B***		4	7	3
3	***C***				2
0	***D***			3	

The improvement indices are:

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 0 = 6$
$I_{BP} = C(BP) - S(B) - D(P) = 4 - 4 - 2 = -2$
$I_{CQ} = C(CQ) - S(C) - D(Q) = 8 - 3 - 0 = 5$
$I_{DP} = C(DP) - S(D) - D(P) = 3 - 0 - 2 = 1$
$I_{DS} = C(DS) - S(D) - D(S) = 4 - 0 - (-1) = 5$

$I_{AS} = C(AS) - S(A) - D(S) = 5 - 0 - (-1) = 6$
$I_{CP} = C(CP) - S(C) - D(P) = 4 - 3 - 2 = -1$
$I_{CR} = C(CR) - S(C) - D(R) = 5 - 3 - 3 = -1$
$I_{DQ} = C(DQ) - S(D) - D(Q) = 1 - 0 - 0 = 1$

As there are negative improvement indices, the solution is not optimal.

Challenge

1 b (continued)

Third iteration

The entering cell is BP as it has the most negative value. Entering θ into cell BP and applying the stepping-stone method:

	P	***Q***	***R***	***S***
A	$120-\theta$		$20+\theta$	
B	θ	160	$120-\theta$	20
C				200
D			10	

The maximum value of θ is 120, making either AP or BR the exiting cell. Choose AP as the exiting cell, but BR must have the value 0 (and not be blank) otherwise the solution would be degenerate. This is the improved solution.

	P	***Q***	***R***	***S***
A			140	
B	120	160	0	20
C				200
D			10	

The cost of this solution = $140\times3+120\times4+160\times4+0\times7+20\times3+200\times2+10\times3=£2030$

The shadow costs of the new solution are:

		0	0	3	–1
		P	***Q***	***R***	***S***
0	***A***			3	
4	***B***	4	4	7	3
3	***C***				2
0	***D***			3	

The improvement indices are:

$I_{AP} = C(AP) - S(A) - D(P) = 2 - 0 - 0 = 2$

$I_{AS} = C(AS) - S(A) - D(S) = 5 - 0 - (-1) = 6$

$I_{CQ} = C(CQ) - S(C) - D(Q) = 8 - 3 - 0 = 5$

$I_{DP} = C(DP) - S(D) - D(P) = 3 - 0 - 0 = 3$

$I_{DS} = C(DS) - S(D) - D(S) = 4 - 0 - (-1) = 5$

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 0 = 6$

$I_{CP} = C(CP) - S(C) - D(P) = 4 - 3 - 0 =$

$I_{CR} = C(CR) - S(C) - D(R) = 5 - 3 - 3 = -1$

$I_{DQ} = C(DQ) - S(D) - D(Q) = 1 - 0 - 0 = 1$

As there is a negative improvement index, the solution may not be optimal.

Challenge

1 b (continued)

Fourth iteration

The entering cell is CR as it has a negative value. Entering θ into cell CR and applying the stepping-stone method:

	P	***Q***	***R***	***S***
A			140	
B	120	160	$0 - \theta$	$20 + \theta$
C			θ	$200 - \theta$
D			10	

The maximum value of θ is 0, making BR the exiting cell.

	P	***Q***	***R***	***S***
A			140	
B	120	160		20
C			0	200
D			10	

The cost of this solution = $140\times3+120\times4+160\times4+0\times5+20\times3+200\times2+10\times3=£2030$

The shadow costs of the new solution are:

		1	1	3	0
		P	***Q***	***R***	***S***
0	***A***			3	
3	***B***	4	4		3
2	***C***			5	2
0	***D***			3	

The improvement indices are:

$I_{AP} = C(AP) - S(A) - D(P) = 2 - 0 - 1 = 1$
$I_{AS} = C(AS) - S(A) - D(S) = 5 - 0 - 0 = 5$
$I_{CP} = C(CP) - S(C) - D(P) = 4 - 2 - 1 = 1$
$I_{DP} = C(DP) - S(D) - D(P) = 3 - 0 - 1 = 2$
$I_{DS} = C(DS) - S(D) - D(S) = 4 - 0 - 0 = 4$

