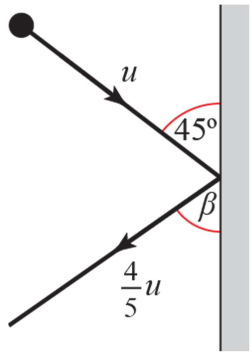
Mixed Exercise 5

1



For motion parallel to the wall:

$$\frac{4u}{5}\cos\beta = u\cos 45°$$

$$\frac{4u}{5}\cos\beta = \frac{\sqrt{2}u}{2} \qquad \textbf{(1)}$$

For motion perpendicular to the wall:

$$\frac{4u}{5}\sin\beta = eu\sin 45°$$

$$\frac{4u}{5}\sin\beta = \frac{eu\sqrt{2}}{2} \qquad \textbf{(2)}$$

Squaring and adding equations **(1)** and **(2)** gives:

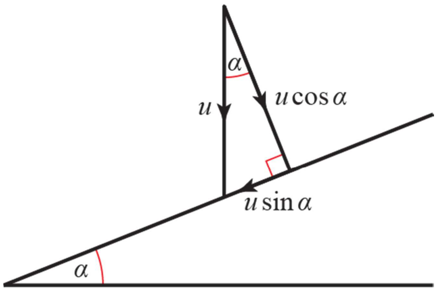
$$\frac{16}{25}u^2\cos^2\beta + \frac{16}{25}u^2\sin^2\beta = \frac{2}{4}u^2 + \frac{2}{4}e^2u^2$$

$$\frac{16}{25}u^2(\cos^2\beta + \sin^2\beta) = \frac{u^2}{2}(1+e^2)$$

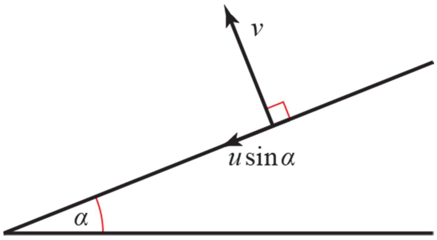
$$\frac{32}{25} = 1 + e^2$$

$$e^2 = \frac{7}{25} \Rightarrow e = \frac{\sqrt{7}}{5}$$

2 Before the collision:



After the collision:



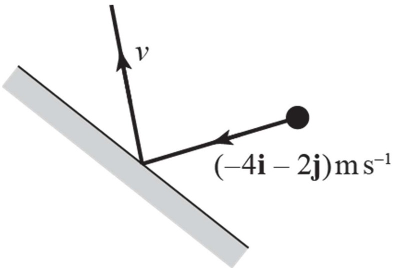
a The component of velocity parallel to the slope is $u\sin\alpha = 5.2\times\frac{5}{13} = 2$

Perpendicular to the slope: $v = eu\cos\alpha = \frac{1}{4}\times 5.2\times\frac{12}{13} = 1.2$

Therefore the speed immediately after impact $= \sqrt{2^2 + 1.2^2} = 2.33\,\text{m s}^{-1}$ (3 s.f.)

b Impulse $= mv - m(-u\cos\alpha) = \frac{1}{2}(1.2-(-4.8)) = 3\,\text{N s}$

3



a Let **v** = **a** + **b**, where **a** is parallel to the wall and **b** is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector in the direction of the wall

and $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

Parallel to the wall:

$$\mathbf{a}=\left[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$
$$=\frac{1}{\sqrt{2}}(4-2)\times\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$
$$=(-\mathbf{i}+\mathbf{j})$$

Perpendicular to the wall:

$$\mathbf{b}=-\frac{1}{2}[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$
$$=-\frac{1}{2}\times\frac{1}{\sqrt{2}}(-4-2)\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$
$$=\frac{3}{2}(\mathbf{i}+\mathbf{j})$$

