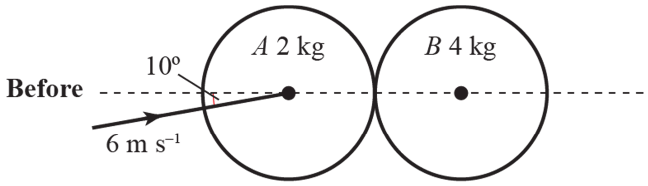
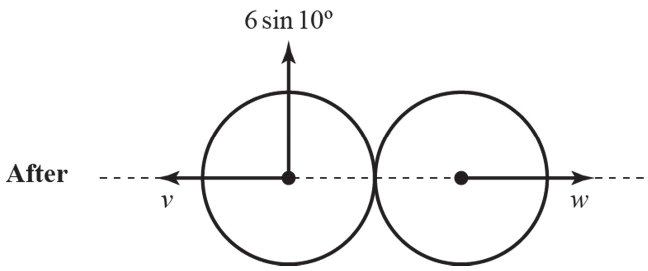
Elastic collisions in two dimensions 5C

1





No change in component of velocity perpendicular to line of centres.
So component of velocity for $A = 6\sin 10°$
Since B is stationary before impact, it will be moving along the line of centres.

Conservation of momentum along the line of centres gives:

$2 \times 6\cos 10° = 4w - 2v$

$\Rightarrow 2w - v = 6\cos 10°$ **(1)**

Using Newton's law of restitution:

$w + v = \frac{1}{2} \times 6\cos 10° = 3\cos 10°$ **(2)**

Adding equations **(1)** and **(2)** gives:

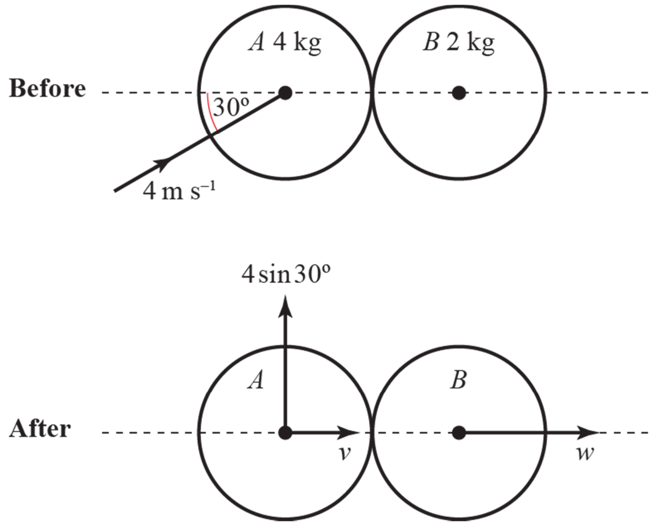
$3w = 9\cos 10° \Rightarrow w = 3\cos 10°$

Substituting in equation **(2)** gives:

$v = 0$

So, after the impact, A has velocity $6\sin 10° = 1.04\,\text{m s}^{-1}$ (3 s.f.) perpendicular to the line of centres, and B has velocity $3\cos 10° = 2.95\,\text{m s}^{-1}$ (3 s.f.) parallel to the line of centres.

2



No change in component of velocity perpendicular to line of centres.
So component of velocity for $A = 4\sin 30° = 2$
Since B is stationary before impact, it will be moving along the line of centres.

Conservation of momentum along the line of centres gives:

$$4 \times 4\cos 30° = 4v + 2w$$

$$\Rightarrow w + 2v = 8\cos 30° \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$w - v = \frac{1}{3} \times 4\cos 30° = \frac{4}{3}\cos 30° \qquad \textbf{(2)}$$

Adding equation **(1)** and 2 × equation **(2)** gives:

$$3w = \left(8 + \frac{8}{3}\right)\cos 30° \Rightarrow w = \frac{32}{9} \times \frac{\sqrt{3}}{2} = \frac{16\sqrt{3}}{9}$$

Substituting in equation **(2)** gives:

