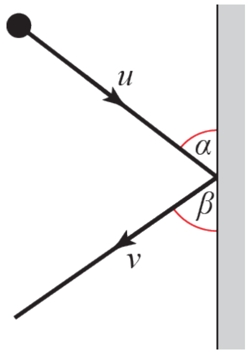
Elastic collisions in two dimensions 5A

1



a $e = \frac{1}{3}$, $\tan\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (from Pythagoras' theorem)

For motion parallel to the wall:

$v\cos\beta = u\cos\alpha \;\Rightarrow v\cos\beta = \frac{4}{5}u$ **(1)**

For motion perpendicular to the wall:

$v\sin\beta = eu\sin\alpha$

$v\sin\beta = \frac{1}{3}\times u\times\frac{3}{5} = \frac{1}{5}u$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$$v^2\cos^2\beta + v^2\sin^2\beta = \frac{16}{25}u^2 + \frac{1}{25}u^2$$

$$v^2(\cos^2\beta + \sin^2\beta) = \frac{17}{25}u^2$$

$$v = \frac{\sqrt{17}u}{5}$$

b Dividing equation **(2)** by equation **(1)** gives:

$$\tan\beta = \frac{\frac{1}{5}u}{\frac{4}{5}u} = \frac{1}{4} = 0.25$$

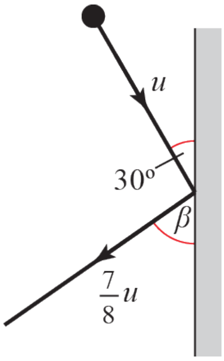
$\beta = 14.04$ (2 d.p.)

$$\tan\alpha = \frac{3}{4} = 0.75$$

$\alpha = 36.87°$ (2 d.p.)

Angle of deflection $= \alpha + \beta = 36.87 + 14.04 = 50.9°$ (1 d.p.)

2



For motion parallel to the wall:

$$\frac{7u}{8}\cos\beta = u\cos 30°$$

$$\frac{7u}{8}\cos\beta = \frac{\sqrt{3}u}{2} \qquad \textbf{(1)}$$

For motion perpendicular to the wall:

$$\frac{7u}{8}\sin\beta = eu\sin 30°$$

$$\frac{7u}{8}\sin\beta = \frac{eu}{2} \qquad \textbf{(2)}$$

Squaring and adding equations **(1)** and **(2)** gives:

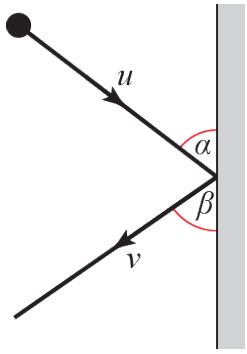
$$\frac{49}{64}u^2\cos^2\beta + \frac{49}{64}u^2\sin^2\beta = \frac{3}{4}u^2 + \frac{1}{4}e^2u^2$$

$$\frac{49}{64}u^2(\cos^2\beta + \sin^2\beta) = u^2\left(\frac{3}{4} + \frac{e^2}{4}\right)$$

$$\frac{49}{16} = 3 + e^2$$

$$e^2 = \frac{1}{16} \Rightarrow e = \frac{1}{4}$$

3



a $e = \frac{3}{5}$, $\tan\alpha = \frac{5}{12} \Rightarrow \cos\alpha = \frac{12}{13}$ and $\sin\alpha = \frac{5}{13}$ (from Pythagoras' theorem)

For motion parallel to the wall:

$$v\cos\beta = u\cos\alpha \Rightarrow v\cos\beta = \frac{12}{13}u \qquad \textbf{(1)}$$

For motion perpendicular to the wall:

$$v\sin\beta = eu\sin\alpha$$

$$v\sin\beta = \frac{3}{5} \times u \times \frac{5}{12} = \frac{3}{12}u \qquad \textbf{(2)}$$

