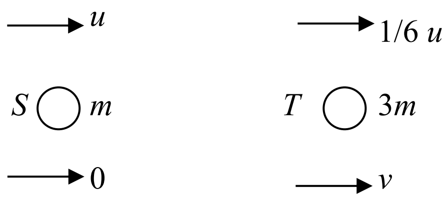
Review Exercise 2

1



$$mu + 3m \times \frac{1}{6}u = 3mv$$ (Conservation of momentum.)

$$\frac{3}{2}u = 3v$$

$$u = 2v$$

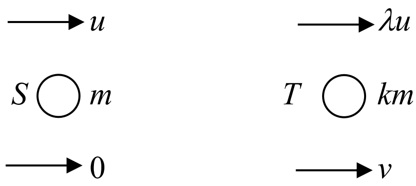
$$e(u - \frac{1}{6}u) = v$$ (Newton's Law of Restitution.)

$$\therefore \quad \frac{5}{6}eu = \frac{1}{2}u$$

$$e = \frac{6}{5} \times \frac{1}{2}$$

$$e = \frac{3}{5}$$

2 a



$$mu + km\lambda u = kmv$$ (Conservation of momentum.)

$$u(1 + k\lambda) = kv \qquad \textbf{(1)}$$

$$v = e(u - \lambda u) = eu(1 - \lambda) \; \textbf{(2)}$$ (Newton's Law of Restitution.)

Eliminate v from **(1)** and **(2)**

$$u(1 + k\lambda) = keu(1 - \lambda)$$

$$e = \frac{1 + k\lambda}{k(1 - \lambda)}$$

b

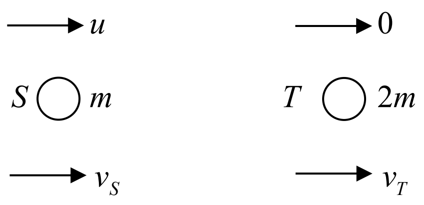
$$e \leqslant 1$$

$$\Rightarrow 1 + k\lambda \leqslant k - k\lambda$$ (Any coefficient of restitution satisfies $0 \leqslant e \leqslant 1$)

$$\frac{1}{1 - 2\lambda} \leqslant k$$

but $0 < \lambda < \frac{1}{2} \Rightarrow 0 < 1 - 2\lambda < 1$ and $k > 1$

3 a



$$mu = mv_S + 2mv_T$$

$$u = v_S + 2v_T \quad \textbf{(1)}$$

Conservation of momentum.

$$eu = v_T - v_S \quad \textbf{(2)}$$

Newton's Law of Restitution.

$$\textbf{(1)}+\textbf{(2)}\ u + eu = 3v_T$$

$$v_T = \frac{1}{3}u(1+e)$$

b i from **(2)**

$$eu = \frac{1}{3}u(1+e) - v_S$$

$$v_S = \frac{1}{3}u(1+e) - eu$$

$$v_S = \frac{1}{3}u(1-2e)$$

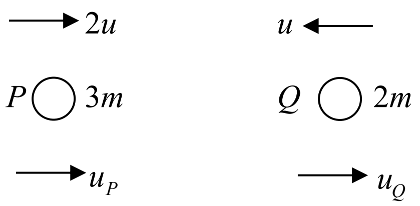
$$\text{but } e > \frac{1}{2} \Rightarrow 1-2e < 0$$

Speed must be positive.

$$\therefore \text{Speed of } S \text{ is } \frac{1}{3}u(2e-1)$$

ii The arrow in the diagram was the wrong way round, as shown in **b i**, so the direction of motion was reversed.

4 a



Speed must be positive.

$$3m \times 2u - 2mu = 3mu_P + 2mu_Q$$

Conservation of momentum.

$$\therefore \quad 4u = 3u_P + 2u_Q \qquad \textbf{(1)}$$

$$e(2u + u) = u_Q - u_P$$

Newton's Law of Restitution.

$$3eu = u_Q - u_P \qquad \textbf{(2)}$$

Eliminating u_P between **(1)** and **(2)**:

$$4u = 3(u_Q - 3eu) + 2u_Q$$

$$4u = 5u_Q - 9eu$$

$$u_Q = \frac{1}{5}u(9e + 4)$$

b Using **(2)**

$$u_P = u_Q - 3eu$$

$$= \frac{1}{5}u(9e + 4) - 3eu$$

$$= \frac{2}{5}u(2 - 3e)$$

But

$$u_P < 0$$

Direction of motion of P is reversed.

$$\therefore 2 - 3e < 0$$

$$e > \frac{2}{3}$$

Use the general condition $0 \leq e \leq 1$

$$\therefore \quad \frac{2}{3} < e \leqslant 1$$

c For Q

$$\frac{32}{5}mu = 2m \times \frac{1}{5}u(9e + 4) + 2mu$$

Impulse = change of momentum.

$$32 = 2(9e + 4) + 10$$

$$18e = 14$$

$$e = \frac{7}{9}$$

5 For the fall, down positive:

$u = 0\ \text{ms}^{-1}$, $a = g$, $t = 2$ s, $v = ?$
$v = u + at$
$v = 0 + 2g = 2g$

Speed after the bounce, v', is given by Newton's law of restitution:

$v' = ev = \frac{6}{7} \times 2g$

For the return, up positive: $u = \frac{12}{7}g\ \text{ms}^{-1}$, $a = -g = -9.8\ \text{ms}^{-2}$, $v = 0\ \text{ms}^{-1}$, $s = ?$

$v^2 = u^2 + 2as$

$0 = \left(\frac{12}{7}g\right)^2 - 2gs$

$2s = \left(\frac{12}{7}\right)^2 g$

$s = \frac{1}{2} \times \frac{144}{49} \times 9.8 = 14.4$

The ball rises to a height of 14.4 m on the first bounce.

6 **a** Distance travelled $= s$, $t_{in} = 2$ s, $t_{out} = 3$ s

When travelling towards the wall, average speed, $u = \frac{s}{t_{in}} = \frac{s}{2}$

When travelling away from the wall, average speed, $v = \frac{s}{t_{out}} = \frac{s}{3}$

Using Newton's law of restitution:
$v = eu$

$\frac{s}{3} = e\frac{s}{2}$

$e = \frac{\frac{s}{3}}{\frac{s}{2}} = \frac{2}{3}$

The coefficient of restitution is $\frac{2}{3}$

b If the plane is rough, then the sphere will experience a frictional force and decelerate as it travels to and from the wall.

If the times it takes to travel between the wall and P are the same as in part **a**, then, although the average speed in each direction remains the same, the sphere hits the wall at a lower speed (u is smaller) and leaves it at a greater speed (v is greater) than the values calculated.

Since the coefficient of restitution is given by $e = \frac{v}{u}$, it would therefore have a bigger value than that calculated in part **a**.

7 a For the fall, down positive: $u = 0\text{ ms}^{-1}$, $a = g$, $s = 50$ m, $v = ?$

$v^2 = u^2 + 2as$

$v^2 = 2g \times 50 = 100g$

Speed after the bounce, v', is given by Newton's law of restitution:

$v' = ev$

$v'^2 = e^2v^2 = 100ge^2$

For the return, up positive: $v = 0\text{ ms}^{-1}$, $u = v'$, $a = -g$, $s = 35$ m

$v^2 = u^2 + 2as$

$0 = 100ge^2 - (2g \times 35)$

$100e^2 = 70$

$e = \frac{\sqrt{70}}{10}$

The coefficient of restitution is $\frac{\sqrt{70}}{10}$

b For the first fall, down positive: $u = 0\text{ ms}^{-1}$, $a = g$, $s = 50$ m, $t = t_1$

$s = ut + \frac{1}{2}at^2$

$50 = \frac{1}{2}gt_1^{\,2}$

$t_1^{\,2} = \frac{100}{g}$

For the second fall, down positive: $u = 0\text{ ms}^{-1}$, $a = g$, $s = 35$ m, $t = t_2$

$35 = \frac{1}{2}gt_2^{\,2}$

$t_2^{\,2} = \frac{70}{g}$

The ball takes the same time to rise to 35 m after the first bounce so total time, t, is given by:

$t = t_1 + 2t_2$

$t = \frac{10}{\sqrt{g}} + 2\frac{\sqrt{70}}{\sqrt{g}}$

If $g = 9.8 \Rightarrow \frac{1}{\sqrt{g}} = \frac{\sqrt{5}}{7}$

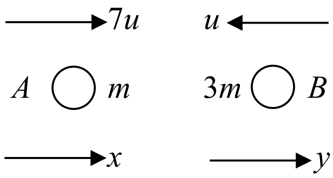
then $t = \frac{10\sqrt{5}}{7} + 2\frac{\sqrt{5}\sqrt{70}}{7}$

$t = \frac{10\sqrt{5}}{7} + 2\frac{\sqrt{5}\sqrt{70}}{7}$

$t = \frac{1}{7}\left(10\sqrt{5} + \sqrt{1400}\right)$

$t = \frac{10}{7}\left(\sqrt{5} + \sqrt{14}\right)$ as required.

8 a



$e = 0.25$

Draw a diagram.

$$7mu - 3mu = mx + 3my$$
$$4u = x + 3y \quad \textbf{(1)}$$

Conservation of momentum.

$$0.25 \times (7u + u) = y - x$$
$$2u = y - x \quad \textbf{(2)}$$

Newton's Law of Restitution.

$$\textbf{(1)} + \textbf{(2)}\ 6u = 4y$$
$$y = \frac{3u}{2}$$

Solve **(1)** and **(2)** simultaneously.

$$\text{In } \textbf{(2)}\ 2u = \frac{3u}{2} - x$$
$$x = -\frac{u}{2}$$

The minus sign shows the arrow in the diagram is pointing in the wrong direction.

A has speed $\frac{u}{2}$

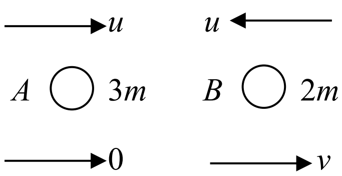
B has speed $\frac{3u}{2}$

Speed is always positive.

b K.E. lost

$$= \frac{1}{2} \times m \times (7u)^2 + \frac{1}{2} \times 3m \times u^2 - \left(\frac{1}{2} m \times \left(\frac{u}{2}\right)^2 + \frac{1}{2} \times 3m \left(\frac{3u}{2}\right)^2\right)$$
$$= \frac{1}{2} m \times 49u^2 + \frac{3}{2} mu^2 - \left(\frac{mu^2}{8} + \frac{27mu^2}{8}\right)$$
$$= \frac{45}{2} mu^2$$

9



a $3mu - 2mu = 2mv$

$$mu = 2mv$$

Conservation of momentum.

$$e(u + u) = v$$
$$2ue = v$$

Newton's Law of Restitution.

Eliminating v:

$$mu = 2m(2ue)$$
$$e = \frac{1}{4}$$

9 b K.E. lost

$$= \frac{1}{2} \times 3mu^2 + \frac{1}{2} \times 2mu^2 - \left(0 + \frac{1}{2} \times 2m\left(\frac{1}{2}u\right)^2\right)$$

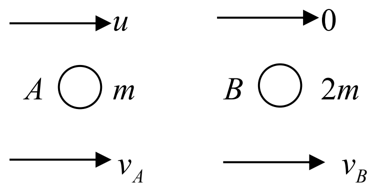
$$= \frac{5}{2}mu^2 - \frac{1}{4}mu^2$$

$$= \frac{9}{4}mu^2$$

$v = 2ue = 2u \times \frac{1}{4}$

$= \frac{1}{2}u$

10 a



$mu = mv_A + 3mv_B$

$y = v_A + 3v_B$ **(1)**

$eu = v_B - v_A$ **(2)**

Conservation of momentum.

Newton's Law of Restitution.