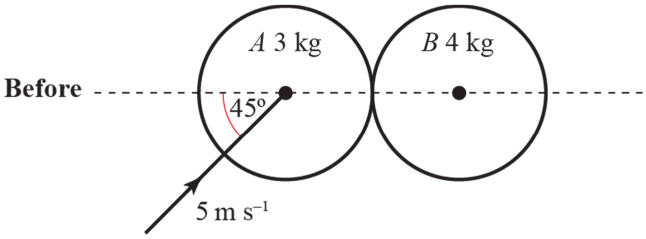
$$\frac{16\sqrt{3}}{9} - v = \frac{4}{3} \times \frac{\sqrt{3}}{2} \Rightarrow v = \frac{10\sqrt{3}}{9}$$

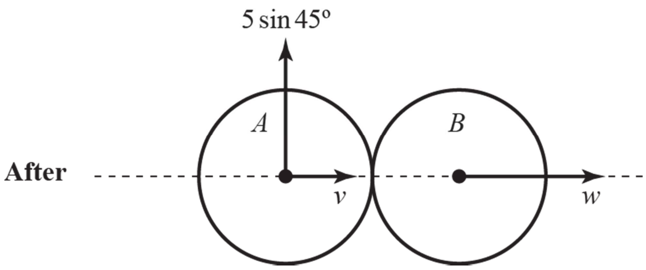
A has speed $\sqrt{(2)^2 + \left(\frac{10\sqrt{3}}{9}\right)^2} = \sqrt{4 + \frac{100}{27}} = \sqrt{\frac{208}{27}} = \frac{4\sqrt{13}}{3\sqrt{3}} = \frac{4\sqrt{39}}{9} \text{ m s}^{-1}$

A is moving at $\arctan\left(\frac{2}{10\frac{\sqrt{3}}{9}}\right) = \arctan\left(\frac{18}{10\sqrt{3}}\right) = \arctan\left(\frac{3\sqrt{3}}{5}\right) = 46.1°$ (3 s.f.) to the line of centres

B has speed $\frac{16\sqrt{3}}{9} \text{ m s}^{-1}$ along the line of centres

3





No change in component of velocity perpendicular to line of centres.

So component of velocity for $A = 5\sin 45° = \frac{5\sqrt{2}}{2}$

Since B is stationary before impact, it will be moving along the line of centres.

Conservation of momentum along the line of centres gives:

$3 \times 5\cos 45° = 3v + 4w$

$\Rightarrow 4w + 3v = \frac{15\sqrt{2}}{2}$ **(1)**

Using Newton's law of restitution:

$w - v = \frac{1}{2} \times 5\cos 45° = \frac{5\sqrt{2}}{4}$ **(2)**

Adding equation **(1)** and 3 × equation **(2)** gives:

$7w = \frac{15\sqrt{2}}{2} + \frac{15\sqrt{2}}{4} \Rightarrow w = \frac{45\sqrt{2}}{28}$

Substituting in equation **(2)** gives:

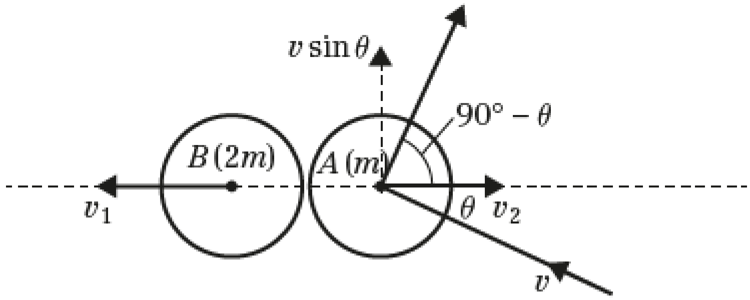
$\frac{45\sqrt{2}}{28} - v = \frac{5\sqrt{2}}{4} \Rightarrow v = \frac{10\sqrt{2}}{28} = \frac{5\sqrt{2}}{14}$