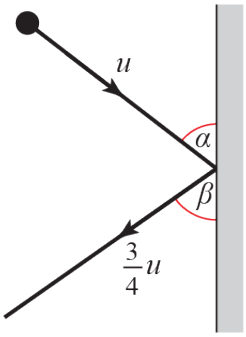
Squaring and adding equations **(1)** and **(2)** gives:

$$v^2\cos^2\beta + v^2\sin^2\beta = \frac{144}{169}u^2 + \frac{9}{169}u^2$$

$$v^2(\cos^2\beta + \sin^2\beta) = \frac{153}{169}u^2$$

$$v = \frac{\sqrt{153}u}{13} = \frac{\sqrt{9 \times 17}u}{13} = \frac{3\sqrt{17}u}{13}$$

4



$\tan\alpha = 2 \Rightarrow \cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$ (from Pythagoras' theorem)

For motion parallel to the wall:

$$\frac{3u}{4}\cos\beta = u\cos\alpha$$

$$\frac{3u}{4}\cos\beta = \frac{u}{\sqrt{5}} \qquad \textbf{(1)}$$

For motion perpendicular to the wall:

$$\frac{3u}{4}\sin\beta = eu\sin\alpha$$

$$\frac{3u}{4}\sin\beta = \frac{2eu}{\sqrt{5}} \qquad \textbf{(2)}$$

Squaring and adding equations **(1)** and **(2)** gives:

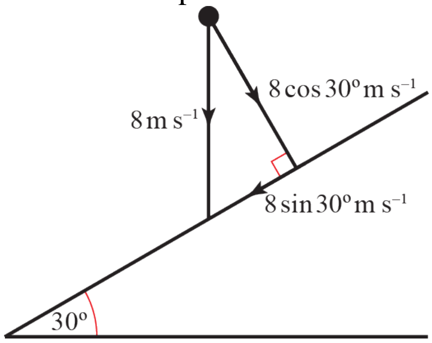
$$\frac{9}{16}u^2\cos^2\beta + \frac{9}{16}u^2\sin^2\beta = \frac{1}{5}u^2 + \frac{4}{5}e^2u^2$$

$$\frac{9}{16}u^2(\cos^2\beta + \sin^2\beta) = u^2\left(\frac{1}{5} + \frac{4e^2}{5}\right)$$

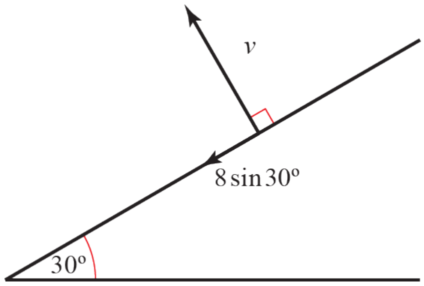
$$\frac{45}{16} = 1 + 4e^2$$

$$e^2 = \frac{29}{64} \Rightarrow e = \frac{\sqrt{29}}{8}$$

5 Before the impact:



After the impact:

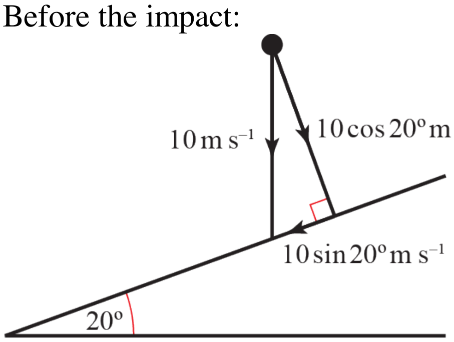


The component of velocity parallel to the slope is $8\sin 30° = 8 \times \frac{1}{2} = 4$

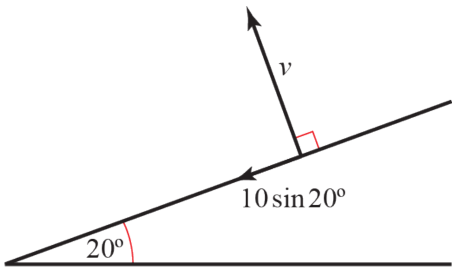
Perpendicular to the slope: $v = e \times 8\cos 30° = \frac{1}{4} \times 8 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Therefore the speed immediately after impact $= \sqrt{4^2 + \sqrt{3}^2} = \sqrt{19}\,\text{m s}^{-1}$

6 Before the impact:



After the impact:



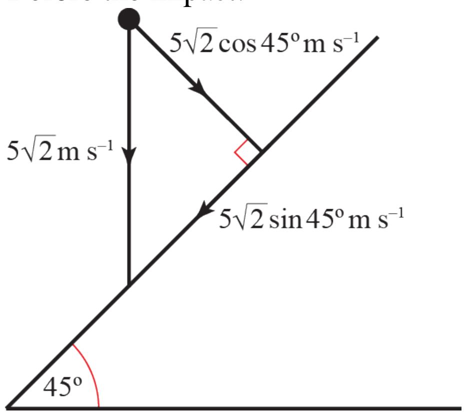
The component of velocity parallel to the slope is $10\sin 20°$

Perpendicular to the slope: $v = e \times 10\cos 20° = \frac{2}{5} \times 10\cos 20° = \cos 20°$

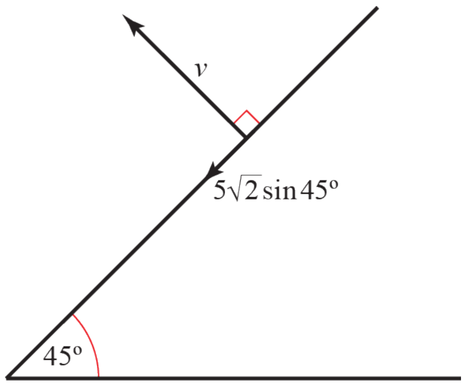
Therefore the speed immediately after impact

$$= \sqrt{(10\sin 20°)^2 + (4\cos 20°)^2} = \sqrt{25.826} = 5.08\,\text{m s}^{-1} \text{ (3 s.f.)}$$

7 a Before the impact:



After the impact:



The component of velocity parallel to the slope is $5\sqrt{2}\sin 45° = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

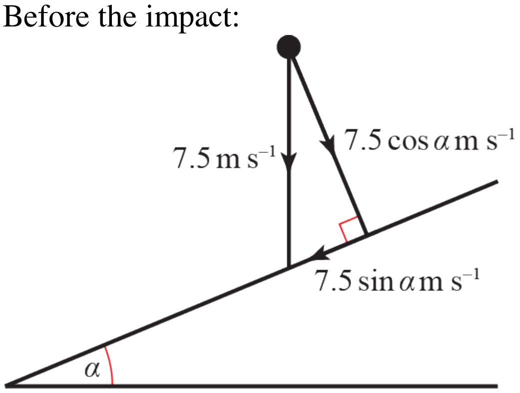
Perpendicular to the slope: $v = \frac{1}{2} \times 5\sqrt{2}\cos 45° = \frac{1}{2} \times 5\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{5}{2} = 2.5$

Therefore the speed immediately after impact $= \sqrt{5^2 + 2.5^2} = \sqrt{31.25} = 5.59\,\text{m s}^{-1}$ (3 s.f.)

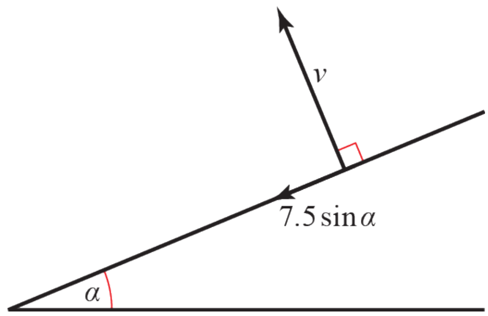
b The impulse is perpendicular to the surface:

$$\text{I} = mv - m(-5\sqrt{2}\cos 45°)$$
$$= \frac{3}{4}\left(\frac{5}{2} - (-5)\right) = \frac{3}{4} \times \frac{15}{2} = \frac{45}{8} = 5.625\,\text{Ns}$$

8 Before the impact:



After the impact:



$\tan\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (from Pythagoras' theorem)

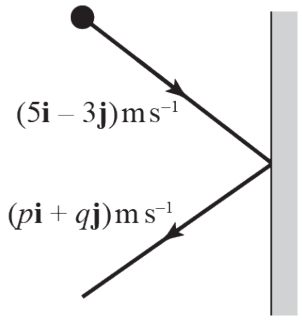
The component of velocity parallel to the slope is $7.5\sin\alpha = 7.5 \times \frac{3}{5} = 4.5$