So $\mathbf{v}=(-\mathbf{i}+\mathbf{j})+\frac{3}{2}(\mathbf{i}+\mathbf{j})=\frac{1}{2}\mathbf{i}+\frac{5}{2}\mathbf{j}$

b Kinetic energy before impact $=\frac{1}{2}\times\frac{1}{2}\times(4^2+2^2)=5$

Kinetic energy after impact $=\frac{1}{2}\times\frac{1}{2}\times\left(\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2\right)=\frac{1}{4}\times\frac{26}{4}=\frac{13}{8}$

Kinetic energy lost $=5-\frac{13}{8}=3.375$ J

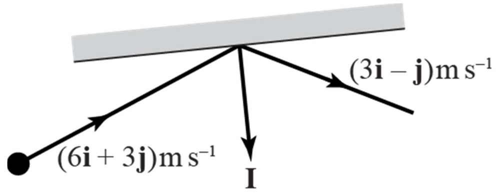
c Using the scalar product to determine the angle of deflection:

$$\cos\theta=\frac{\mathbf{u}.\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\cos\theta=\frac{(-4\times\frac{1}{2})+(-2\times\frac{5}{2})}{\sqrt{20}\sqrt{\frac{26}{4}}}=\frac{-2-5}{\sqrt{130}}=-0.61394$$

$\theta=128°$ (3 s.f.)

4



If the component of the initial velocity perpendicular to the wall is u and that of the final velocity is v, then the coefficient of restitution, e is given by:

$$e = \frac{v}{u}$$

Since neither **i** nor **j** component is unchanged, wall is not along one of the axes.

Impulse vector, **I** is perpendicular to the wall:

$$\mathbf{I} = m\mathbf{v} - m\mathbf{u}$$

$$\mathbf{I} = m(2\mathbf{i} - 2\mathbf{j}) - m(6\mathbf{i} + 3\mathbf{j})$$

$$\mathbf{I} = m(-4\mathbf{i} - 5\mathbf{j})$$

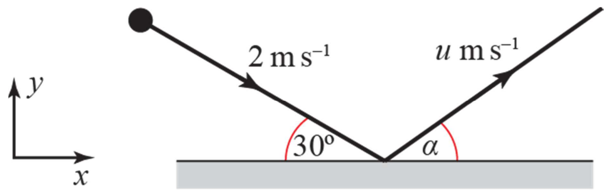
The component of each velocity perpendicular to the wall is given by the scalar product of the velocity and the impulse vector:

$$u = \mathbf{u.I} = m(6\mathbf{i} + 3\mathbf{j}).(-4\mathbf{i} - 5\mathbf{j}) = m(6 \times -4) + (3 \times -5) = -39m$$

$$v = \mathbf{v.I} = m(2\mathbf{i} - 2\mathbf{j}).(-4\mathbf{i} - 5\mathbf{j}) = m(2 \times -4) + (-2 \times -5) = 2m$$

$$\Rightarrow e = \frac{2m}{39m} = \frac{2}{39}$$

5 a



First collision: $e = 0.5$

For motion parallel to the wall:

$u\cos\alpha = 2\cos 30°$ **(1)**

For motion perpendicular to the wall:

$u\sin\alpha = 2e\sin 30°$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$$u^2\cos^2\alpha + u^2\sin^2\alpha = 4\cos^2 30° + 4e^2\sin^2 30°$$

$$u^2(\cos^2\alpha + \sin^2\alpha) = 4\left(\frac{3}{4} + 0.25 \times \frac{1}{4}\right)$$

$$u^2 = 3.25$$

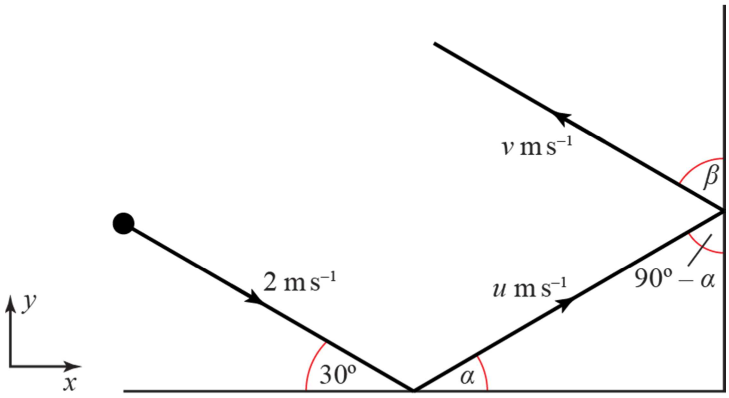
$$\Rightarrow u = 1.80\,\text{m s}^{-1} \text{ (3 s.f.)}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\alpha = 0.5\tan 30° = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \alpha = 16.1° \text{ (3 s.f.)}$$