A has speed $\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{14}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{1 + \left(\frac{1}{7}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{\frac{50}{49}} = \frac{5\sqrt{25}}{7} = \frac{25}{7}\,\text{m s}^{-1}$

A is moving at $\arctan\left(\frac{\frac{5\sqrt{2}}{2}}{\frac{5\sqrt{2}}{14}}\right) = \arctan 7 = 81.9°$ (3 s.f.) to the line of centres

B has speed $\frac{45\sqrt{2}}{28}\,\text{m s}^{-1}$ along the line of centres

4



No change in component of velocity perpendicular to line of centres.
So for A, the component perpendicular to the line of centres is $v\sin\theta$.

Conservation of momentum along the line of centres gives:

$mv\cos\theta = 2mv_1 - mv_2 \Rightarrow v\cos\theta = 2v_1 - v_2$ **(1)**

Using Newton's law of restitution:

$v_1 + v_2 = ev\cos\theta \Rightarrow v_1 = ev\cos\theta - v_2$ **(2)**

Substituting for v_1 in equation **(1)** gives:

$v\cos\theta = 2(ev\cos\theta - v_2) - v_2 = 2ev\cos\theta - 3v_2$

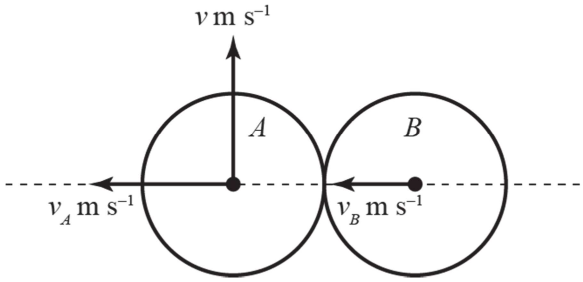
$v_2 = \dfrac{v\cos\theta(2e-1)}{3}$

$\tan(90° - \theta) = \dfrac{1}{\tan\theta} = \dfrac{v\sin\theta}{v_2} = \dfrac{3v\sin\theta}{v\cos\theta(2e-1)}$

So $\dfrac{1}{\tan\theta} = \dfrac{3\tan\theta}{2e-1}$

$\Rightarrow \tan^2\theta = \dfrac{2e-1}{3}$

5 After impact:



No change in component of velocity perpendicular to line of centres.

Conservation of momentum along the line of centres gives:

$mv = mv_A + mv_B \Rightarrow v = v_A + v_B$ **(1)**

Using Newton's law of restitution:

$v_A - v_B = \frac{2}{3}v$ **(2)**

Adding equations **(1)** and **(2)** gives: $2v_A = \frac{5}{3}v \Rightarrow v_A = \frac{5}{6}v$

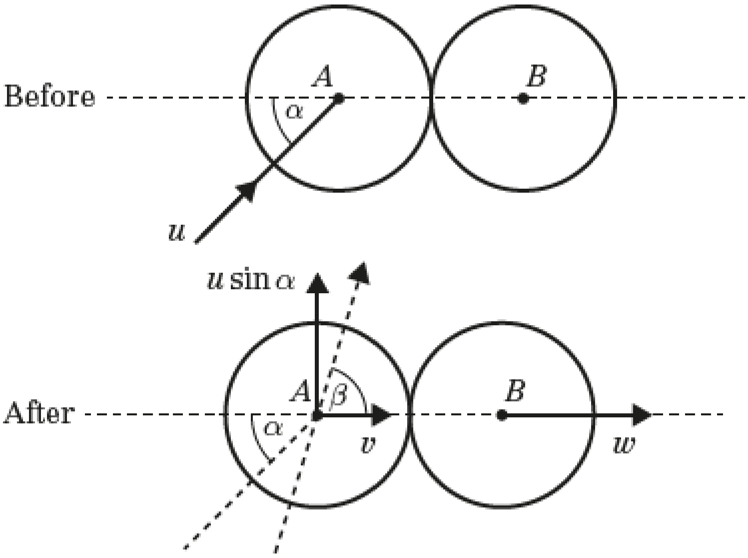
And by substitution $v_B = \frac{1}{6}v$

A has speed $\sqrt{1^2 + \left(\frac{5}{6}\right)^2}\,v = \sqrt{\frac{61}{36}}\,v = \frac{\sqrt{61}}{6}v\,\text{m s}^{-1}$

A is moving at $\arctan\frac{1}{\left(\frac{5}{6}\right)} = \arctan\frac{6}{5} = 50.2°$ (3 s.f.) to the line of centres.