Perpendicular to the slope: $v = e \times 7.5\cos a = e \times 7.5 \times \frac{4}{5} = 6e$

The speed immediately after impact is $5\,\text{m s}^{-1}$, so

$$5^2 = 4.5^2 + (6e)^2$$
$$25 = 20.25 + 36e^2$$
$$e^2 = \frac{4.75}{36} = 0.13194\ldots$$
$$e = 0.36 \text{ (2 s.f.)}$$

9



a Let the velocity v of the ball immediately after impact be $p\mathbf{i} + q\mathbf{j}$

Parallel to the wall: $q = -3$

Perpendicular to the wall: $-p = e \times 5 = 2.5$

So $v = (-2.5\mathbf{i} - 3\mathbf{j})\,\text{m s}^{-1}$

b Kinetic energy before impact $= \frac{1}{2} \times 0.8 \times (5^2 + 3^2) = 13.6$

Kinetic energy after impact $= \frac{1}{2} \times 0.8 \times (2.5^2 + 3^2) = 6.1$

Kinetic energy lost $= 13.6 - 6.1 = 7.5$ J

c Using the scalar product to determine the angle of deflection:

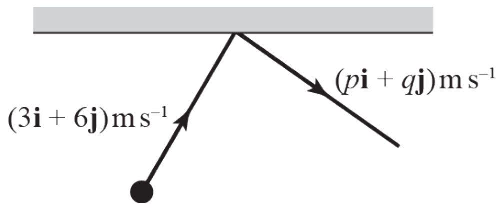
$$\cos\theta = \frac{\mathbf{u.v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta = \frac{5(-2.5) + (-3)(-3)}{\sqrt{5^2 + 3^2}\sqrt{2.5^2 + 3^2}} = \frac{-3.5}{\sqrt{34}\sqrt{15.25}} = \frac{-3.5}{22.771} = -0.15371$$

$$\theta = 98.8° \text{ (3 s.f.)}$$

The angle of deflection is 98.8° (3 s.f.).

10



a Let the velocity v of the ball immediately after impact be $p\mathbf{i} + q\mathbf{j}$

Parallel to the wall: $p = 3$

Perpendicular to the wall: $-q = e \times 6 = 2$

So $v = (3\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$

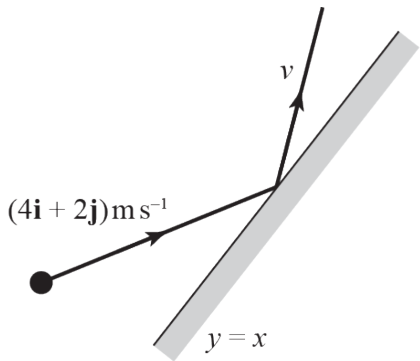
Speed $= \sqrt{3^2 + 2^2} = \sqrt{13}\,\text{m s}^{-1}$

b Kinetic energy before impact $= \frac{1}{2} \times 1 \times (3^2 + 6^2) = 22.5$

Kinetic energy after impact $= \frac{1}{2} \times 1 \times 13 = 6.5$

Kinetic energy lost $= 22.5 - 6.5 = 16$ J

11



a Let v = **a** + **b**, where **a** is parallel to the wall and **b** is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector in the direction of the wall

and $\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

Parallel to the wall:

$$\mathbf{a}=\left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})=\frac{1}{\sqrt{2}}\times 6\times\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})=3\mathbf{i}+3\mathbf{j}$$

Perpendicular to the wall:

$$\mathbf{b}=\frac{1}{3}\left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})=\frac{1}{3}\times\frac{1}{\sqrt{2}}\times(4-2)\times\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})=\frac{1}{3}(-\mathbf{i}+\mathbf{j})$$

So $v=3\mathbf{i}+3\mathbf{j}-\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}=\left(\frac{8}{3}\mathbf{i}+\frac{10}{3}\mathbf{j}\right)\text{m s}^{-1}$