5 b



Second collision: $e = 0.5$

For motion parallel to the wall:

$v\cos\beta = u\cos(90° - \alpha) = u\sin\alpha$ **(3)**

For motion perpendicular to the wall:

$v\sin\beta = 0.5 \times u\sin(90° - \alpha) = 0.5u\cos\alpha$ **(4)**

Squaring and adding equations **(3)** and **(4)** gives:

$v^2\cos^2\beta + v^2\sin^2\beta = u^2\sin^2\alpha + 0.25u^2\cos^2\alpha$

As $\tan\alpha = \dfrac{1}{2\sqrt{3}}$, by Pythagoras $\sin\alpha = \dfrac{1}{\sqrt{13}}$ and $\cos\alpha = \dfrac{2\sqrt{3}}{\sqrt{13}}$

$v^2(\cos^2\beta + \sin^2\beta) = u^2\left(\dfrac{1}{13} + \dfrac{3}{13}\right) = \dfrac{13}{4} \times \dfrac{4}{13} = 1$

$v = 1\,\text{m s}^{-1}$

Dividing equation **(4)** by equation **(3)** gives:

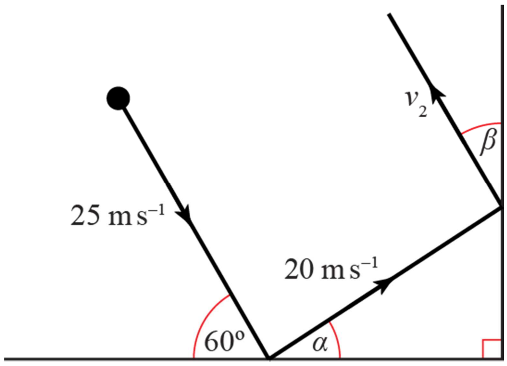
$\tan\beta = \dfrac{0.5}{\tan\alpha} = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$\Rightarrow \beta = 60°$

So the ball travels parallel to the original direction, but in the opposite direction.

c The resistance to the motion due to the floor not being smooth will result in the final speed of the ball being slower, but the angle of motion will remain the same.

6



a First collision

For motion parallel to the wall:

$20\cos\alpha = 25\cos 60°$ **(1)**

For motion perpendicular to the wall:

$20\sin\alpha = 25e\sin 60°$ **(2)**

From equation **(1)**:

$20\cos\alpha = 25\cos 60°$

$\cos\alpha = \frac{25}{20}\times\frac{1}{2} = \frac{25}{40} = \frac{5}{8}$

By Pythagoras' theorem

$\tan\alpha = \frac{\sqrt{39}}{5}$ and $\sin\alpha = \frac{\sqrt{39}}{8}$

Dividing equation **(2)** by equation **(1)** gives:

$\tan\alpha = e\tan 60°$

$\Rightarrow e = \frac{\tan\alpha}{\tan 60°} = \frac{\sqrt{39}}{5\sqrt{3}} = \frac{\sqrt{13}}{5} = 0.721$ (3 s.f.)