(1) + **(2)** $u + eu = 4v_B$

$$v_B = \frac{1}{4}(1+e)u$$

b Using **(2)**:

$v_A = v_B - eu$

$$= \frac{1}{4}(1+e)u - eu$$

$$= \frac{1}{4}(1-3e)u$$

10 c K.E. after impact

$$= \frac{1}{2}mv_A^2 + \frac{1}{2} \times 3mv_B^2$$

$$= \frac{1}{2}m\left(\frac{1}{4}(1-3e)u\right)^2 + \frac{3}{2}m\left(\frac{1}{4}(1+e)u\right)^2$$

$$= \frac{1}{2}m\frac{u^2}{16}(1-6e+9e^2) + \frac{3}{2}m\frac{u^2}{16}(1+2e+e^2)$$

$$= \frac{mu^2}{32}(1-6e+9e^2+3+6e+3e^2)$$

$$= \frac{mu^2}{32}(4+12e^2)$$

$$= \frac{mu^2}{8}(1+3e^2)$$

K.E. after impact $= \frac{1}{6}mu^2$

$$\therefore \quad \frac{1}{8}(1+3e^2) = \frac{1}{6}$$

$$6+18e^2 = 8$$

$$18e^2 = 2$$

$$e^2 = \frac{1}{9}$$

$$e = \frac{1}{3} \quad (e > 0)$$

d $v_A = \frac{u}{4}(1-3e)$

$$= \frac{u}{4}(1-3\times\frac{1}{3})$$

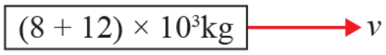
$$= 0$$

$\therefore$ A is at rest.

11 Initially: $m = 8 \times 10^3$ kg, $v = 4$ ms^{-1}, kinetic energy = E_{ki}

Before

After



$$E_k = \frac{1}{2}mv^2$$

$$E_{ki} = \frac{1}{2} \times 8 \times 10^3 \times 4^2 = 64 \times 10^3$$

Finally: $m = (8 \times 10^3 + 12 \times 10^3) = 20 \times 10^3$ kg, $v = 1.5$ ms^{-1}

$$E_{kf} = \frac{1}{2} \times 20 \times 10^3 \times 1.5^2 = 22.5 \times 10^3$$

Change in kinetic energy:

$$\Delta E_k = E_{ki} - E_{kf}$$

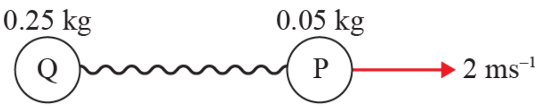
$$\Delta E_k = 64 \times 10^3 - 22.5 \times 10^3$$

$$\Delta E_k = 41.5 \times 10^3$$

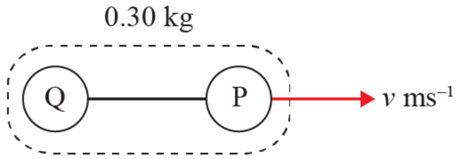
The loss in kinetic energy is 41.5 kJ.

12 Initially: $m = 0.05$ kg, $v = 2$ ms^{-1}, kinetic energy = E_{ki}

Before



After



$$E_k = \frac{1}{2}mv^2$$

$$E_{ki} = \frac{1}{2} \times \frac{1}{20} \times 2^2 = 0.1$$

Once string is taut, speed of the particles, v, is found using conservation of momentum:

$$0.05 \times 2 = (0.05 + 0.25)v$$
$$0.1 = 0.3v$$
$$v = \frac{1}{3}$$

and the total kinetic energy E_{kf} is

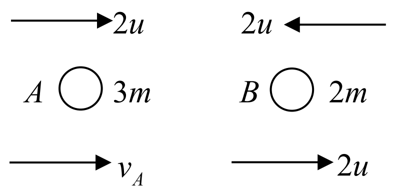
$$E_{kf} = \frac{1}{2} \times \frac{3}{10} \times \frac{1}{3}^2 = \frac{1}{60}$$

Change in kinetic energy:

$$\Delta E_k = E_{ki} - E_{kf}$$
$$\Delta E_k = \frac{1}{10} - \frac{1}{60}$$
$$\Delta E_k = \frac{1}{12}$$

The loss in kinetic energy is $\frac{1}{12}$ J

13 a



Direction of motion of *B* is reversed but speed is unchanged.

$$3m\times 2u-2m\times 2u=3mv_A+2m\times 2u$$
$$2u=3v_A+4u$$
$$v_A=-\frac{2}{3}u$$

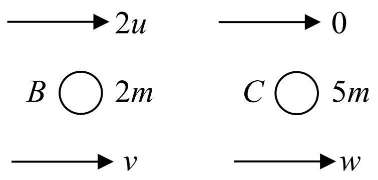
Conservation of momentum.

$$e(2u+2u)=2u-v_A$$
$$4eu=2u-v_A$$

Newton's Law of Restitution.

$$\therefore \quad 4eu=2u+\frac{2}{3}u$$
$$4eu=\frac{8}{3}u$$
$$e=\frac{2}{3}$$

b



$$2m\times 2u=2mv\times 5mw$$
$$4u=2v+5w \quad \textbf{(3)}$$

Conservation of momentum.

$$\frac{3}{5}\times 2u=w-v \quad \textbf{(4)}$$

Newton's Law of Restitution.

Eliminate w from **(3)** and **(4)**

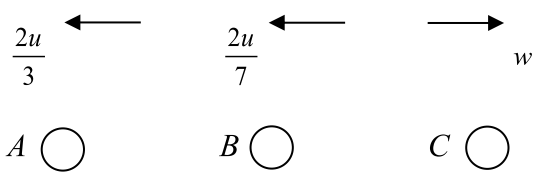
$$4u=2v+5\left(\frac{6u}{5}+v\right)$$
$$4u=2v+6u+5v$$
$$7v=-2u$$
$$v=-\frac{2}{7}u$$

B will not hit *C* again. You must investigate the possibility of it hitting *A* again.

From **a**

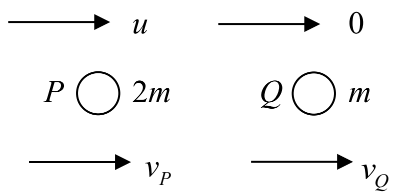
$$v_A=-\frac{2}{3}u$$

After the collision between *B* and *C*:



As speed *A* > speed *B* there will be no further collisions.

14 a



$2mu = 2mv_P + mv_Q$ ← Conservation of momentum.

$2u = 2v_P + v_Q$ **(1)**

$\frac{1}{3}u = v_Q - v_P$ **(2)** ← Newton's Law of Restitution.

(1) $+ 2 \times$ **(2)** $\frac{8u}{3} = v_Q + 2v_Q$

$3v_Q = \frac{8u}{3}$

$v_Q = \frac{8u}{9}$

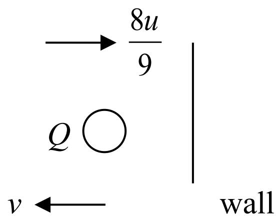
Using **(2)**

$v_P = v_Q - \frac{1}{3}u$

$v_P = \frac{8u}{9} - \frac{1}{3}u$

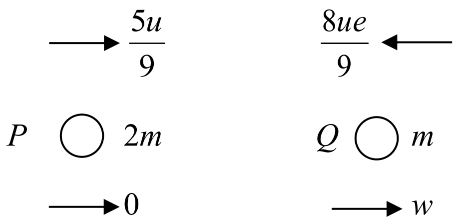
$v_P = \frac{5u}{9}$

14 b



$$v = \frac{8ue}{9}$$

Newton's Law of Restitution.



Here, $e = \frac{1}{3}$ again.

$$2m \times \frac{5u}{9} - m \times \frac{8ue}{9} = mw$$

Conservation of momentum.

$$10u - 8ue = 9w \quad \textbf{(3)}$$

Newton's Law of Restitution.

$$\frac{1}{3}\left(\frac{5u}{9} + \frac{8ue}{9}\right) = w$$

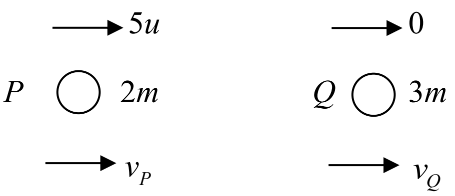
$$5u + 8ue = 27w \quad \textbf{(4)}$$

Eliminating w between **(3)** and **(4)**:

$$3(10u - 8ue) = 5u + 8ue$$
$$30u - 24ue = 5u + 8ue$$
$$32ue = 25u$$
$$e = \frac{25}{32}$$

c Q is now moving towards the wall once more. After Q hits the wall, it will return to collide with P once more.

15 a



$2m \times 5u = 2mv_P + 3mv_Q$

$10u = 2v_P + 3v_Q$ **(1)**

$e \times 5u = v_Q - v_P$ **(2)**

Conservation of momentum.

Newton's Law of Restitution.

$\mathbf{(1)} + 2 \times \mathbf{(2)}$:

$10u + 10eu = 3v_Q + 2v_Q$

$10u + 10eu = 5v_Q$

$v_Q = 2u + 2eu = 2(1+e)u$

b From **(2)**

Find direction of motion for P, as if P is moving towards the wall there must be a second collision between P and Q.

$v_P = v_Q - 5eu$

$= 2(1+e)u - 5eu$

$v_P = 2 \times 1.4u - 5 \times 0.4u$

$= 0.8u$

$e = 0.4$ in **b**.

$v_P > 0 \therefore$ P moves towards the wall and will collide with Q after Q rebounds from the wall.

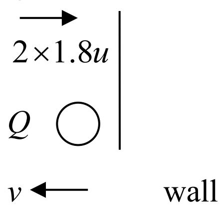
c $e = 0.8$

$e = 0.8$

$v_P = 2 \times 1.8u - 5 \times 0.8u$

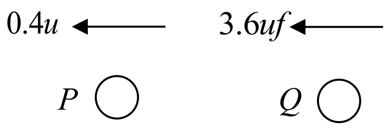
$= -0.4u$

Q hits the wall:



$v = 3.6uf$

Newton's Law of Restitution.



For a second collision

$3.6uf > 0.4u$

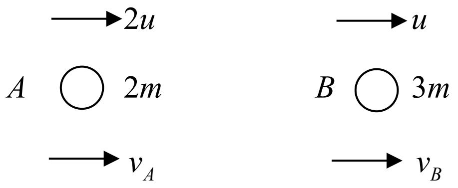
$f > \frac{0.4}{3.6} = \frac{1}{9}$

Range for f is

$\frac{1}{9} < f \leqslant 1$

All coefficients of restitution are less than or equal to 1.

16 a



$$2m \times 2u + 3m \times u = 2mv_A + 3mv_B$$

$$7u = 2v_A + 3v_B \quad \textbf{(1)}$$

$$e(2u - u) = v_B - v_A$$

$$eu = v_B - v_A \quad \textbf{(2)}$$

Conservation of momentum.

Newton's Law of Restitution.

(1) + 2 × **(2)**

$$7u + 2eu = 3v_B + 2v_B$$

$$v_B = \frac{1}{5}u(7 + 2e)$$

b Using **(2)**

$$v_A = v_B - eu$$

$$= \frac{1}{5}u(7 + 2e) - eu$$

$$= \frac{1}{5}u(7 - 3e)$$

c

$$\frac{1}{5}u(7 - 3e) = \frac{11u}{10}$$

$$14u - 6eu = 11u$$

$$6eu = 3u \quad e = \frac{1}{2}$$

d For B:

Distance to barrier = d

$$\text{Speed} = \frac{1}{5}u(7 + 1) = \frac{8u}{5}$$

Use $e = \frac{1}{2}$ now.

$$\therefore \text{Time to barrier} = d + \frac{8u}{5} = \frac{5d}{8u}$$

Distance moved by A in this time:

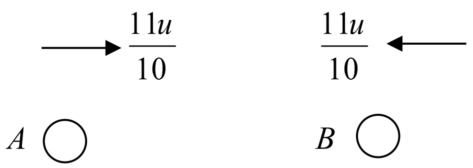
$$= \frac{1}{5}u\left(7 - \frac{3}{2}\right) \times \frac{5d}{8u}$$

$$= \frac{11u}{5 \times 2} \times \frac{5d}{8u} = \frac{11d}{16}$$

$$\therefore A \text{ is } d - \frac{11d}{16} = \frac{5d}{16} \text{ from the barrier.}$$

16 e After B hits the barrier:

$$\text{Speed of } B = \frac{11}{16} \times \frac{8u}{5} = \frac{11u}{10}$$



Equal speeds, opposite directions.