B is moving along the line of centres with speed $\frac{1}{6}v\,\text{m s}^{-1}$.

6



a Perpendicular to the line of centres, component of velocity of A is $u\sin\alpha$

Conservation of momentum along the line of centres gives:

$mu\cos\alpha = mv + mw \Rightarrow u\cos\alpha = v + w$ **(1)**

Using Newton's law of restitution:

$w - v = eu\cos\alpha$ **(2)**

Solving equations **(1)** and **(2)**:

$2v = u\cos\alpha - eu\cos\alpha = u\cos\alpha(1-e)$

$\Rightarrow \tan\beta = \dfrac{u\sin\alpha}{v} = \dfrac{2u\sin\alpha}{u\cos\alpha(1-e)} = \dfrac{2\tan\alpha}{1-e}$

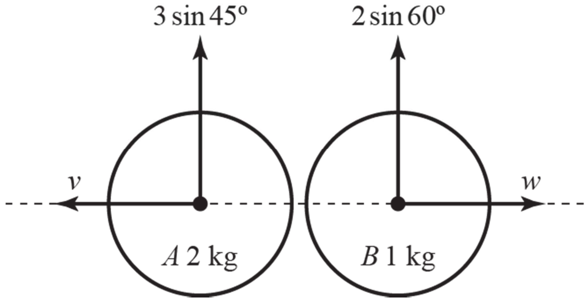
b The path of A has been deflected through an angle equal to $\beta - \alpha$

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta} = \frac{\frac{2\tan\alpha}{1-e} - \tan\alpha}{1+\tan\alpha\,\frac{2\tan\alpha}{1-e}}$$

$$= \frac{2\tan\alpha - (1-e)\tan\alpha}{1-e+2\tan^2\alpha} = \frac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}$$

Hence $\beta - \alpha = \arctan\left(\dfrac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}\right)$

7 a After impact:



No change in the components of velocity perpendicular to the line of centres.

Conservation of momentum along the line of centres gives:

$$1\times 2\cos 60° - 2\times 3\cos 45° = 2v - w \Rightarrow 2v - w = 1 - 3\sqrt{2} \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$v + w = \frac{\sqrt{2}}{3}(3\cos 45° + 2\cos 60°) \Rightarrow v + w = 1 + \frac{\sqrt{2}}{3} \qquad \textbf{(2)}$$

Solving equations **(1)** and **(2)**:

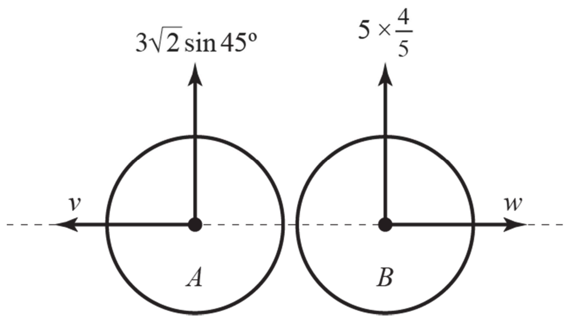
$$3v = 2 - \frac{8\sqrt{2}}{3} \Rightarrow v = \frac{2}{3} - \frac{8\sqrt{2}}{9} = -0.590\,\text{m s}^{-1} \text{ (3 s.f.)}$$

$$w = 1 + \frac{\sqrt{2}}{3} - \frac{2}{3} + \frac{8\sqrt{2}}{9} = \frac{1}{3} + \frac{11\sqrt{2}}{9} = 2.062\,\text{m s}^{-1} \text{ (4 s.f.)}$$