$I_{AQ} = C(AQ) - S(A) - D(Q) = 6 - 0 - 1 = 5$
$I_{BR} = C(BR) - S(B) - D(R) = 7 - 3 - 3 = 1$
$I_{CQ} = C(CQ) - S(C) - D(Q) = 8 - 2 - 1 = 5$
$I_{DQ} = C(DQ) - S(D) - D(Q) = 1 - 0 - 1 = 0$

As there are no negative improvement indices, the solution of 140 boxes A to R, 120 boxes B to P, 160 boxes B to Q, 20 boxes B to S, 200 boxes C to S with 10 boxes of unmet demand at R is optimal. The cost of this solution is £2030.

Note that as there is an improvement index of zero (cell DQ), an alternative optimal solution exists. This can be found by using DQ as the entering cell and carrying out another iteration.

Challenge

2 Worker A should be selected as the worker that works on two tasks as this worker has the lowest costs associated with two tasks (1 and 3) and no other worker has the lowest costs for two tasks. To allocate two tasks to worker A, we replicate the costs associated with worker A to worker E, thus ensuring a $n \times n$ matrix to apply the Hungarian algorithm as follows:

	1	**2**	**3**	**4**	**5**
A	53	62	41	55	68
B	59	55	57	46	60
C	62	58	–	40	62
D	–	59	64	49	58
E	53	62	41	55	68

Replacing the '–' forbidden allocations with the value of infinity (100) makes the assignments 'unattractive'. Therefore, the matrix becomes:

	1	**2**	**3**	**4**	**5**
A	53	62	41	55	68
B	59	55	57	46	60
C	62	58	100	40	62
D	100	59	64	49	58
E	53	62	41	55	68

The smallest numbers in rows 1, 2, 3, 4 and 5 are 41, 46, 40, 49 and 41. Reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	1	**2**	**3**	**4**	**5**
A	12	21	0	14	27
B	13	9	11	0	14
C	22	18	60	0	22
D	51	10	15	0	9
E	12	21	0	14	27

The smallest numbers in columns 1, 2, 3, 4 and 5 are 12, 9, 0, 0 and 9. Reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	1	**2**	**3**	**4**	**5**
A	0	12	0	14	18
B	1	0	11	0	5
C	10	9	60	0	13
D	39	1	15	0	0
E	0	12	0	14	18

It requires five lines to cover the zeros, so the solution is optimal. Worker C must be allocated task 4 (as there is only one zero in row C); so worker B must be allocated task 2 and worker D task 5. Worker A (E) is allocated tasks 1 and 3.
The minimum cost is 53 + 55 + 41 + 40 + 58 = £247

Challenge

3 a Flow through the network = flow leaving the source = 43 + 61 + 56 = 160
Flow through the network = flow entering the sink = 70 + 90 = 160

The cut through *AD*, *BD*, *BE* and *CF* is a minimum cut as these are all saturated arcs.
The value of the minimum cut = 25 + 29 + 28 + 78 = 160

As the minimum cut value equals the maximum flow value, the flow is maximal, as proved by the maximum flow – minimum cut theorem.

b Of the three arcs leaving the source, the flow through arc *SC* can be increased by the most (by 9 from 56 to 65). Therefore, giving arc *CF* unlimited capacity would allow a flow of 65 + 22 = 87 along it. This would increase the flow along arc *FT* to 99 (87 + 12) and the total flow entering the sink to 169 (99 + 70). Therefore, upgrade *CF* to have maximum effect – an increased flow of 9.

c The two arcs that can be increased the most in terms of flow are *ED*, from its current value of 16 up to a maximum of 80 and arc *EF*, from its current value of 12 up to a maximum of 50. The maximum flow that can be sent along arc *ED* is 100 – (25 + 29) = 46 and the maximum flow that can be sent along arc *EF* is 120 – 78 = 42. So if arcs *SB* and *BE* have unlimited capacity, then arcs *ED* and *EF* can be increased by 46 and 42 respectively. This increases the flow along *DT* to 100 (25 + 29 + 46) and the flow along arc *FT* to 120 (42 + 78). Therefore, the flow has increased by 60 from 160 to 220.