$\therefore$ A and B will collide at mid-point of the distance from A to the barrier at the instant B hits the barrier, i.e. they collide at distance $\frac{5d}{32}$ from the barrier.

17 a For the initial fall, down positive:

$u = 0\text{ ms}^{-1}$, $a = g$, $s = 2$ m, $v = ?$

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + (2\times 2g) = 4g$$

Speed after the first bounce, v_1, is given by Newton's law of restitution:

$$v_1 = ev$$

$$v_1^2 = e^2v^2$$

$$v_1^2 = 4ge^2$$

For the return, up positive: $v = 0\text{ ms}^{-1}$, $u = v_1$, $a = -g$, and it reaches a height $s = s_1$

$$v^2 = u^2 + 2as$$

$$0 = 4ge^2 - 2gs$$

$$s_1 = 2e^2$$

When it hits the ground for a second time, it has travelled a distance $(2 + 2s_1)$ m $= 2(1 + 2e^2)$ m and is again travelling at $v_1\text{ ms}^{-1}$

Speed after the bounce, v_2, is given by Newton's law of restitution:

$$v_2 = ev_1$$

$$v_2^2 = e^2v_1^2$$

$$v_2^2 = e^2e^2v^2 = e^4v^2$$

$$v_2^2 = 4ge^4$$

For the second return, up positive: $v = 0\text{ ms}^{-1}$, $u = v_2$, $a = -g$, and it reaches a height $s = s_2$

$$v^2 = u^2 + 2as$$

$$0 = 4ge^4 - 2gs_2$$

$$s_2 = 2e^4$$

When it hits the ground for a third time, it has travelled a distance $2(1 + 2e^2 + 2e^4)$ m

In general, when it hits the ground for the $(n + 1)$th time, it will have travelled a distance s where

$$s = 2 + 4\sum_{k=1}^{k=n} e^{2k}$$

The second of these terms includes the sum of a geometric series where $a = r = e^2$

Since e < 1, the bounce height approaches zero asymptotically and there are, theoretically, an infinite number of bounces. Using the formula for the sum of an infinite geometric series:

$$S_\infty = \frac{a}{1-r}$$

$$s = 2 + 4\left(\frac{e^2}{1-e^2}\right)$$

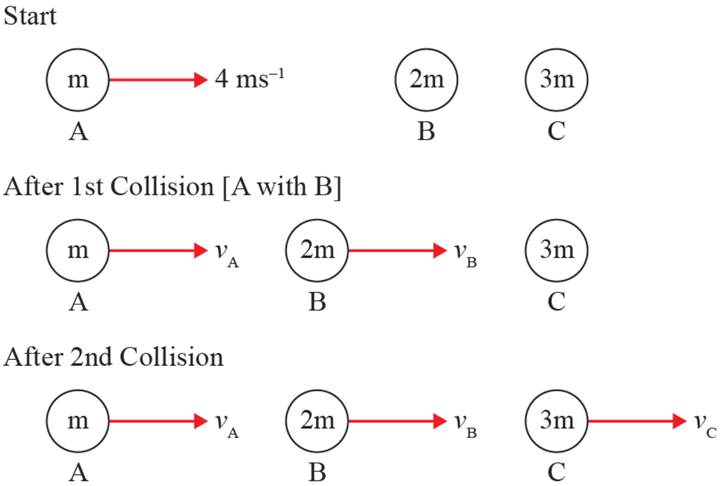
$$s = 2 + 4\left(\frac{0.8^2}{1-0.8^2}\right)$$

$$s = 2 + \left(4\times\frac{16}{9}\right) = \frac{82}{9}$$

The total distance travelled by the ball before it comes to rest is $\dfrac{82}{9}$ m

b The model is unrealistic because the ball is not a particle, so there will not be an infinite number of bounces; it will stop once the bounce height is less than its radius. There will also be factors such as additional energy losses following each 'bounce'.

18 After A hits B, A moves with speed v_A, and B moves with speed v_B



Using conservation of momentum:

$$4m = mv_A + 2mv_B$$

$$4 = v_A + 2v_B \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$v = eu$$

$$v_B - v_A = 0.7 \times 4 = 2.8 \qquad \textbf{(2)}$$

(1) + **(2)** to find v_B, then substituting back into **(2)** for v_A:

$$2v_B + v_B = 4 + 2.8$$

$$v_B = \frac{6.8}{3} = \frac{68}{30} = \frac{34}{15}$$

$$v_B - v_A = \frac{28}{10}$$

$$v_A = \frac{68}{30} - \frac{28}{10}$$

$$v_A = \frac{68 - 84}{30} = -\frac{16}{30}$$

Since A is travelling in the opposite direction to B (and the opposite to that shown in the diagram), it will not collide with B again. However, B is moving in the right direction to collide with C. After B hits C, B moves with speed $v_{B'}$, and C moves with speed v_C and their relative speed is $v_C - v_{B'}$

Using conservation of momentum:

$$\frac{34}{15} \times 2m = 2mv_B' + 3mv_C$$

$$\frac{68}{15} = 2v_B' + 3v_C \qquad \textbf{(3)}$$

Using Newton's law of restitution:

$$v_C - v_B' = 0.4 \times \frac{34}{15} = \frac{68}{75} \qquad \textbf{(4)}$$

18 continued

(3) + 2 × **(4)** to find v_C, then substituting back into **(4)** for $v_{B'}$:

$$3v_C + 2v_C = \frac{68}{15} + 2\frac{68}{75}$$

$$5v_C = \frac{476}{75}$$

$$v_C = \frac{476}{375}$$

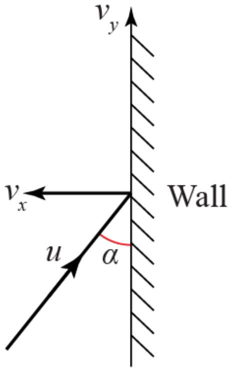
$$v_C - v_B' = \frac{68}{75}$$

$$v_B' = \frac{476}{375} - \frac{68}{75}$$

$$v_B' = \frac{476-340}{375} = \frac{136}{375}$$

B and *C* move off in the same direction and neither can subsequently collide with A as it is moving in the opposite direction. Since $v_{B'} < v_C$, *B* cannot reach *C* to collide with it again and so there are no further collisions.

19



Let the components of the velocity perpendicular and parallel to the wall immediately after the collision be v_x and v_y respectively.

Parallel to the wall

$v_y = u\cos\alpha$

The impulse is perpendicular to the wall and so the component of the velocity parallel to the wall is unchanged.

Perpendicular to the wall

Newton's law of restitution

$v_x = eu\sin\alpha$

Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, the component after collision $= e \times$ the component before collision

The component of the velocity perpendicular to the wall before collision is $u\sin\alpha$.

The kinetic energy of S after the collision is given by

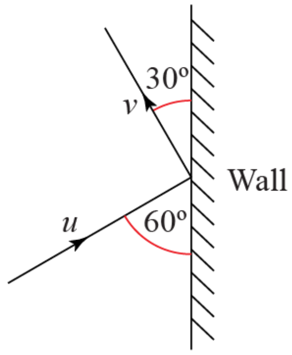
$$\frac{1}{2}m(v_x^2 + v_y^2)$$

$$= \frac{1}{2}m(e^2u^2\sin^2\alpha + u^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2(e^2\sin^2\alpha + \cos^2\alpha)$$

If v is the velocity after collision, the kinetic energy of S after the collision is $\frac{1}{2}mv^2$ and $v^2 = v_x^2 + v_y^2$

20 a



Let the speed of the ball before impact be u m s^{-1} and the speed of the ball after impact be v m s^{-1}

Parallel to the wall

$u\cos 60^\circ = v\cos 30^\circ$

$$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow v = \frac{u}{\sqrt{3}}$$

As the impulse of the wall on the ball is perpendicular to the wall, parallel to the wall the component of the velocity of the ball is unchanged.

The kinetic energy lost is

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}m\left(\frac{u}{\sqrt{3}}\right)^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{6}mu^2 = \frac{1}{3}mu^2$$

Substituting $v = \frac{u}{\sqrt{3}}$

The fraction of the kinetic energy lost is

$$\frac{\frac{1}{3}mu^2}{\frac{1}{2}mu^2} = \frac{2}{3}$$

You find the $\frac{\text{loss in kinetic energy}}{\text{originsal kinetic energy}}$

b Perpendicular to the wall

Newton's law of restitution

$v\sin 30^\circ = eu\cos 60^\circ$

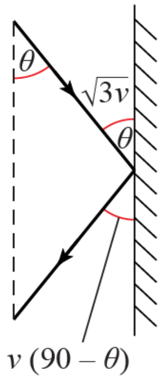
$$\frac{1}{2}v = \frac{\sqrt{3}}{2}eu$$

As $v = \frac{u}{\sqrt{3}}$

$$\frac{1}{2}\times\frac{u}{\sqrt{3}} = \frac{\sqrt{3}}{2}eu \Rightarrow e = \frac{1}{\sqrt{3}\times\sqrt{3}} = \frac{1}{3}$$

Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, component after collision = $e\times$ component before collision.

21 a



Using conservation of momentum in the direction parallel to the wall:

$$\sqrt{3}mv\cos\theta = mv\cos(90-\theta)$$
$$\sqrt{3}\cos\theta = \sin\theta$$
$$\tan\theta = \sqrt{3}$$

The sphere is initially travelling at 60° to the wall.

b Considering the components of motion perpendicular to the wall and using Newton's law of restitution:

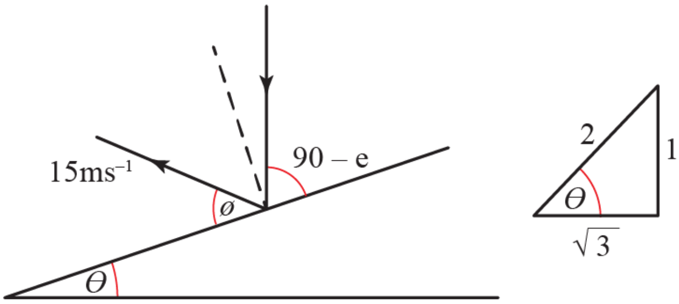
$$v\sin(90-\theta) = e\sqrt{3}v\sin\theta$$
$$\cos\theta = e\sqrt{3}\sin\theta$$
$$e = \frac{1}{\sqrt{3}\tan\theta} = \frac{1}{\sqrt{3}\times\sqrt{3}}$$

So the coefficient of restitution is $\frac{1}{3}$, as required.

22 $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$

For the fall, down positive: $u = 0\text{ ms}^{-1}$, $a = g$, $s = 20$ m, $v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 2g \times 20 = 40g$$



Momentum is conserved parallel to the plane so:

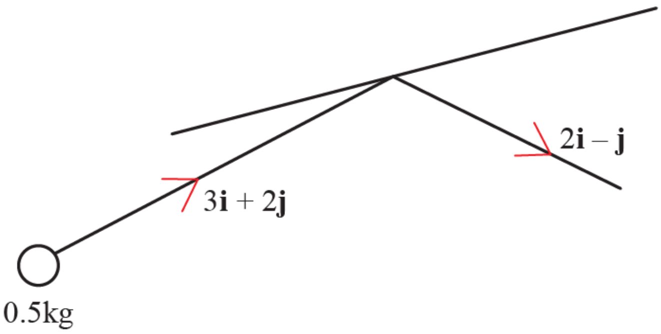
$$m\sqrt{40g}\cos(90-\theta) = m \times 15\cos\phi$$
$$\sqrt{40g}\sin\theta = 15\cos\phi$$
$$\cos\phi = \frac{\sqrt{40g} \times \frac{1}{2}}{15} = \frac{\sqrt{40g}}{30}$$
$$\phi = 48.702...$$
$$\sin\phi = 0.75129...$$

Considering the components of movement perpendicular to the plane and using Newton's law of restitution:

$$15\sin\phi = e\sqrt{40g}\cos\theta$$
$$e = \frac{15 \times 0.75129...}{\sqrt{40g} \times \frac{\sqrt{3}}{2}}$$
$$e = \frac{11.269...}{\sqrt{30g}} = 0.65724...$$

The coefficient of restitution is 0.657 (3 s.f.)