$$\begin{aligned}\text{Kinetic energy lost in the impact} &= \frac{1}{2}\times 2\times((3\cos 45°)^2 - v^2) + \frac{1}{2}\times 1\times((2\cos 60°)^2 - w^2)\\ &= \frac{1}{2}\times 2\times((3\cos 45°)^2 - 0.590^2) + \frac{1}{2}\times 1\times((2\cos 60°)^2 - 2.062^2)\\ &= 4.1519 - 1.6259 = 2.53\text{ J (3 s.f.)}\end{aligned}$$

b Impulse on $B = 1(w + 2\cos 60°) = 3.06\,\text{N s}$ (3 s.f.)

8 a After impact:



No change in the components of velocity perpendicular to the line of centres. So after the collision the components of velocity perpendicular to the line of centres are $3\,\text{m s}^{-1}$ and $4\,\text{m s}^{-1}$.

Conservation of momentum along the line of centres gives:

$$m\times 3\sqrt{2}\cos 45° - m\times 5\times\frac{3}{5} = mw - mv \Rightarrow w - v = 0 \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$v + w = \frac{2}{3}\left(3\sqrt{2}\cos 45° + 5\times\frac{3}{5}\right) \Rightarrow v + w = 4 \qquad \textbf{(2)}$$

Solving equations **(1)** and **(2)** gives $v = w = 2$

A has speed $\sqrt{2^2+3^2} = \sqrt{13}\,\text{m s}^{-1}$

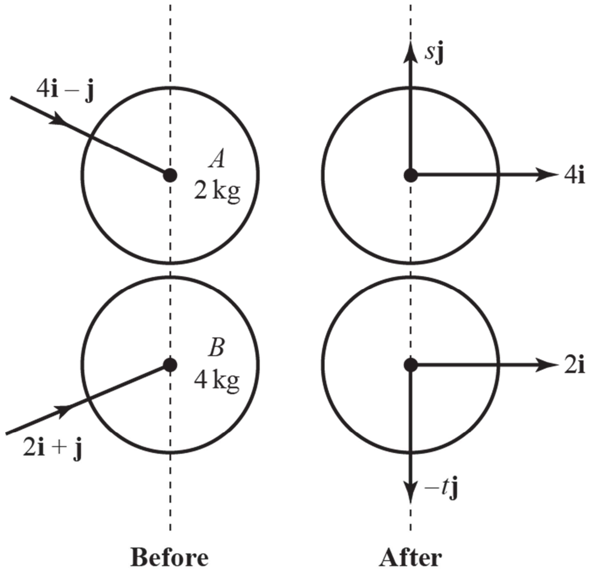
B has speed $\sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}\,\text{m s}^{-1}$

b Total kinetic energy before impact $= \frac{1}{2}\times m\times(3\sqrt{2})^2 + \frac{1}{2}\times m\times 5^2 = \frac{43}{2}m$ J

Total kinetic energy after impact $= \frac{1}{2}\times m\times(\sqrt{13})^2 + \frac{1}{2}\times m\times(2\sqrt{5})^2 = \frac{33}{2}m$ J

Fraction lost $= \frac{43-33}{43} = \frac{10}{43}$

9



Line of centres parallel to **j** so no change in the components of velocity parallel to **i**

Conservation of momentum along the line of centres gives:

$$-2\times 1+4\times 1=2\times s-4\times t\Rightarrow s-2t=1 \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$s+t=\frac{1}{2}(1+1)\Rightarrow s+t=1 \qquad \textbf{(2)}$$

Solving equations **(1)** and **(2)**:

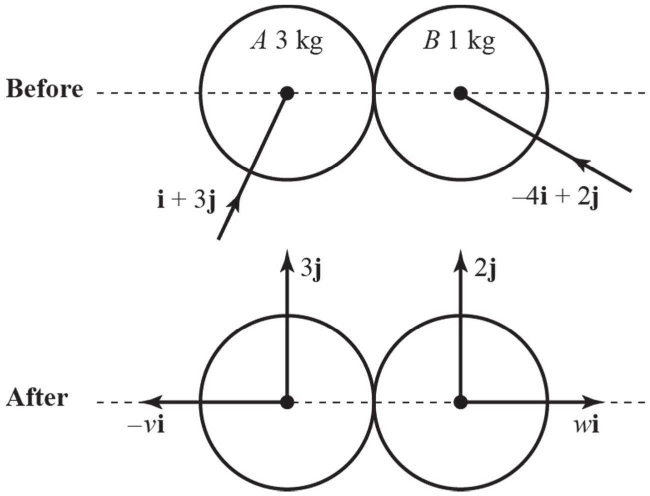
$3s=3\Rightarrow s=1$

And by substitution $t=0$

Velocity of A is $4\mathbf{i}+\mathbf{j}$ m s^{-1}

Velocity of B is $2\mathbf{i}$ m s^{-1}

10



Line of centres parallel to **i** so no change in the components of velocity parallel to **j**

Conservation of momentum along the line of centres gives:

$3\times1-1\times4=w-3v\Rightarrow w-3v=-1$ **(1)**

Using Newton's law of restitution:

$v+w=\frac{3}{4}(4+1)\Rightarrow 4v+4w=15$ **(2)**

Solving equations **(1)** and **(2)**:

$4w-12v=-4$

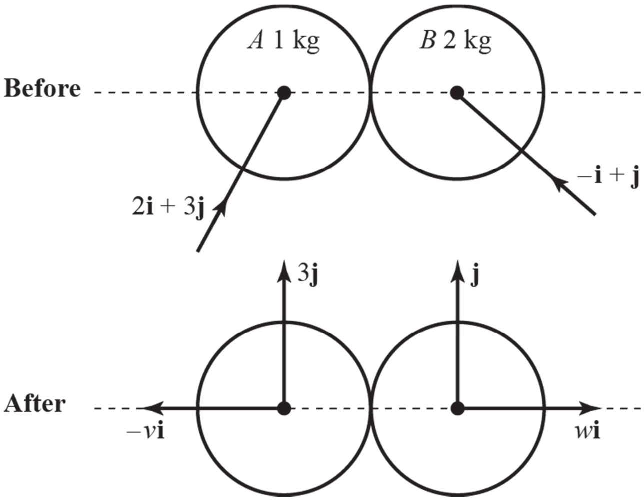
$\Rightarrow 16v=19$

$\Rightarrow v=\frac{19}{16},\ w=\frac{41}{16}$

After the impact, speed of $A=\sqrt{3^2+\left(\frac{19}{16}\right)^2}=3.23\,\text{m}\,\text{s}^{-1}$ (3 s.f.)

Speed of $B=\sqrt{2^2+\left(\frac{41}{16}\right)^2}=3.25\,\text{m}\,\text{s}^{-1}$ (3 s.f)

11



Line of centres parallel to **i** so no change in the components of velocity parallel to **j**

Conservation of momentum along the line of centres gives:

$1\times 2-2\times 1=2w-v\Rightarrow 2w-v=0$ **(1)**

Using Newton's law of restitution:

$v+w=\frac{3}{5}(2+1)\Rightarrow 5v+5w=9$ **(2)**

Solving equations **(1)** and **(2)**:

$15w=9\Rightarrow w=\frac{3}{5}$ and $v=\frac{6}{5}$

As components of velocity unchanged parallel to **j** all kinetic energy lost is parallel to **i**

$$\text{Kinetic energy lost}=\frac{1}{2}\times 1\times\left(2^2-\left(\frac{6}{5}\right)^2\right)+\frac{1}{2}\times 2\times\left(1^2-\left(\frac{3}{5}\right)^2\right)=\frac{48}{25}=1.92\,\text{J}$$