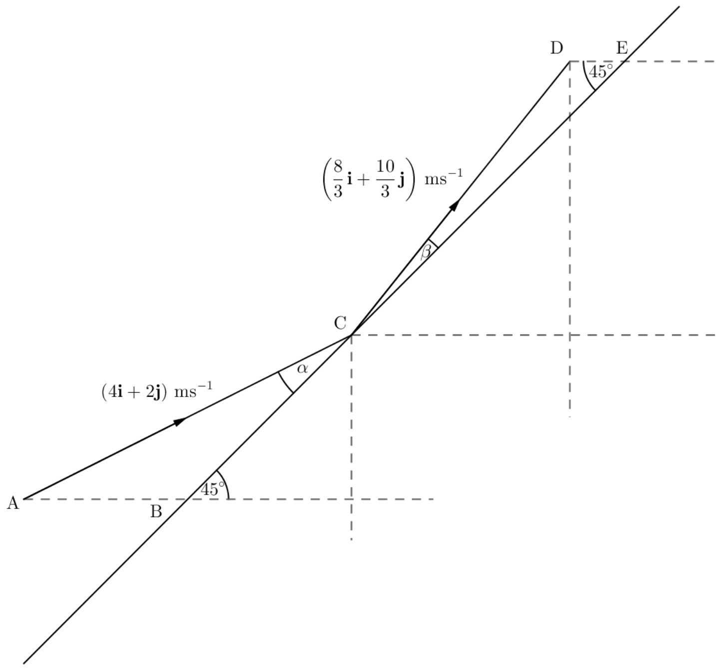
b Kinetic energy before impact $=\frac{1}{2}\times 2\times(4^2+2^2)=20$

Kinetic energy after impact $=\frac{1}{2}\times 2\times\left(\frac{64}{9}+\frac{100}{9}\right)=\frac{164}{9}$

Kinetic energy lost $=20-\frac{164}{9}=\frac{16}{9}$ J

Proportion of kinetic energy lost $=\frac{\frac{16}{9}}{20}\times 100=8.89\%$ (3 s.f.)

11 c

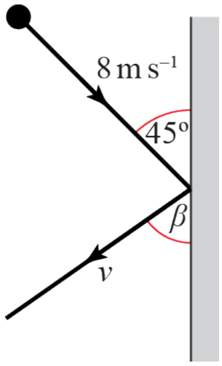


Angle of deflection = $\alpha + \beta$
Consider triangle ABC
Angle $ABC = 135°$
Angle $CAB = \arctan\frac{2}{4} = 26.56505...°$
$\alpha = 180° - 135° - 26.56505...° = 18.43494...°$

Consider triangle CDE
Angle $DEC = 45°$
Angle $CDE = 90° +$ angle $CDF = 90 + \arctan\frac{8}{10} = 128.54980...°$
$\beta = 180° - 45° - 128.54980...° = 6.34019...°$

$\alpha + \beta = 18.43494...° + 6.34019...° = 24.8°$ (3 s.f.)

12



For motion parallel to the cushion:

$$v\cos\beta = 8\cos 45° \Rightarrow v\cos\beta = \frac{8}{\sqrt{2}} \qquad \textbf{(1)}$$

For motion perpendicular to the wall:

$$v\sin\beta = e \times 8\sin 45°$$

$$v\sin\beta = \frac{2}{5}\times 8\times\frac{1}{\sqrt{2}} = \frac{16}{5\sqrt{2}} \qquad \textbf{(2)}$$

Squaring and adding equations **(1)** and **(2)** gives:

$$v^2\cos^2\beta + v^2\sin^2\beta = \frac{64}{2}+\frac{256}{50}$$

$$v^2(\cos^2\beta+\sin^2\beta) = \frac{1856}{50} = 37.12$$

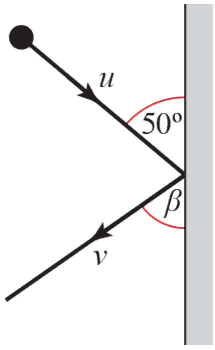
$$v = \sqrt{37.12} = 6.09\,\text{m s}^{-1}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\beta = \frac{16}{5\sqrt{2}}\times\frac{\sqrt{2}}{8} = \frac{2}{5} = 0.4$$

$$\beta = 21.8° \text{ (3 s.f.)}$$

13



a For motion parallel to the cushion:

$v\cos\beta = u\cos 50°$ **(1)**

For motion perpendicular to the cushion:

$v\sin\beta = eu\sin 50°$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:

$\tan\beta = \dfrac{eu\sin 50°}{u\cos 50°} = e\tan 50°$

Hence β is independent of u.