6 b Second collision: $e = 0.5$

For motion parallel to the wall:

$$v_2 \cos\beta = 20\cos(90° - \alpha) = 20\sin\alpha \qquad \textbf{(3)}$$

For motion perpendicular to the wall:

$$v_2 \sin\beta = 20e\sin(90° - \alpha) = 20e\cos\alpha \qquad \textbf{(4)}$$

Squaring and adding equations **(3)** and **(4)** gives:

$$v_2^2\cos^2\beta + v_2^2\sin^2\beta = 20^2\sin^2\alpha + 20^2 e^2\cos^2\alpha$$

$$v_2^2(\cos^2\beta + \sin^2\beta) = 20^2\left(\frac{39}{64} + \frac{13}{25} \times \frac{25}{64}\right) = 400 \times \frac{52}{64} = 325$$

$$v_2 = 18.0\,\text{m s}^{-1} \text{ (3 s.f.)}$$

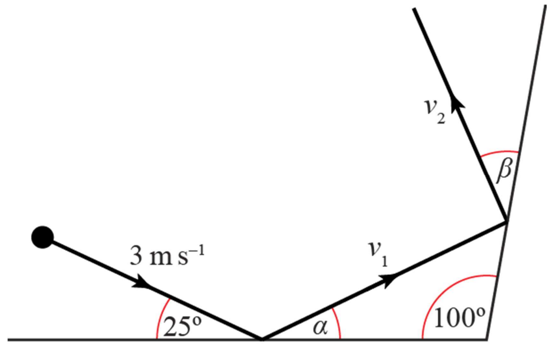
Dividing equation **(4)** by equation **(3)** gives:

$$\tan\beta = \frac{e}{\tan\alpha} = \frac{\sqrt{13}}{5} \times \frac{5}{\sqrt{39}} = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \beta = 30°$$

So the ball travels parallel to the original direction, but in the opposite direction.

7



a First collision: $e = 0.2$

For motion parallel to the wall:

$v_1 \cos\alpha = 3\cos 25°$ **(1)**

For motion perpendicular to the wall:

$v_1 \sin\alpha = 0.2 \times 3\sin 25°$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$v_1^2 \cos^2\alpha + v_1^2 \sin^2\alpha = 9\cos^2 25° + 0.36\sin^2 25°$

$v_1^2(\cos^2\alpha + \sin^2\alpha) = 7.45683$

$v_1 = 2.7307\text{ m s}^{-1}$ (5 s.f.)

Dividing equation **(2)** by equation **(1)** gives:

$\tan\alpha = 0.2\tan 25° = 0.09326$

$\Rightarrow \alpha = 5.3281°$ (5 s.f.)

$$\text{Loss of kinetic energy} = \frac{1}{2}m(u^2 - v_1^{\,2})$$
$$= \frac{1}{2} \times 0.5(3^2 - 2.7307^2)$$
$$= 0.386\text{ J (3 s.f.)}$$

b Second collision: $e = 0.4$

For motion parallel to the wall:

$v_2 \cos\beta = v_1 \cos(80° - \alpha)$ **(3)**

For motion perpendicular to the wall:

$v_2 \sin\beta = 0.4 v_1 \sin(80° - \alpha)$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:

$\tan\beta = 0.4\tan(80° - \alpha) = 0.4\tan(80° - 5.3281°) = 0.4\tan 74.6719° = 1.4593$

$\Rightarrow \beta = 55.6°$ (3 s.f.)

Substituting in equation **(3)** gives:

$v_2 \times \cos 55.579° = 2.7307 \times \cos 74.6719°$

$v_2 = 1.28\text{ m s}^{-1}$ (3 s.f.)

8 a Let velocity of B immediately after the collision be $\mathbf{v}$

Using conservation of momentum:

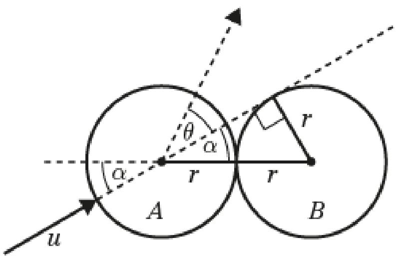
$$4m(2\mathbf{i}+3\mathbf{j})+m(3\mathbf{i}-\mathbf{j})=4m(3\mathbf{i}+2\mathbf{j})+m\mathbf{v}$$

$$\mathbf{v}=\mathbf{i}(4\times2+1\times3-4\times3)+\mathbf{j}(4\times3-1\times1-4\times2)=-\mathbf{i}+3\mathbf{j}$$

b Impulse on A $=4m(3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+3\mathbf{j})=4m(\mathbf{i}-\mathbf{j})$

$\Rightarrow \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$ is a unit vector parallel to the line of centres.