23 a



Impulse = change in momentum

$= \frac{1}{2}(2\mathbf{i}-\mathbf{j}) - \frac{1}{2}(3\mathbf{i}+2\mathbf{j})$

$= \frac{1}{2}(-\mathbf{i}-3\mathbf{j})$

$= \frac{\sqrt{10}}{2}\left(\frac{1}{\sqrt{10}}(-\mathbf{i}-3\mathbf{j})\right)$

The impulse is of magnitude $\frac{1}{2}\sqrt{10}$ Ns and acts parallel to the unit vector $\frac{1}{\sqrt{10}}(-\mathbf{i}-3\mathbf{j})$

b Kinetic energy $E_k = \frac{1}{2}mv^2$ so kinetic energy lost:

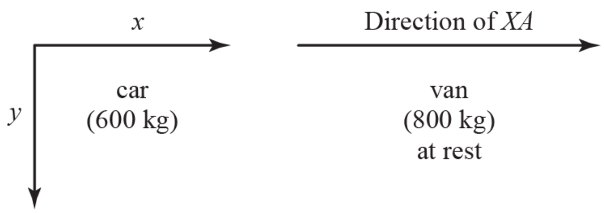
$\Delta E_k = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

$\Delta E_k = \frac{1}{2}m\left(\left(3^2+2^2\right)-\left(2^2+1^2\right)\right)$

$\Delta E_k = \frac{1}{2}\times\frac{1}{2}(13-5) = \frac{8}{4} = 2$

The kinetic energy lost in the collision is 2 J.

24 a Let the components of the velocity of the car before the collision, with all components in m s^{-1}, be



As the van is at rest, after the collision it must travel along the line of centres of the car and the van. In the diagram in the question, XA must be the line of centres, so you consider the components of velocity perpendicular and parallel to XA.

Let the components of the velocity of the car and van after the collision, with all components in m s^{-1}, be

After the collision, the van is moving along XA and the car is moving perpendicular to XA.

Parallel to XA

Conservation of linear momentum

$$600x = 800w \Rightarrow w = \frac{3}{4}x \quad *$$

Newton's law of restitution

velocity of separation = e × velocity of approach

$$w = ex$$

Hence

$$\frac{3}{4}x = ex$$

$$e = \frac{3}{4}$$

24 b For the van

$$\mathbf{F} = m\mathbf{a}$$
$$-500 = 800a \Rightarrow a = -0.625$$
$$v^2 = u^2 + 2as$$
$$0^2 = w^2 - 2 \times 0.625 \times 45$$
$$w^2 = 56.25 \Rightarrow w = 7.5$$

For the car

$$\mathbf{F} = m\mathbf{a}$$
$$-300 = 600a \Rightarrow a = -0.5$$
$$v^2 = u^2 + 2as$$
$$0^2 = y^2 - 2 \times 0.5 \times 21$$
$$y^2 = 21 \Rightarrow y = \sqrt{21}$$

To find w (and hence x) and y, you need to use both Newton's second law and the kinematic equation for constant acceleration, $v^2 = u^2 + 2as$

Conservation of momentum in the direction XA $(\rightarrow)$

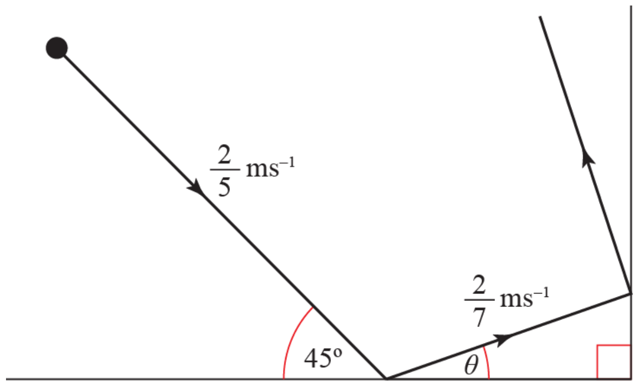
$$600x = 800 \times 7.5$$

$$x = \frac{800 \times 7.5}{600} = 10$$

So the speed of the car before the collision is given by

$$\sqrt{x^2 + y^2} = \sqrt{100 + 21} = \sqrt{121} = 11 \text{ m s}^{-1}$$

25 a Using conservation of momentum for component of motion parallel to the wall:



$$\frac{2}{5}\cos 45 = \frac{2}{7}\cos\theta$$

$$\cos\theta = \frac{7}{2}\times\frac{2}{5}\times\frac{1}{\sqrt{2}} = \frac{7}{5\sqrt{2}}$$

$$\theta = 8.1301...$$

After the first impact the sphere moves at an angle of 8.13° (3 s.f.) to the first wall.

$$\sin^2\theta = 1-\cos^2\theta$$

$$\sin^2\theta = 1-\frac{49}{50}$$

$$\sin\theta = \sqrt{\frac{1}{50}} = \frac{1}{\sqrt{50}}$$

b Using Newton's law of restitution for components of movement perpendicular to the first wall:

$$\frac{2}{7}\sin\theta = e\times\frac{2}{5}\sin 45$$

$$e = \frac{\frac{2}{7}\times\frac{1}{\sqrt{50}}}{\frac{2}{5}\times\frac{1}{\sqrt{2}}}$$

$$e = \frac{10\sqrt{2}}{14\sqrt{50}} = \frac{10}{14\times 5} = \frac{1}{7}$$

The coefficient of restitution has a value of $\frac{1}{7}$

25 c From conservation of momentum, component of velocity parallel to second wall, v_v, is unchanged by the second collision:

$$v_v = \frac{2}{7}\cos(90-\theta)$$
$$v_v = \frac{2}{7}\sin\theta = \frac{2}{7}\sqrt{\frac{1}{50}}$$

Using Newton's law of restitution for components of movement perpendicular to the first wall, perpendicular component of velocity v_h:

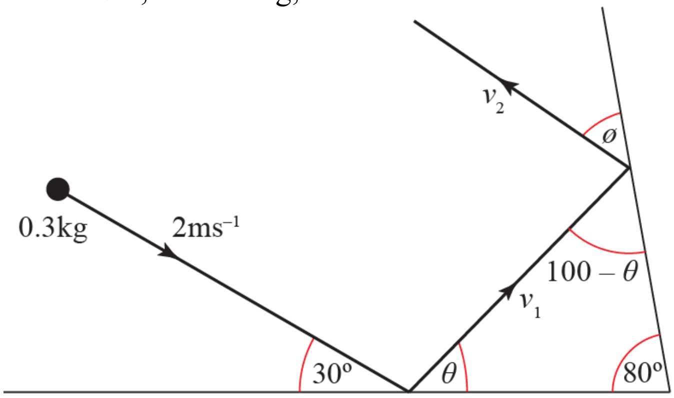
$$v_h = eu_h$$
$$v_h = \frac{1}{7}\times\frac{2}{7}\cos\theta$$
$$v_h = \frac{1}{7}\times\frac{2}{7}\times\frac{7}{5\sqrt{2}}$$
$$v_h = \frac{\sqrt{2}}{35}$$

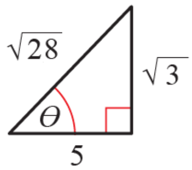
Resultant speed, v, is given by $v^2 = v_h^2 + v_v^2$ and the final kinetic energy is therefore:

$$E_k = \frac{1}{2}mv^2$$
$$E_k = \frac{1}{2}\times 0.8\times\left(\left(\frac{2}{7\sqrt{50}}\right)^2+\left(\frac{\sqrt{2}}{35}\right)^2\right)$$
$$E_k = \frac{1}{2}\times\frac{4}{5}\times\left(\frac{4}{49\times 50}+\frac{2}{35\times 35}\right)$$
$$E_k = \frac{2}{5}\times\frac{2}{49}\times\left(\frac{2}{50}+\frac{1}{25}\right)$$
$$E_k = \frac{4}{245}\times\frac{2}{25} = \frac{8}{6125} = \frac{8}{6125} = 0.0013061...$$

After the second collision, the kinetic energy of the sphere is 1.31 mJ (3 s.f.).

26 $u = 2\text{ ms}^{-1}$, $m = 0.3$ kg, $e = 0.6$





First collision

Considering components of motion parallel to wall and using conservation of momentum:

$2\cos 30 = v_1 \cos\theta$ **(1)**

Considering components of motion perpendicular to wall and using Newton's law of restitution:

$0.6 \times 2\sin 30 = v_1 \sin\theta$ **(2)**

(2) ÷ (1)

$0.6\tan 30 = \tan\theta$

$$\tan\theta = \frac{3}{5} \times \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{5}$$

$$\Rightarrow \cos\theta = \frac{5}{\sqrt{28}}$$

Substituting into **(1)**

$$2\frac{\sqrt{3}}{2} = \frac{5}{\sqrt{28}} v_1$$

$$v_1 = \frac{\sqrt{84}}{5}$$

Second collision

Using conservation of momentum for components of motion parallel to wall:

$v_1 \cos(100-\theta) = v_2 \cos\phi$ **(3)**

Using Newton's law of restitution for components of motion perpendicular to wall:

$0.6 \times v_1 \sin(100-\theta) = v_2 \sin\phi$ **(4)**

(4) ÷ (3)

$\tan\phi = 0.6 \times \tan(100 - 19.106...)$

$\tan\phi = 3.7431...$

$\phi = 75.042...$

$\cos\phi = 0.25810...$

Substituting into **(3)**

$\frac{\sqrt{84}}{5} \times 0.15827... = v_2 \times 0.25810...$

$v_2 = 1.1240...$

26 continued

Kinetic energy $E_k = \frac{1}{2}mv^2$ so kinetic energy lost:

$$\Delta E_k = \frac{1}{2}mu^2 - \frac{1}{2}mv_2^{\,2}$$

$$\Delta E_k = \frac{1}{2}\times\frac{3}{10}\left(2^2 - 1.1240...^2\right)$$

$$\Delta E_k = 0.41047...$$

The total kinetic energy lost is 0.41 J (2 s.f.)