12 a Let velocity of B immediately after the collision be $\mathbf{v}$
Using conservation of momentum:
$m(2\mathbf{i}+5\mathbf{j})+2m(3\mathbf{i}-\mathbf{j})=m(3\mathbf{i}+2\mathbf{j})+2m\mathbf{v}$
$2\mathbf{v}=\mathbf{i}(2+2\times3-3)+\mathbf{j}(5-2\times1-2)=5\mathbf{i}+\mathbf{j}$
$\mathbf{v}=\frac{5}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$

b Impulse on $A=m((3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+5\mathbf{j}))=m(\mathbf{i}+3\mathbf{j})$

Therefore the line of centres is parallel to $\frac{1}{\sqrt{10}}(\mathbf{i}-3\mathbf{j})$

13 a Let velocity of B immediately after the collision be $\mathbf{v}$
Using conservation of momentum:
$3m(3\mathbf{i}-5\mathbf{j})+m(4\mathbf{i}+\mathbf{j})=3m(4\mathbf{i}-4\mathbf{j})+m\mathbf{v}$
$\mathbf{v}=\mathbf{i}(3\times3+4-3\times4)+\mathbf{j}(-3\times5+1+3\times4)=\mathbf{i}-2\mathbf{j}$
Speed of B is $\sqrt{1^2+2^2}=\sqrt{5}\,\text{m s}^{-1}$

b Kinetic energy lost $=\frac{3m}{2}((3^2+5^2)-(4^2+4^2))+\frac{m}{2}((4^2+1^2)-5)$
$=\frac{m}{2}(3(34-32)+(17-5))=\frac{m}{2}(6+12)=9m$ J

14 a Let velocity of B immediately after the collision be $\mathbf{v}$
Using conservation of momentum:
$2m(2\mathbf{i}+5\mathbf{j})+m(2\mathbf{i}-2\mathbf{j})=2m(3\mathbf{i}+4\mathbf{j})+m\mathbf{v}$
$\mathbf{v}=\mathbf{i}(2\times2+2-2\times3)+\mathbf{j}(2\times5-2-2\times4)=0$

b B is brought to a halt in the collision, therefore the line of centres must be parallel to the original direction of motion of B, i.e. $\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$

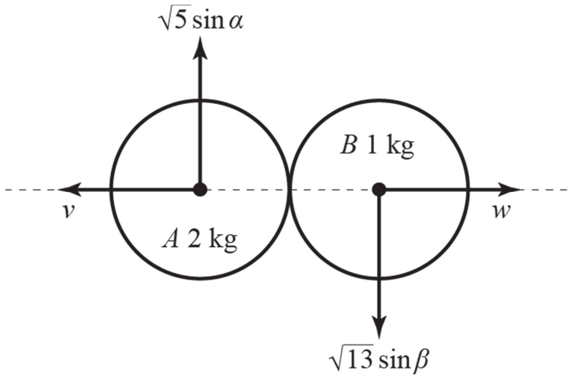
In this direction, speed of A before impact $=(2\mathbf{i}+5\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})=\frac{\sqrt{2}}{2}(2-5)=-3\frac{\sqrt{2}}{2}$

Speed of A after impact $=(3\mathbf{i}+4\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})=\frac{\sqrt{2}}{2}(3-4)=-\frac{\sqrt{2}}{2}$

Speed of B before $=2\sqrt{2}$
Speed of B after $=0$

Therefore the impact law gives $e=\frac{\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2}+2\sqrt{2}}=\frac{1}{7}$

15 After impact:



Before collision, components of velocity of A are $1\,\text{m s}^{-1}$ perpendicular to the lines of centres and $2\,\text{m s}^{-1}$ parallel to the line. The components of the velocity of B are $3\,\text{m s}^{-1}$ perpendicular to the line, and $2\,\text{m s}^{-1}$ parallel to it.