9



Tangent perpendicular to radius $\Rightarrow \sin\alpha=\frac{1}{2}$, so $\cos\alpha=\frac{\sqrt{3}}{2}$ and $\tan\alpha=\frac{\sqrt{3}}{3}$

Initial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

Using conservation of momentum:

$mu\cos\alpha=mv+mw\Rightarrow v+w=u\cos\alpha$ **(1)**

where v is the velocity of A along the line of centres and w the velocity of B along the line of centres immediately after the collision.

Using Newton's law of restitution:

$w-v=eu\cos\alpha$ **(2)**

Solving equations **(1)** and **(2)**:

$2v=u\cos\alpha-eu\cos\alpha,$

$$\Rightarrow v=\frac{u\cos\alpha(1-\frac{1}{3})}{2}=\frac{u\times\frac{\sqrt{3}}{2}\times\frac{1}{3}}{2}=\frac{u\sqrt{3}}{12}$$

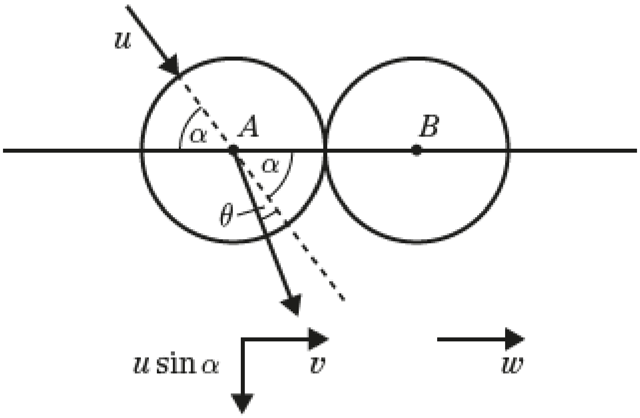
$$\tan(\theta+\alpha)=\frac{u\sin\alpha}{v}=\frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{12}\right)}=\frac{6}{\sqrt{3}}=2\sqrt{3}$$

$$\tan\theta=\tan((\theta+\alpha)-\alpha)=\frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}$$

$$=\frac{2\sqrt{3}-\frac{1}{\sqrt{3}}}{1+2\sqrt{3}\times\frac{1}{\sqrt{3}}}=\frac{\left(\frac{6-1}{\sqrt{3}}\right)}{(1+2)}=\frac{5}{3\sqrt{3}}$$

$$=\frac{5\sqrt{3}}{9}$$

10



Using conservation of momentum:

$mu\cos\alpha = mv + mw \Rightarrow v + w = u\cos\alpha$ **(1)**

Using Newton's law of restitution:

$w - v = eu\cos\alpha$ **(2)**

Solving equations **(1)** and **(2)**:

$2v = u\cos\alpha - eu\cos\alpha$

As $\tan\alpha = \frac{3}{4} \Rightarrow \cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$\Rightarrow v = \frac{u\cos\alpha(1-e)}{2} = \frac{2u(1-e)}{5}$

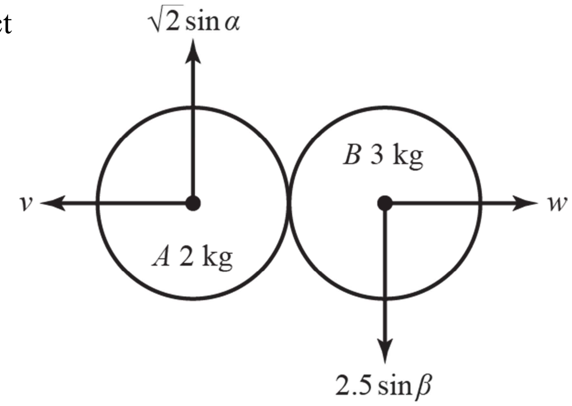
$\tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{u\sin\alpha}{\left(\frac{2u(1-e)}{5}\right)} = \frac{3}{2(1-e)}$