27a First collision: $u = 4$ m s^{-1}, $v = 3$ m s^{-1}

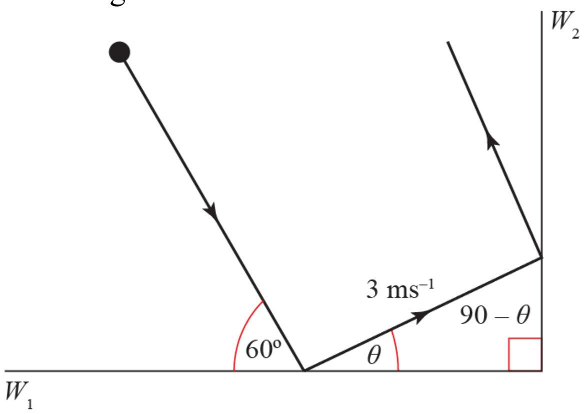
Considering components of motion parallel to wall and using conservation of momentum:

$$4\cos 60 = 3\cos\theta$$

$$\cos\theta = \frac{4}{3}\times\frac{1}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{5}}{3} \text{ and } \tan\theta = \frac{\sqrt{5}}{2}$$

Considering components of motion perpendicular to wall and using Newton's law of restitution:



$$3\sin\theta = e\times 4\sin 60$$

$$e = \frac{3\times\frac{\sqrt{5}}{3}}{4\times\frac{\sqrt{3}}{2}}$$

$$e = \frac{\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{15}}{6}$$

The coefficient of restitution is $\frac{\sqrt{15}}{6}$

b Second collision: $u = 3$ ms^{-1}, $e = 0.35$

Considering components of motion parallel to wall and using conservation of momentum:

$3\sin\theta = v\cos\phi$ **(1)**

Considering components of motion perpendicular to wall and using Newton's law of restitution:

$e\times 3\cos\theta = v\sin\phi$ **(2)**

27 continued

(2) ÷ (1)

$$\tan\phi = \frac{e}{\tan\theta} = \frac{0.35}{\frac{\sqrt{5}}{2}}$$

$$\tan\phi = \frac{7}{10\times\sqrt{5}}$$

$$\phi = 17.382...$$

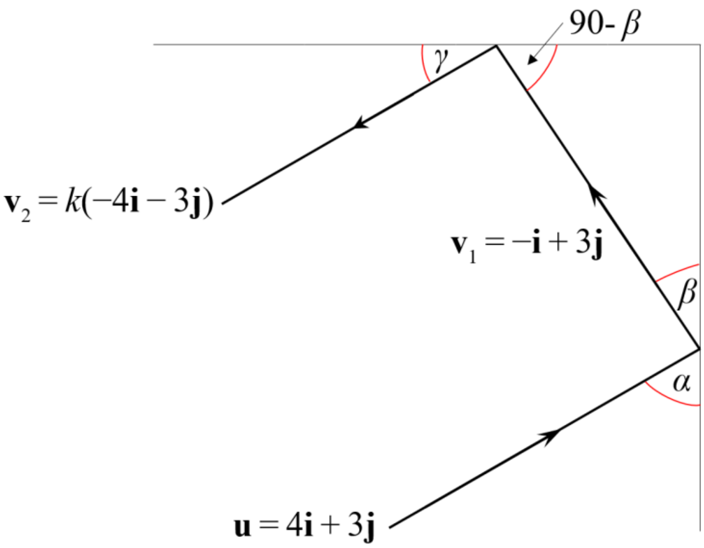
Substituting into **(1)**

$$3\times\frac{\sqrt{5}}{3} = v\times 0.95433...$$

$$v = \frac{\sqrt{5}}{0.95433...} = 2.3430...$$

After the second collision, the sphere travels at 2.34 ms^{-1} at an angle of 17.4° to W_2 (both 3 s.f.)

28



Initial values:

$$u = \sqrt{4^2 + 3^2} = 5$$

$$v_1 = \sqrt{1^2 + 3^2} = \sqrt{10}$$

First collision:

COM: $5\cos\alpha = \sqrt{10}\cos\beta$

NLR: $e(5\sin\alpha) = \sqrt{10}\sin\beta$

$\Rightarrow e\tan\alpha = \tan\beta$

Second collision:

COM: $\sqrt{10}\sin\beta = v\cos\gamma$

NLR: $e\sqrt{10}\cos\beta = v\sin\gamma$

$$\Rightarrow \frac{e}{\tan\beta} = \tan\gamma$$

28 continued

Substituting for $\tan\beta$:

$$\tan\gamma = \frac{e}{e\tan\alpha} = \frac{1}{\tan\alpha}$$

i.e. $\alpha = 90 - \gamma$ so the final motion is antiparallel to the initial motion (this is a general result for equal values of e and perpendicular walls)

$\mathbf{v} = k(-4\mathbf{i} - 3\mathbf{j})$ where $k < 1$

Using $\mathbf{i}$ component of velocity in first collision to calculate e

$$e = \frac{|v|}{|u|} = \frac{1}{4}$$

Using $\mathbf{j}$ component of velocity in second collision to calculate k

$$e = \frac{|v|}{|u|} = \frac{3k}{3} = \frac{1}{4}$$

$$k = \frac{1}{4}$$

Now calculate v_2

$$v_2 = \sqrt{1^2 + \left(\frac{3}{4}\right)^2} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

The kinetic energy lost is therefore

$$\Delta E_k = \frac{1}{2}mu^2 - \frac{1}{2}mv_2{}^2$$

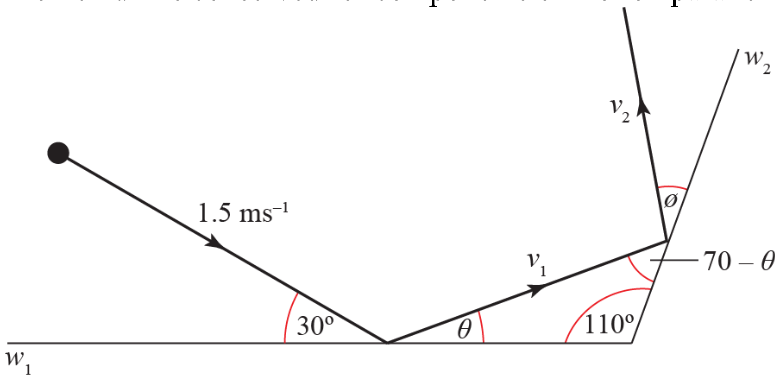
$$\Delta E_k = \frac{1}{2}m\left(5^2 - \left(\frac{5}{4}\right)^2\right)$$

$$\Delta E_k = \frac{m}{2} \times \frac{375}{16} = 11.71875m$$

The kinetic energy lost is 11.7 m J (3 s.f.)

29 a First collision: $u = 1.5\text{ ms}^{-1}$, $e = 0.8$

Momentum is conserved for components of motion parallel to W_1:



$1.5\cos 30 = v_1 \cos\theta$ **(1)**

Considering components of motion perpendicular to W_1 and using Newton's law of restitution:

$e \times 1.5\sin 30 = v_1 \sin\theta$ **(2)**

(2) ÷ **(1)**

$0.8\tan 30 = \tan\theta$

$\tan\theta = \frac{4}{5\sqrt{3}}$ $\left(\Rightarrow \cos\theta = \frac{5\sqrt{3}}{\sqrt{91}}\right)$

$\theta = \tan^{-1}\left(\frac{4}{5\sqrt{3}}\right) = 24.791...$

Substituting into **(1)**

$$1.5 \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{\sqrt{91}} v_1$$

$$\frac{3}{4} = \frac{5}{\sqrt{91}} v_1$$

$$v_1 = \frac{3\sqrt{91}}{20} = 1.4309...$$

After the first collision the sphere moves at 1.43 ms^{-1} at an angle of 24.8° to W_1 (both 3 s.f.)

29 b After second collision, kinetic energy, $E_k = 1.35$ J, $m = 1.6$ kg, speed = v_2, so:

$$E_k = \frac{1}{2}mv^2$$
$$1.35 = \frac{1}{2}\times 1.6{v_2}^2$$
$$v_2 = \sqrt{\frac{2.7}{1.6}} = \frac{3\sqrt{3}}{4}$$

Considering components of motion parallel to W_2 and using conservation of momentum:

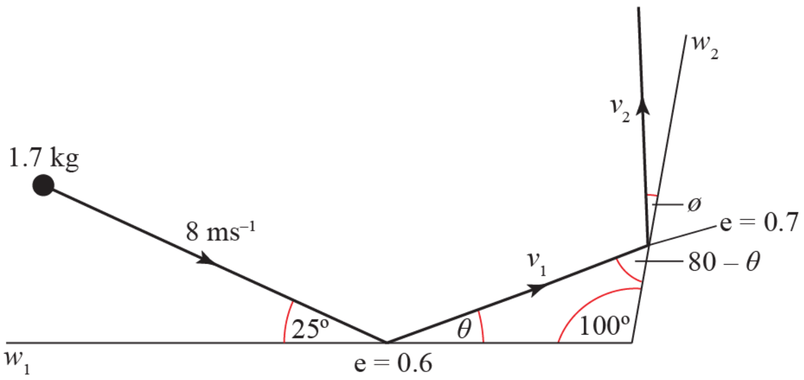
$$v_1\cos(70-\theta) = v_2\cos\phi$$
$$\cos\phi = \frac{\frac{3\sqrt{91}}{20}\cos(70-24.791...)}{\frac{3\sqrt{3}}{4}}$$
$$\cos\phi = \frac{\sqrt{91}\times 0.70452...}{5\sqrt{3}} = 0.77604...$$
$$\text{and } \sin^2\phi = 1-\cos^2\phi$$
$$\sin\phi = 0.63067...$$

Considering components of motion perpendicular to W_2 and using Newton's law of restitution:

$$e\times v_1\sin(70-\theta) = v_2\sin\phi$$
$$e = \frac{v_2\sin\phi}{v_1\sin(70-\theta)}$$
$$e = \frac{\frac{3\sqrt{3}}{4}\times 0.63067...}{\frac{3\sqrt{91}}{20}\sin(70-24.791...)}$$
$$e = \frac{5\sqrt{3}\times 0.63067...}{\sqrt{91}\times 0.70967...}$$
$$e = 0.80678...$$

The coefficient of restitution between the sphere and W_2 is 0.807 (3s.f.)

30



First collision
Considering components of motion parallel to wall and using conservation of momentum:
$8\cos 25 = v_1 \cos\theta$ **(1)**

Considering components of motion perpendicular to wall and using Newton's law of restitution:
$0.6\times 8\sin 25 = v_1 \sin\theta$ **(2)**

(2)÷ (1)

$0.6\tan 25 = \tan\theta$

$\tan\theta = 0.27978...$

$\theta = 15.630...$

Substituting into **(1)**

$8\cos 25 = v_1 \cos 15.630...$

$$v_1 = \frac{8\cos 25}{\cos 15.630...} = 7.5288...$$

Second collision
Using conservation of momentum for components of motion parallel to wall:
$v_1 \cos(80-\theta) = v_2 \cos\phi$ **(3)**

Using Newton's law of restitution for components of motion perpendicular to wall:
$0.7\times v_1 \sin(80-\theta) = v_2 \sin\phi$ **(4)**

(4) ÷ (3)
$\tan\phi = 0.7\times\tan(80-15.630...)$

$\tan\phi = 1.4589...$

$\phi = 55.573...$

30 continued

Substituting into **(3)**

$$v_2 = \frac{v_1 \cos(80-\theta)}{\cos\phi}$$

$$v_2 = \frac{7.5288...\times\cos(80-15.630...)}{0.56532...}$$

$$v_2 = 5.7606...$$

Kinetic energy $E_k = \frac{1}{2}mv^2$ so kinetic energy lost:

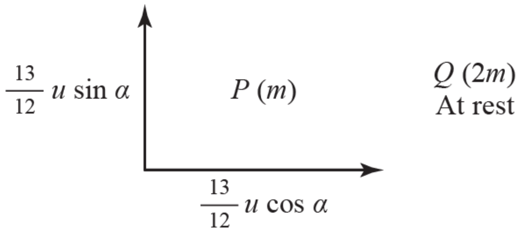
$$\Delta E_k = \frac{1}{2}mu^2 - \frac{1}{2}m{v_2}^2$$

$$\Delta E_k = \frac{1}{2}\times 1.7\left(8^2 - 5.7606...^2\right)$$

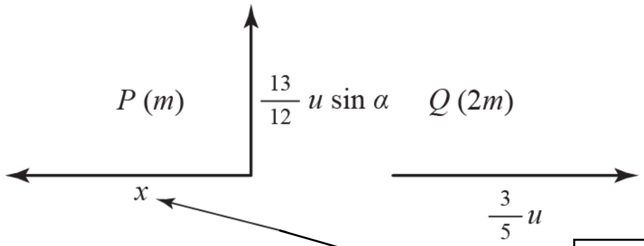
$$\Delta E_k = 26.193...$$

The total kinetic energy lost is 26.2 J (3 s.f.)