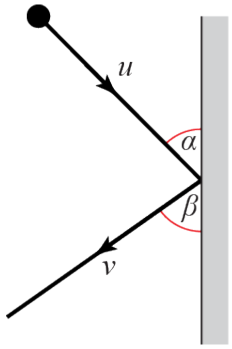
b If the ball rebounds at right angles from its original direction: $\beta = 40°$

From part **a**:

$\tan\beta = \tan 40° = e\tan 50°$

$\Rightarrow e = \dfrac{\tan 40°}{\tan 50°} = 0.704$ (3 s.f.)

14



a $\tan\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (from Pythagoras' theorem)

$\tan\beta = \frac{5}{12} \Rightarrow \cos\beta = \frac{12}{13}$ and $\sin\beta = \frac{5}{13}$ (from Pythagoras' theorem)

For motion parallel to the cushion:

$v\cos\beta = u\cos\alpha \Rightarrow v = \frac{13}{12} \times \frac{4}{5}u = \frac{13}{15}u$

Kinetic energy before impact $= \frac{1}{2}mu^2$

Kinetic energy lost $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2} \times \frac{169}{225}mu^2 = \frac{1}{2}mu^2\left(1 - \frac{169}{225}\right) = \frac{1}{2}mu^2 \times \frac{56}{225}$

Fraction of kinetic energy lost $= \frac{\text{Kinetic energy lost}}{\text{Kinetic energy before impact}} = \frac{56}{225}$

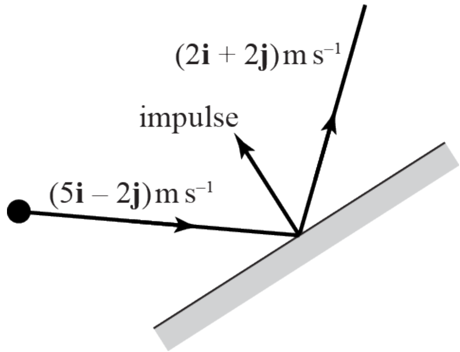
b For motion perpendicular to the cushion:

$v\sin\beta = eu\sin\alpha$

$\frac{13}{15}u \times \frac{5}{13} = eu \times \frac{3}{5}$

$\Rightarrow e = \frac{5 \times 5}{15 \times 3} = \frac{5}{9}$

15



a Impulse = $mv - mu$

$\mathbf{I} = m(2\mathbf{i}+2\mathbf{j}) - m(5\mathbf{i}-2\mathbf{j})$

$\mathbf{I} = m(-3\mathbf{i}+4\mathbf{j})$

$\mathbf{I} = 5m\left(-\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)$

The impulse acts in the direction of the unit vector $\left(-\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right) = \frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

b Component of ($5\mathbf{i} - 2\mathbf{j}$) in the direction of the impulse

$$= \left[(5\mathbf{i}-2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})\right]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-15-8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{-23}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$$

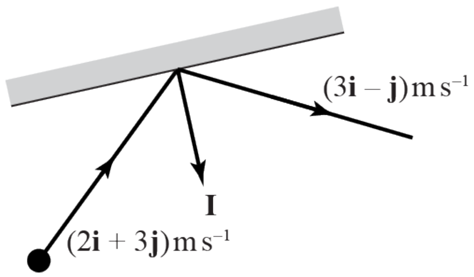
Component of ($2\mathbf{i} + 2\mathbf{j}$) in the direction of the impulse

$$= \left[(2\mathrm{i}+2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})\right]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-6+8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{2}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$$

By Newton's law of restitution:

$\frac{2}{5} = e\times\frac{23}{5} \Rightarrow e = \frac{2}{23}$

16



a Impulse = *mv* – *mu*

$\mathbf{I} = m(3\mathbf{i} - \mathbf{j}) - m(2\mathbf{i} + 3\mathbf{j})$

$\mathbf{I} = 2(\mathbf{i} - 4\mathbf{j})$

The impulse has magnitude $2\sqrt{17}$ N s in the direction parallel to the unit vector $\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})$

b Component of (2**i** + 3**j**) in the direction of the impulse

$$= \left[(2\mathbf{i} + 3\mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})\right]\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{1}{\sqrt{17}}(2 - 12)\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = -\frac{10}{17}(\mathbf{i} - 4\mathbf{j})$$

Component of (3**i** – **j**) in the direction of the impulse

$$= \left[(3\mathbf{i} - \mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})\right]\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{1}{\sqrt{17}}(3 + 4)\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{7}{17}(\mathbf{i} - 4\mathbf{j})$$

By Newton’s law of restitution:

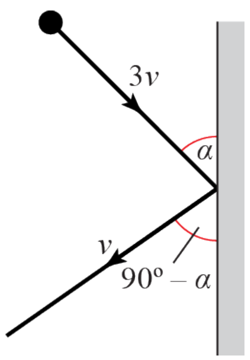
$\frac{7}{17} = e \times \frac{10}{17} \Rightarrow e = \frac{7}{10} = 0.7$

c Kinetic energy before impact $= \frac{1}{2} \times 2 \times (2^2 + 3^2) = 13$

Kinetic energy after impact $= \frac{1}{2} \times 2 \times (3^2 + 1^2) = 10$

Kinetic energy lost $= 13 - 10 = 3$ J

17



a For motion parallel to the wall:

$v\cos(90° - \alpha) = 3v\cos\alpha$

$\Rightarrow \sin\alpha = 3\cos\alpha$ (as $\cos(90° - \alpha) = \sin\alpha$)

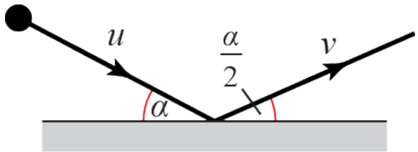
$\Rightarrow \tan\alpha = 3$

17 b For motion perpendicular to the wall:

$$v\sin(90° - \alpha) = eu\sin\alpha$$

$$\Rightarrow e = \frac{v\cos\alpha}{3v\sin\alpha} = \frac{1}{3\tan\alpha} = \frac{1}{9}$$

18



For motion parallel to wall:

$$v\cos\frac{\alpha}{2} = u\cos\alpha \qquad \textbf{(1)}$$

For motion perpendicular to wall:

$$v\sin\frac{\alpha}{2} = eu\sin\alpha = 0.4u\sin\alpha \qquad \textbf{(2)}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\frac{\alpha}{2} = 0.4\tan\alpha \qquad \textbf{(3)}$$

Using the identity:

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \text{ with } \theta = \phi = \frac{\alpha}{2}$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}}$$

Writing $\tan\frac{\alpha}{2}$ as t, and substituting into equation **(3)**

$$t = 0.4 \times \frac{2t}{1 - t^2} \qquad \text{if } t \neq 0$$

$$1 - t^2 = 0.8$$

$$t^2 = 0.2$$

So $t = \tan\frac{\alpha}{2} = \sqrt{0.2}$

$$\Rightarrow \frac{\alpha}{2} = 24.09° \Rightarrow \alpha = 48.2° \text{ (3 s.f.)}$$

Challenge
Let angle of rebound from W_1 be β and let the point where W_1 meets W_2 be O

For motion parallel to wall W_1:
$v\cos\beta = u\cos\alpha$ **(1)**

For motion perpendicular to wall W_1:
$v\sin\beta = eu\sin\alpha$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\beta = e\tan\alpha \Rightarrow e\tan\alpha = \frac{OQ}{1}$$

Using Pythagoras' theorem:

$$PQ^2 = OQ^2 + 1^2$$
$$PQ^2 = e^2\tan^2\alpha + 1$$

So $PQ = \sqrt{e^2\tan^2\alpha + 1}$