$\tan\theta = \tan((\theta+\alpha)-\alpha) = \frac{\tan(\theta+\alpha) - \tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}$

$= \frac{\frac{3}{2(1-e)} - \frac{3}{4}}{1 + \frac{3}{2(1-e)} \times \frac{3}{4}} = \frac{12-6(1-e)}{8(1-e)+9}$

$= \frac{6+6e}{17-8e}$

11 After impact



$\tan\alpha = 1 \Rightarrow \sin\alpha = \cos\alpha = \dfrac{1}{\sqrt{2}}$

$\tan\beta = \dfrac{3}{4} \Rightarrow \sin\beta = \dfrac{3}{5}$ and $\cos\beta = \dfrac{4}{5}$

Using Newton's law of restitution:

$v + w = \dfrac{2}{3}\left(\sqrt{2}\cos\alpha + 2.5\cos\beta\right)$

$v + w = 2 \Rightarrow v = 2 - w$

Using conservation of momentum:

$\left(2\times\sqrt{2}\cos\alpha\right) - \left(3\times 2.5\cos\beta\right) = 3w - 2v \Rightarrow -4 = 3w - 2v$

Substituting for v gives:

$-4 = 3w - 2(2 - w)$

$-4 = -4 + 5w$

$w = 0$

$v = 2 - w = 2$

Let speed of A after collision be v_A

$v_A^{\,2} = \left(\sqrt{2}\sin\alpha\right)^2 + v^2$

$v_A^{\,2} = 1 + 4$

$v_A = \sqrt{5}$

Let speed of B after collision be v_B

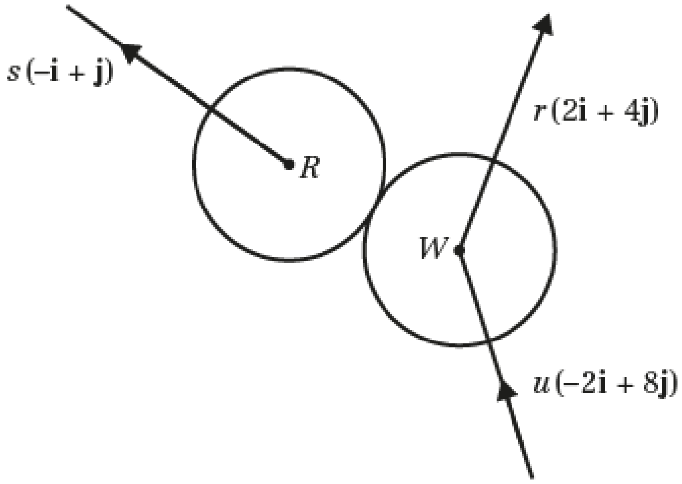
$v_B^{\,2} = \left(2.5\sin\beta\right)^2 + w^2$

$v_B^{\,2} = \left(\dfrac{3}{2}\right)^2 + 0$

$v_B = 1.5$

After the collision, the speeds of A and B are $\sqrt{5}$ m s^{-1} and 1.5 m s^{-1} respectively.

12



Using conservation of momentum:

$mu(-2\mathbf{i}+8\mathbf{j}) = ms(-\mathbf{i}+\mathbf{j}) + mr(2\mathbf{i}+4\mathbf{j})$

So $-2u = -s + 2r$

and $8u = s + 4r$

Adding $6u = 6r$

$\Rightarrow u = r$ and $s = 4u$

The red ball moves along the line of centres after impact, so line of centres is parallel to ($-\mathbf{i} + \mathbf{j}$).