31 a Components of the velocities before the collision

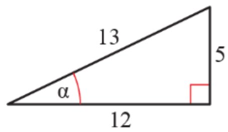


Let the components of the velocities after the collision be



The direction of the component of the velocity of P along the line of centres, here called x, is not obvious. If you put it in the opposite direction to that shown here, you would get a negative value of x and your solution would still be valid.

$$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}, \cos\alpha = \frac{12}{13}$$



This sketch illustrates that, as $5^2 + 12^2 = 13^2$, if $\tan\alpha = \frac{5}{12}$, then $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$

Perpendicular to the line of centres CB

In this direction, the component of the velocity of P is unchanged and is

$$\frac{13}{12}u\sin\alpha \frac{13}{12}u \times \frac{5}{13} = \frac{5}{12}u$$, as required.

Perpendicular to the line of centres CB

Conservation of linear momentum

$$m \times \frac{13}{12}u\cos\alpha = -mx + 2m \times \frac{3}{5}u$$

$$x = \frac{6}{5}u - \frac{13}{12}u \times \frac{12}{13} = \frac{6}{5}u - u = \frac{1}{5}u, \text{as required}$$

31 b Newton's law of restitution

velocity of separation = e × velocity of approach

$$x+\frac{3}{5}u=e\frac{13}{12}u\cos\alpha$$
$$\frac{1}{5}u+\frac{3}{5}u=eu$$
$$e=\frac{4}{5}$$

$\frac{13}{12}u\cos\alpha=\frac{13}{12}u\times\frac{12}{13}=u$

c Let the time after the collision for Q to reach C be t_1

distance = speed × time

$$d_1=\frac{3}{5}ut_1\Rightarrow t_1=\frac{5d_1}{3u}$$

Perpendicular to W, in time t_1, P travels a distance s given by
distance = speed × time

$$s=\frac{1}{5}u\times t_1=\frac{1}{5}u\times\frac{5d_1}{3u}=\frac{1}{3}d_1$$

Perpendicular to W, the component of the velocity of P after the collision is $\frac{1}{5}u$

To find the distance of P from W, you need consider only this component.

The distance of P from W, is

$$d_1+s=d_1+\frac{1}{3}d_1=\frac{4}{3}d_1\text{, as required}$$

d Before hitting W, Q, has speed $\frac{3}{5}u$

After hitting W, Q has speed $e\frac{3}{5}u=\frac{1}{2}\times\frac{3}{5}u=\frac{3}{10}u$

In the direction CB, the velocity of Q relative to P is

$$\frac{3}{10}u-\frac{1}{5}u=\frac{1}{10}u$$

The time, t_2, for Q to travel from C to the point of the second collision is given by

$$t_2=\frac{\frac{4}{3}d_1}{\frac{1}{10}u}=\frac{40d_1}{3u}$$

In the direction CB, the time is given by the distance of Q relative to $P\left(\frac{4}{3}d_1\right)$ divided by the velocity of Q relative to $P\left(\frac{1}{10}u\right)$

The time between the two collision is

$$t_1+t_2=\frac{5d_1}{3u}+\frac{40d_1}{3u}=\frac{45d_1}{3u}=\frac{15d_1}{u}\text{, as required}$$

31 e Before hitting the perpendicular wall, P has a component velocity $\frac{5}{12}u$ perpendicular to CB.

After hitting the wall, this component becomes
$$e\frac{5}{12}u = \frac{1}{2}\times\frac{5}{12}u = \frac{5}{24}u$$

If t_3 is the time for P to move from B to the wall and t_4 is the time for P to move from the wall back to CB, then

As Q moves along CB the second collision must occur on CB. So you need to find the time it takes for P to move to the wall and return to CB.

$$t_3 = \frac{d_2}{\frac{5}{12}u} = \frac{12d_2}{5u}$$

and

$$t_4 = \frac{d_2}{\frac{5}{24}u} = \frac{24d_2}{5u}$$

$$t_3 + t_4 = \frac{15d_1}{u}$$

Q is moving along CB. So, for the second collision, P must travel from CB to the wall, which is perpendicular to W, and back to the line CB, in time $\frac{15d_1}{u}$

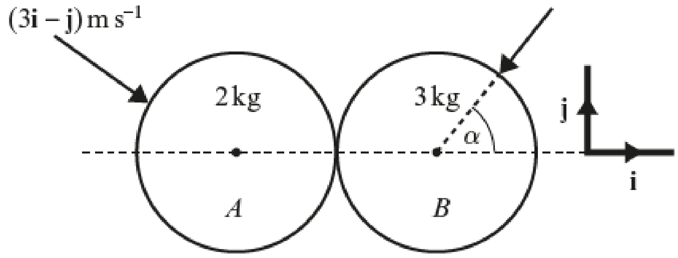
$$\frac{12d_2}{5u} + \frac{24d_2}{5u} = \frac{15d_1}{u}$$

$$36d_2 = 75d_1$$

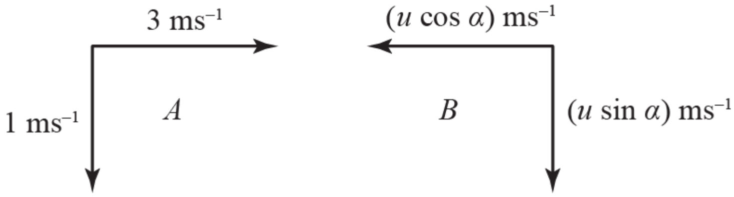
So $\frac{d_1}{d_2} = \frac{36}{75} = \frac{12}{25}$

i.e. the ratio $d_1 : d_2 = 12:25$

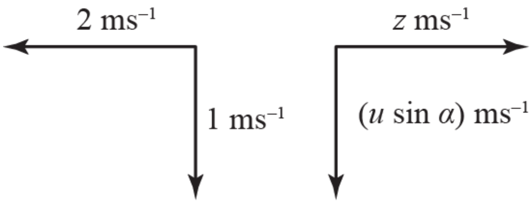
32 Let the speed of B before the collision be u m s^{-1}



Components of velocity before collision



Let the components of velocity after collision be



Parallel to **j**, the components of the velocity are unchanged and you can just write down.

Parallel to **i**

Conservation of linear momentum

$$2\times 3-3\times u\cos\alpha = 2\times(-2)+3z$$

$$6-3u\cos\alpha = -4+3z$$

$$3u\cos\alpha + 3z = 10 \qquad \textbf{(1)}$$

Newton's law of restitution

velocity of separation = e × velocity of approach

$$2+z=\frac{1}{2}(3+u\cos\alpha)$$

$$u\cos\alpha - 2z = 1 \qquad \textbf{(2)}$$

Equations (**1**) and (**2**) are a pair of simultaneous equations in $u\cos\alpha$ and z. The question asks you to find the velocity of B before the collision. You do not need to know z, so eliminate it.

(1)×2

$$6u\cos\alpha + 6z = 20 \qquad \textbf{(3)}$$

(2)×3

$$3u\cos\alpha - 6z = 3 \qquad \textbf{(4)}$$

(3)+**(4)**

$$9u\cos\alpha = 23$$

$$u\cos\alpha = \frac{23}{9}$$

$$\tan\alpha = \frac{u\sin\alpha}{u\cos\alpha} = 2$$

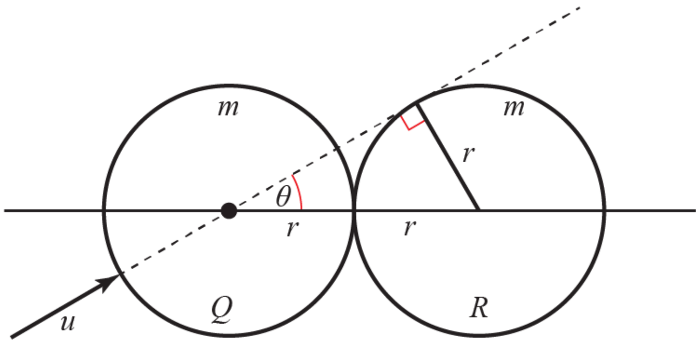
The question gives you that $\tan\alpha = 2$ and, as you have found $u\cos\alpha$, you can use this result to find $u\sin\alpha$.

$$u\sin\alpha = 2u\cos\alpha = 2\times\frac{23}{9} = \frac{46}{9}$$

The velocity of B before the collision is

$$(-u\cos\alpha\mathbf{i} - u\sin\alpha\mathbf{j})\text{m s}^{-1} = \left(-\frac{23}{9}\mathbf{i} - \frac{46}{9}\mathbf{j}\right)\text{m s}^{-1}$$

33



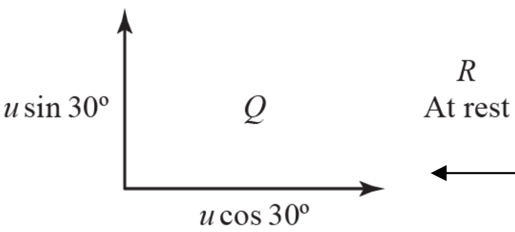
Let the mass of each of the spheres be m and the radius of each of the spheres be r.
Let the angle the direction of motion of Q makes with the line of centres before the impact be θ

$\sin\theta = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = 30^\circ$

Hence, the angle the direction of motion of Q makes with the line of centres, after the impact, is 60°.

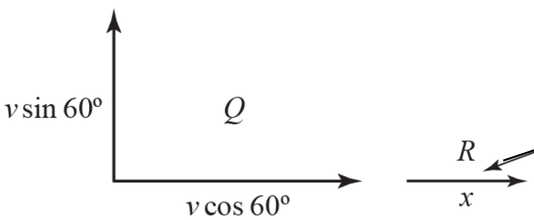
Let the speed of Q immediately after the impact be v and the speed of R immediately after the impact be x.

Components of velocity before the collision



Q is turned through 30° so, after the collision, it makes an angle of $30^\circ + 30^\circ = 60^\circ$ with the line of centres.

Components of velocity after the collision



Initially, R is at rest and the impulse of Q on R acts along the line of centres. So, after the impact, R moves along the line of centres.

Perpendicular to the line of centres

For Q

$u\sin 30^\circ = v\sin 60^\circ$

$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow u = v\sqrt{3}$ **(1)**

As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

33 continued

Parallel to the line of centres

Conservation of linear momentum

$$mu\cos 30^\circ = mv\cos 60^\circ + mx$$

$$x = \frac{\sqrt{3}}{2}u - \frac{1}{2}v \qquad \textbf{(2)}$$

Newton's law of restitution

velocity of separation = e × velocity of approach

$$x - v\cos 60^\circ = eu\cos 30^\circ$$

From equation **(1)**, $v = \frac{u}{\sqrt{3}}$

Substituting into equation **(2)** gives

$$x = \frac{u\sqrt{3}}{2} - \frac{u}{2\sqrt{3}}$$

Now $x - \frac{u\cos 60^\circ}{\sqrt{3}} = eu\cos 30^\circ$

So $\frac{u\sqrt{3}}{2} - \frac{u}{2\sqrt{3}} - \frac{u\cos 60^\circ}{\sqrt{3}} = eu\cos 30^\circ$

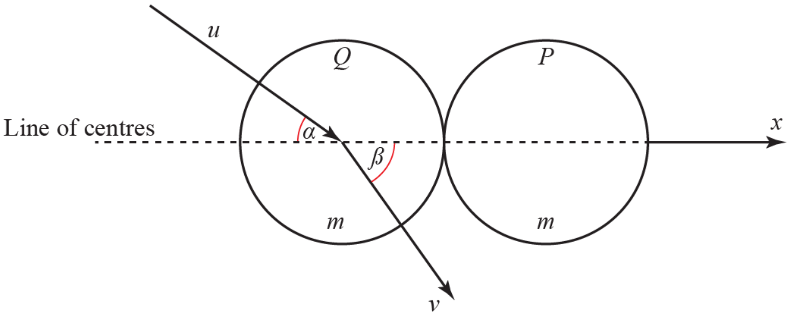
$$\frac{u\sqrt{3}}{2} - \frac{u}{2\sqrt{3}} - \frac{u}{2\sqrt{3}} = \frac{eu\sqrt{3}}{2}$$

$$3u - u - u = 3eu$$

$$u = 3ue$$

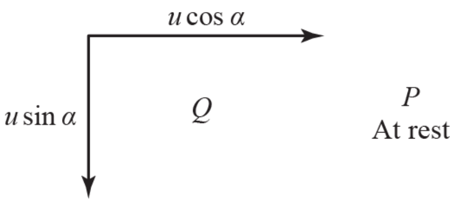
So $e = \frac{1}{3}$

34 Let the mass of each of the spheres be m.
Let the speed of Q immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of P immediately after the collision be x.