Conservation of momentum along the line of centres gives:

$2\times 2-1\times 2=1\times w-2\times v\Rightarrow w-2v=2$ **(1)**

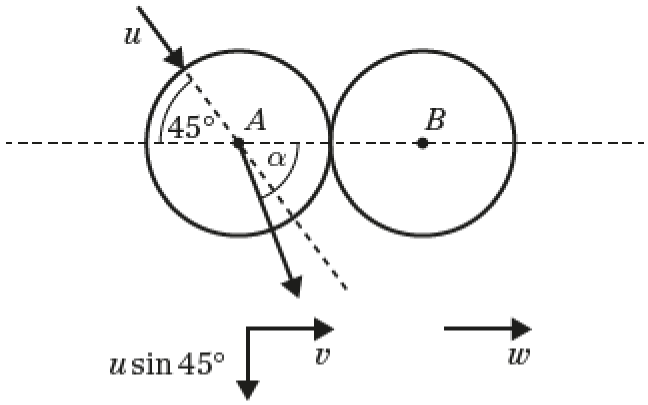
Using Newton's law of restitution:

$w+v=e(2+2)=2\Rightarrow w+v=2$ **(2)**

Solving equations **(1)** and **(2)**: $\Rightarrow w=2, v=0$

After the collision, the speed of A is $1\,\text{m s}^{-1}$ and speed of B is $\sqrt{3^2+2^2}=\sqrt{13}\,\text{m s}^{-1}$

16



Parallel to the line of centres, using conservation of momentum and the law of restitution gives:
$mu\cos 45° = mv + mw$ and $w - v = eu\cos 45°$

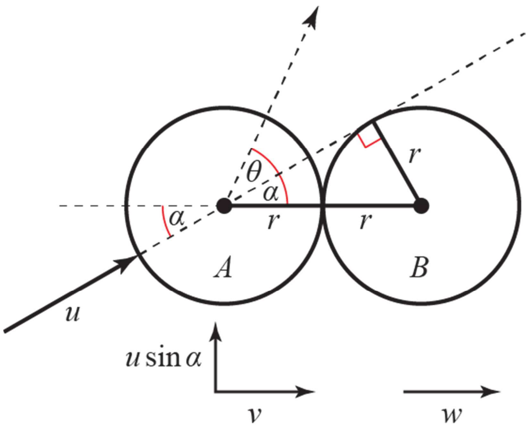
By subtracting:
$2v = u\cos 45°(1-e)$
$\Rightarrow v = \dfrac{u\sqrt{2}(1-e)}{4}$

So $\tan\alpha = \dfrac{u\sin 45°}{\left(\frac{u\sqrt{2}(1-e)}{4}\right)} = \dfrac{2}{1-e}$

$\theta = \alpha - 45°$

$\Rightarrow \tan\theta = \dfrac{\tan\alpha - \tan 45°}{1+\tan\alpha\tan 45°} = \dfrac{\frac{2}{1-e}-1}{1+\frac{2}{1-e}} = \dfrac{2-1+e}{1-e+2} = \dfrac{1+e}{3-e}$

Challenge



Tangent perpendicular to radius $\Rightarrow \sin\alpha = \frac{1}{2}$

Initial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

Conservation of momentum along the line of centres gives:

$mu\cos\alpha = mv + mw \Rightarrow u\cos\alpha = v + w$ **(1)**

Using Newton's law of restitution: **(2)**

$w - v = eu\cos\alpha$

Subtracting equation **(2)** from equation **(1)** gives:

$2v = u\cos\alpha - eu\cos\alpha$

$$\Rightarrow v = \frac{u\cos\alpha\left(1-\frac{1}{2}\right)}{2} = \frac{u \times \frac{\sqrt{3}}{2} \times \frac{1}{2}}{2} = \frac{u\sqrt{3}}{8}$$

$$\tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{8}\right)} = \frac{4}{\sqrt{3}}$$

$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1 + \tan(\theta+\alpha)\tan\alpha} = \frac{\frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \frac{4}{\sqrt{3}} \times \frac{1}{\sqrt{3}}} = \frac{\left(\frac{3}{\sqrt{3}}\right)}{\left(\frac{3+4}{3}\right)} = \frac{9}{7\sqrt{3}} = \frac{3\sqrt{3}}{7}$$