Component of initial velocity along this line = 0

Component of final velocity along this line = $4u$

For the white ball, component of initial velocity along this line is the scalar product of this vector and its motion:

$= u(-2\mathbf{i}+8\mathbf{j}).(-\mathbf{i}+\mathbf{j}) = u\big((-2\times -1)+(8\times 1)\big) = 10u$

And, similarly, component of final velocity along this line is:

$= u(2\mathbf{i}+4\mathbf{j}).(-\mathbf{i}+\mathbf{j}) = u\big((2\times -1)+(4\times 1)\big) = 2u$

Using Newton's law of restitution:

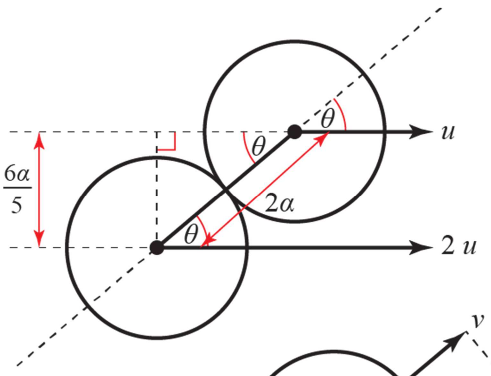
$4u + 2u = e \times 10u$

$6 = 10e$

$\Rightarrow e = \frac{3}{5}$

13 Angle between line of centres and direction of travel is θ where:

$$\sin\theta = \frac{\frac{6a}{5}}{2a} = \frac{3}{5} \Rightarrow \cos\theta = \frac{4}{5}$$

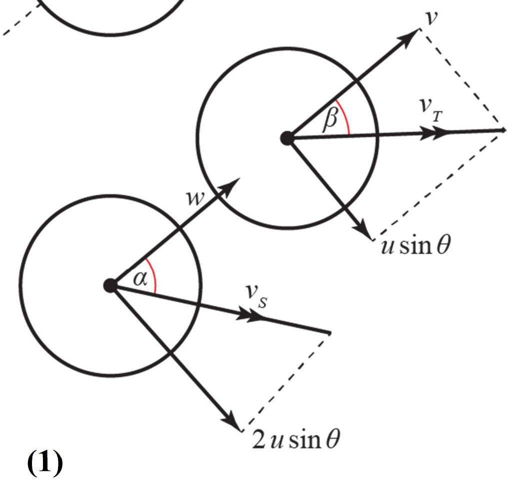


Perpendicular to line of centres the components of the initial speeds are:

$u\sin\theta = 0.6u$

$2u\sin\theta = 1.2u$

These remain unchanged.



Parallel to line of centres the components of the initial speeds are:

$u\cos\theta = 0.8u$

$2u\cos\theta = 1.6u$

And the components of the final speeds v_T and v_B along these lines are v (for the top sphere) and w (for the bottom sphere)

Using conservation of momentum along the line of centres:

$mv + mw = 2mu\cos\theta + mu\cos\theta \Rightarrow v + w = 1.6u + 0.8u = 2.4u$ **(1)**

Using Newton's law of restitution:

$v - w = \frac{3}{4}(0.8u + 1.6u) = 0.6u$ **(2)**

Solving equations (1) and (2) gives:

$2v = 0.6u + 2.4u \Rightarrow v = 1.5u$

$1.5u + w = 2.4u \Rightarrow w = 0.9u$

So final speeds are:

$$v_T{}^2 = (0.6u)^2 + (1.5u)^2 = (0.36 + 2.25)u^2 = 2.61u^2 = \frac{261}{100}u^2$$

$$v_T = \frac{3\sqrt{29}}{10}u$$

$$v_B{}^2 = (1.2u)^2 + (0.9u)^2 = (1.44 + 0.81)u^2 = 2.25u^2$$

$$v_B = 1.5u$$

Angle between paths is the difference between the angles each makes with the line of centres:

$$\tan\alpha = \frac{2u\sin\theta}{w} = \frac{1.2u}{0.9u} = \frac{4}{3} \text{ and } \tan\beta = \frac{u\sin\theta}{v} = \frac{0.6u}{1.5u} = \frac{2}{5}$$