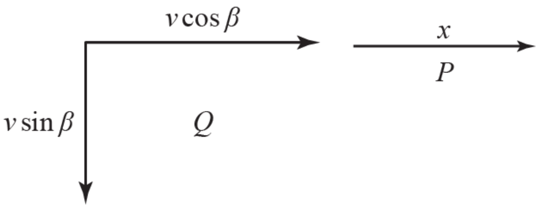


You need to introduce a number of variables to solve this question and you need to make clear to an examiner what the variables stand for. You can do this with a clearly labelled diagram.

Components of velocity before the collision



Components of velocity after the collision



Perpendicular to the line of centres
For Q

$u\sin\alpha = v\sin\beta$ **(1)**

As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

Along line of centres

Conservation of linear momentum

$\cancel{m}u\cos\alpha = \cancel{m}v\sin\beta + \cancel{m}x$

$x = u\cos\alpha - v\cos\beta$ **(2)**

Newton's law of restitution
velocity of separation = e × velocity of approach

$x - v\cos\beta = eu\cos\alpha$

$x = eu\cos\alpha + v\cos\beta$ **(3)**

Use these two equations to eliminate x, the speed of P.

Eliminating x between **(2)** and **(3)**

$u\cos\alpha - v\cos\beta = eu\cos\alpha + v\cos\beta$

$(1-e)u\cos\alpha = 2v\cos\beta$ **(4)**

34 continued

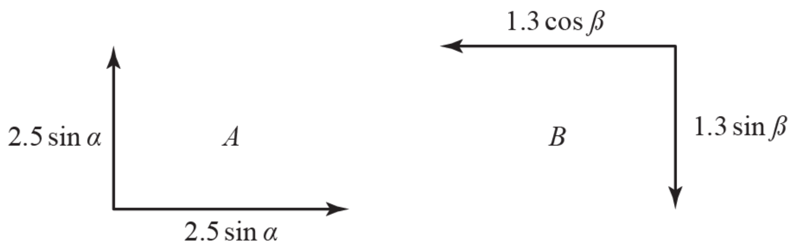
Dividing **(1)** by **(4)**

$$\frac{u\sin\alpha}{(1-e)u\cos\alpha} = \frac{v\sin\beta}{2v\cos\beta}$$

$$\frac{\tan\alpha}{1-e} = \frac{\tan\beta}{2}$$

$$(1-e)\tan\beta = 2\tan\alpha, \text{as required}$$

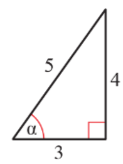
35 a Components of the velocity before the collision.
All velocity are in m s^{-1}



The component of the velocity of A perpendicular to the line of centres immediately before the collision is

$$2.5\sin\alpha \text{ m s}^{-1} = 2.5\times\frac{4}{5}\text{ m s}^{-1} = 2\text{ m s}^{-1}$$

This sketch illustrates that, as $3^2+4^2=5^2$, if $\tan\alpha = \frac{4}{3}$, then $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$



The component of the velocity of A parallel to the line of centres immediately before the collision is

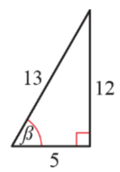
$$2.5\cos\alpha \text{ m s}^{-1} = 2.5\times\frac{3}{5}\text{ m s}^{-1} = 1.5\text{ m s}^{-1}$$

The component of the velocity of B perpendicular to the line of centres immediately before the collision is

$$1.3\sin\beta \text{ m s}^{-1} = 1.3\times\frac{12}{13}\text{ m s}^{-1} = 1.2\text{ m s}^{-1}$$

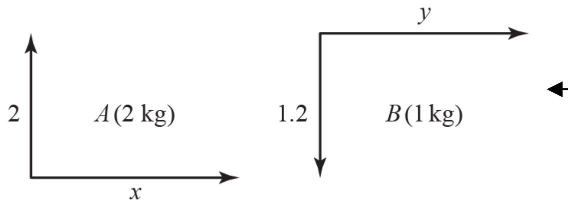
The component of the velocity of B parallel to the line of centres immediately before the collision is

$$1.3\cos\beta \text{ m s}^{-1} = 1.3\times\frac{5}{13}\text{ m s}^{-1} = 0.5\text{ m s}^{-1}$$



This sketch illustrates that, as $5^2+12^2=13^2$, if $\tan\beta = \frac{12}{5}$, then $\sin\beta = \frac{12}{13}$ and $\cos\beta = \frac{5}{13}$

35 b Let the components of the velocity after the collision be, with all velocities in m s^{-1},



The components perpendicular to the line of centres are unchanged.

Parallel to the line of centres
Conservation of linear momentum

$2x + y = 2\times1.5 - 1\times0.5$

$2x + y = 2.5$ **(1)**

Newton's law of restitution horizontally:
velocity of separation = e × velocity of approach

$y - x = \frac{1}{2}(1.5 - (-0.5))$

$y - x = 1$ **(2)**

Solving equations **(1)** and **(2)** simultaneously:

$2x + (1 + x) = 2.5$

So $x = \frac{1}{2}$ and $y = \frac{3}{2}$

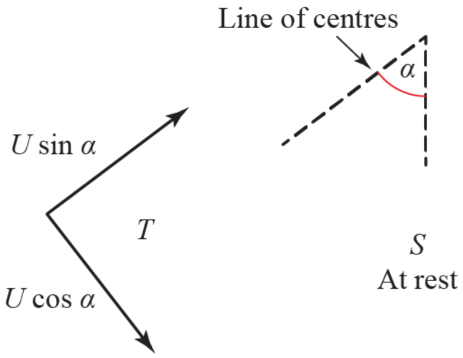
Therefore the speed of A is given by

$\sqrt{\left(\frac{1}{2}\right)^2 + 2^2} = 2.1 \text{ m s}^{-1}$

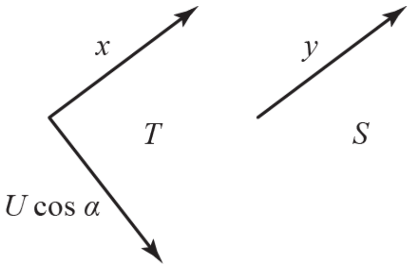
and the speed of B is given by

$\sqrt{\left(\frac{3}{2}\right)^2 + 1.2^2} = 1.9 \text{ m s}^{-1}$

36 a The components of velocity before the collision perpendicular and parallel to the line of centres are



Let the components of velocities after the collision be



Let the mass of each sphere be m

Perpendicular to the line of centres

The component of velocity is unchanged, so the component of the velocity of T after the impact perpendicular to the line of centres is $U \cos \alpha$, as required.

Parallel to the line of centres

Conservation of linear momentum

$$mU \sin \alpha = mx + my$$

$$x + y = u \sin \alpha \qquad \textbf{(1)}$$

Newton's law of restitution
velocity of separation = e × velocity of approach

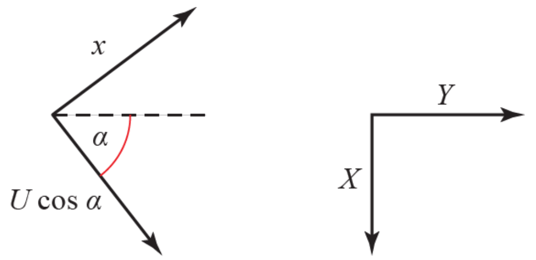
$$y - x = eU \sin \alpha \qquad \textbf{(2)}$$

(1) − (2)

$$2x = U \sin \alpha - eU \sin \alpha = U(1-e)\sin \alpha$$

$$x = \frac{1}{2}U(1-e)\sin \alpha, \text{as required}$$

36 b Let the components of the velocity of T after the impact, parallel and perpendicular to the wall be X and Y respectively



$$R(\downarrow)X = U\cos\alpha\sin\alpha - x\cos\alpha$$

$$= U\cos\alpha\sin\alpha - \frac{1}{2}U(1-e)\sin\alpha\cos\alpha$$

$$= U\cos\alpha\sin\alpha\left(1 - \frac{1}{2} + \frac{1}{2}e\right) = U\cos\alpha\sin\alpha\left(\frac{1}{2} + \frac{1}{2}e\right)$$

$$= \frac{1}{2}U(1+e)\cos\alpha\sin\alpha, \text{ as required}$$

$$R(\rightarrow)Y = U\cos\alpha\cos\alpha + x\sin\alpha$$

$$= U\cos^2\alpha + \frac{1}{2}U(1-e)\sin\alpha\sin\alpha$$

$$= U(1-\sin^2\alpha) + \frac{1}{2}U(1-e)\sin^2\alpha$$

$$= \frac{1}{2}U(2 - 2\sin^2\alpha + \sin^2\alpha - e\sin^2\alpha)$$

$$= \frac{1}{2}U(2 - \sin^2\alpha - e\sin^2\alpha)$$

$$= \frac{1}{2}U[2 - (1+e)\sin^2\alpha], \text{ as required}$$

36 c $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

You can just write these down but, if you can't remember these relations, you can find the sine and cosine by sketching a 3, 4, 5 triangle.

With $e = \frac{2}{3}$, the components in part **b** become

$$X = \frac{1}{2}U\left(1+\frac{2}{3}\right)\times\frac{3}{5}\times\frac{4}{5} = \frac{2}{5}U$$

$$Y = \frac{1}{2}U\left[2-\left(1+\frac{2}{3}\right)\times\frac{9}{25}\right] = \frac{1}{2}U\left(2-\frac{3}{5}\right) = \frac{7}{10}U$$

From part **a**), the speed of S following the collision is given by

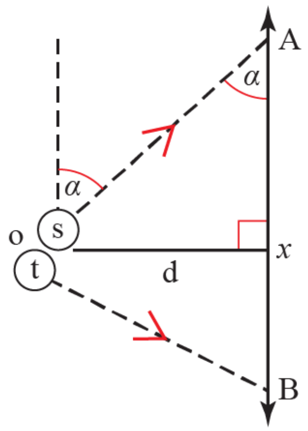
$$y = eU\sin\alpha + x$$

$$y = eU\sin\alpha + \frac{U(1-e)\sin\alpha}{2}$$

Substituting $e = \frac{2}{3}$ and $\sin\alpha = \frac{3}{5}$ into this equation gives

$$y = \frac{2U\sin\alpha}{3} + \frac{U\sin\alpha}{6}$$

$$y = \frac{2U}{5} + \frac{U}{10} = \frac{U}{2}$$



The horizontal component of the velocity of S is given by $\frac{U\sin\alpha}{2} = \frac{U}{2}\left(\frac{3}{5}\right) = \frac{3U}{10}$

The vertical component of the velocity of S is given by $\frac{U\cos\alpha}{2} = \frac{U}{2}\left(\frac{4}{5}\right) = \frac{2U}{5}$

Considering horizontal motion, the time taken for S to travel from

$O \to A$ is $\frac{\text{distance}}{\text{speed}} = \frac{d}{\frac{3U}{10}} = \frac{10d}{3U}$

So considering the vertical motion of S, $AX = \text{speed} \times \text{time} = \frac{2U}{5}\left(\frac{10d}{3U}\right) = \frac{20d}{15} = \frac{4d}{3}$

From part **b**, the horizontal component of the velocity of T is given by $\frac{7U}{10}$

Also from part **b**, the vertical component of the velocity of T is given by $\frac{2U}{5}$

Considering horizontal motion, the time taken for T to travel from

$O \to B$ is $\frac{\text{distance}}{\text{speed}} = \frac{d}{\frac{7U}{10}} = \frac{10d}{7U}$