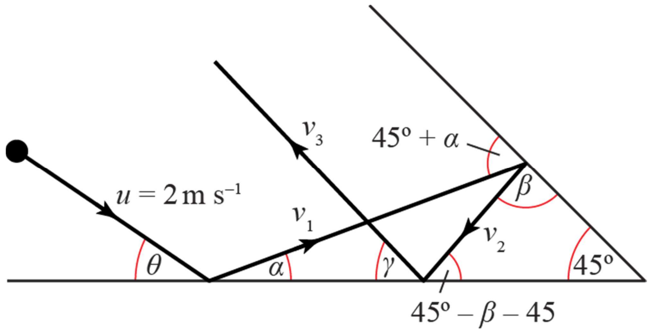
Using $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\frac{4}{3} - \frac{2}{5}}{1 + \frac{4\times 2}{3\times 5}} = \frac{\frac{14}{15}}{\frac{15+8}{15}} = \frac{14}{23}$

Immediately after impact, the speeds of the spheres are $\frac{3\sqrt{29}}{10}u \text{ m s}^{-1}$ and $1.5u \text{ m s}^{-1}$

and the angle between their paths is $\arctan\frac{14}{23}$

Challenge

a



From earlier questions know that in a collision: tan(*angle of departure*) = *e*tan (*angle of approach*)

For the first collision:

$\tan\alpha = e\tan\theta = \frac{1}{2}\times\frac{10}{7} = \frac{5}{7}$

$\alpha = 35.54°$ (4 s.f.)

Since this is less than 45°, the bounce is internal (see diagram) and, for the second collision:

$\tan\beta = e\tan(45°+\alpha) = \frac{1}{2}\times\tan(80.54°) = 3$

$\beta = 71.57°$

Since this is greater than 45°, the bounce is external (see diagram) and, for the third collision:

$\tan\gamma = e\tan(180°-45°-\beta) = \frac{1}{2}\times\tan(63.43°) = 1$

$\gamma = 45°$

In other words, the ball has rebounded from the first wall and is now travelling parallel to the second wall so will not hit either again.

Challenge

b From earlier questions know that in a collision: *v*cos (*angle of departure*) = *u*cos (*angle of approach*)

For the first collision:

$$v_1 \cos\alpha = 2\cos\theta$$

$$v_1 \frac{7}{\sqrt{74}} = 2 \times \frac{7}{\sqrt{149}} = \frac{2\sqrt{74}}{\sqrt{149}}$$

For the second collision:

$$v_2 \cos\beta = \frac{2\sqrt{74}}{\sqrt{149}} \cos(45° + \alpha)$$

$$v_2 \frac{1}{\sqrt{10}} = \frac{2\sqrt{74}}{\sqrt{149}} \times \frac{1}{\sqrt{37}} = \frac{2\sqrt{20}}{\sqrt{149}}$$

For the third collision:

$$v_3 \cos\gamma = \frac{2\sqrt{20}}{\sqrt{149}} \cos(180° - 45° - \beta)$$

$$v_3 \frac{1}{\sqrt{2}} = \frac{2\sqrt{20}}{\sqrt{149}} \times \frac{1}{\sqrt{5}} = \frac{2\sqrt{8}}{\sqrt{149}}$$

$$v_3^2 = \frac{4 \times 8}{149} = \frac{32}{149}$$

Initial kinetic energy $= \frac{1}{2}mu^2$

Loss in kinetic energy $= \frac{1}{2}mu^2 - \frac{1}{2}mv_3^2 = \frac{1}{2}m(u^2 - v_3^2)$

Proportion loss of kinetic energy $= \frac{\frac{1}{2}m(u^2 - v_3^2)}{\frac{1}{2}mu^2} = \frac{u^2 - v_3^2}{u^2}$

$$= \frac{4 - \frac{32}{149}}{4} = \frac{141}{149}$$

The percentage of kinetic energy lost is 94.6% (to 3s.f.)