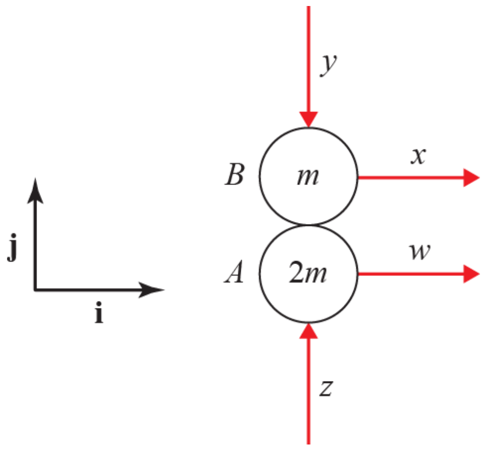
36 c continued

So considering the vertical motion of T, $BX = \text{speed} \times \text{time} = \frac{2U}{5}\left(\frac{10d}{7U}\right) = \frac{20d}{35} = \frac{4d}{7}$

Therefore $AB = AX + XB = \frac{4d}{3} + \frac{4d}{7} = \frac{28d + 12d}{21} = \frac{40d}{21}$

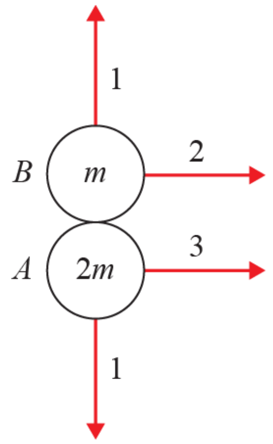
37 a Let the velocity of B before the collision be $(x\mathbf{i} - y\mathbf{j})\,\text{m s}^{-1}$ and the velocity of A before the collision be $(w\mathbf{i} + z\mathbf{j})\,\text{m s}^{-1}$

Before the collision



The components of the velocities are in m s^{-1}

After the collision



Parallel to **i**
$x = 2, w = 3$

As the impulse is in the direction of **j**, the components of the velocities of both A and B in the direction of **l** are unchanged.

Parallel to **j**
Conservation of linear momentum

$-my + 2mz = m \times 1 - 2m \times 1$

$-y + 2z = -1$ **(1)**

37 a continued

Newton's law of restitution

velocity of separation = e × velocity of approach

$$1-(-1)=\frac{1}{2}(y+z)$$

$$y+z=4 \qquad \textbf{(2)}$$

(1) + **(2)**

$3z=3 \Rightarrow z=1$

Substituting z = 1 into **(2)**

$y+1=4 \Rightarrow y=3$

The velocity of A is (3**i** + **j**) m s^{-1}

The velocity of B is (2**i** – 3**j**) m s^{-1}

b Consider the change in momentum of A in the direction of **j**.

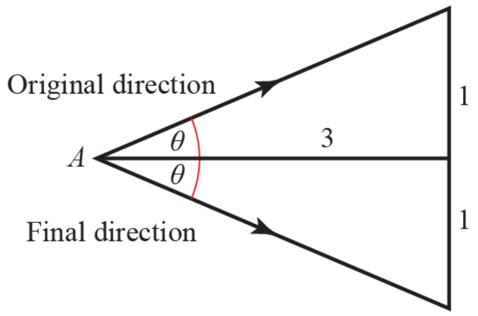
$I=2mv-2mu$

$=2m\times 1-2m\times(-1)=4m$

The mass of A is $2m$.

The magnitude of the impulse is $4m$ N s.

c

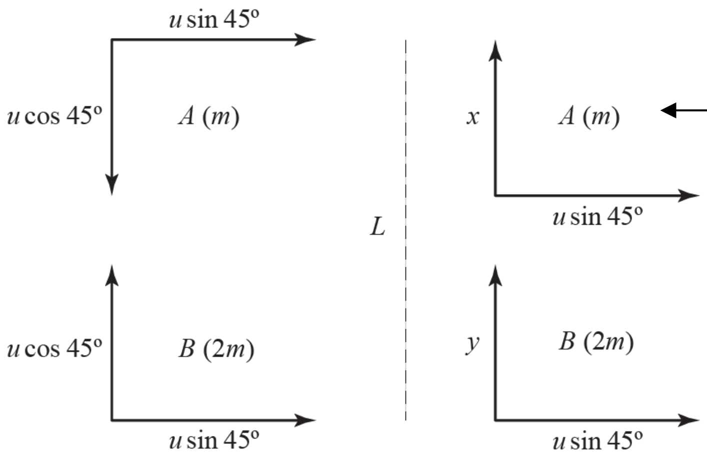


A is deflected from the direction of (3**i** + **j**) to the direction of (3**i** – **j**).

The angle of deflection is given by

$2\theta=2\arctan\frac{1}{3}=37^{\circ}$ (nearest degree)

38 a Components before collision Components after the collision



The components perpendicular to L are unchanged by the collision.
In this case, all of the components perpendicular to L are $u\sin 45^\circ = \frac{u}{\sqrt{2}}$

Parallel to L

Conservation of linear momentum $(\uparrow)$

$$2mu\cos 45^\circ - mu\cos 45^\circ = mx + 2my$$

$$x + 2y = \frac{u}{\sqrt{2}} \quad \textbf{(1)}$$

Newton's law of restitution
velocity of separation = e × velocity of approach

$$x - y = \frac{1}{2}(u\cos 45^\circ + u\cos 45^\circ)$$

$$x - y = \frac{u}{\sqrt{2}} \quad \textbf{(2)}$$

(1) – **(2)**

$$3y = 0 \Rightarrow y = 0$$

As $y = 0$, after the collision B is travelling perpendicular to L.
You will need this to solve part **b**.

Hence $x = \frac{u}{\sqrt{2}}$

The impulse on A is given by
final momentum of A – initial momentum of A

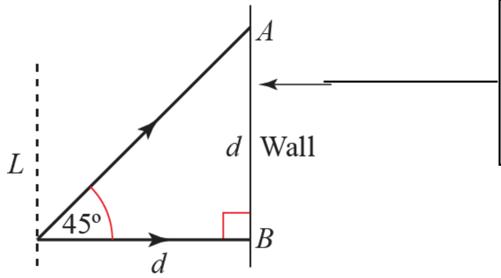
$$(\uparrow) \quad I = \frac{mu}{\sqrt{2}} - m(-u\cos 45^\circ)$$

$$= \frac{mu}{\sqrt{2}} + \frac{mu}{\sqrt{2}} = \frac{2mu}{\sqrt{2}} = \sqrt{2}mu$$

The magnitude of the impulse which acts on A in the collision is $\sqrt{2}mu$

38 b The direction of motion of A is given by

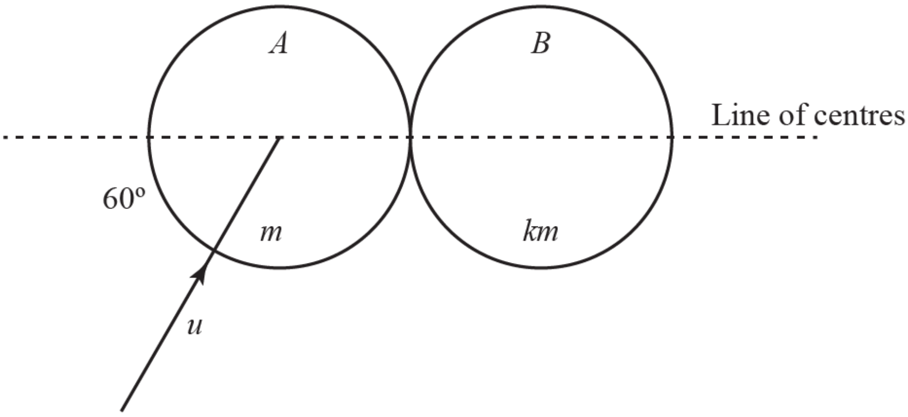
$$\tan\theta = \frac{x}{u\sin 45^\circ} = \frac{\frac{u}{\sqrt{2}}}{\frac{u}{\sqrt{2}}} = 1 \Rightarrow \theta = 45^\circ$$



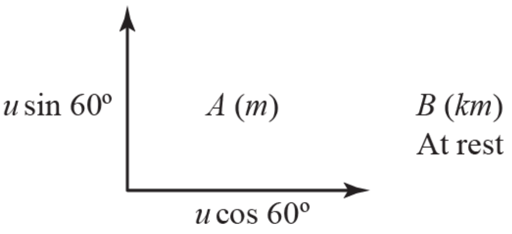
This diagram shows the paths of A and B from the point of collision to the points where they hit the wall. The triangle is isosceles and the required distance is d.

The distance between the points at which the spheres first strike the wall is d.

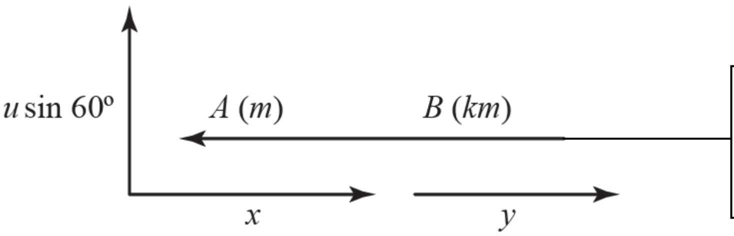
39 a Let the speed of A immediately before the collision be u.



Components of the velocities before the collision



Let the components of the velocities after collision be



As the impulse in along the line of centres, the component of the velocity of A perpendicular to the line of centres is unchanged.

Parallel to the line of centres
Conservation of linear momentum

$$mu\cos 60^\circ = mx + kmy$$

$$x + ky = \frac{1}{2}u \quad \textbf{(1)}$$

As B moves along the line of centres, the component, y, of the velocity of B along the line of centres is the velocity of B. So to solve part **a**, you must find y from this pair of simultaneous equations.

Newton's law of restitution
velocity of separation = $e \times$ veloctiy of approach

$$y - x = \frac{1}{2}u\cos 60^\circ = \frac{1}{4}u \quad \textbf{(2)}$$

$$ky + y = \frac{3u}{4}$$

$$y = \frac{3u}{4(k+1)}, \text{as required}$$

39 b From **(2)**

$$x = y - \frac{u}{4} = \frac{3u}{4(k+1)} - \frac{u}{4} = \frac{3u - u(k+1)}{4(k+1)}$$

$$= \frac{(2-k)u}{4(k+1)}$$

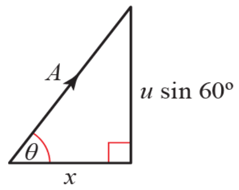
The direction of motion of A is given by

$$\tan\theta = \frac{u\sin 60^\circ}{x} = \frac{\frac{u\sqrt{3}}{2}}{\frac{(2-k)u}{4(k+1)}}$$

$$= \frac{4(k+1)\sqrt{3}}{2(2-k)} = 2\sqrt{3}$$

$$k+1 = 2-k$$

$$2k = 1 \Rightarrow k = \frac{1}{2}, \text{as required}$$



The question gives you that the direction of motion of A makes an angle arctan $\left(2\sqrt{3}\right)$ with the line of centres, so $\tan\theta = 2\sqrt{3}$. This gives an equation that you can solve for k.

c If $k = \frac{1}{2}$

$$y = \frac{3u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{2}u$$

$$x = \frac{\left(2-\frac{1}{2}\right)u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{4}u$$

The kinetic energy of the system after the collision is

$$\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right) + \frac{1}{2}kmy^2$$

$$= \frac{1}{2}m\left(\frac{u^2}{16} + \frac{3u^2}{4}\right) + \frac{1}{4}m \times \frac{1}{4}u^2$$

$$= \frac{1}{2}mu^2\left(\frac{1}{16} + \frac{3}{4} + \frac{1}{8}\right) = \frac{15}{32}mu^2$$

After the collision the velocity of A has components x and $u\sin 60^\circ$. So the kinetic energy of A after the collision is $\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right)$

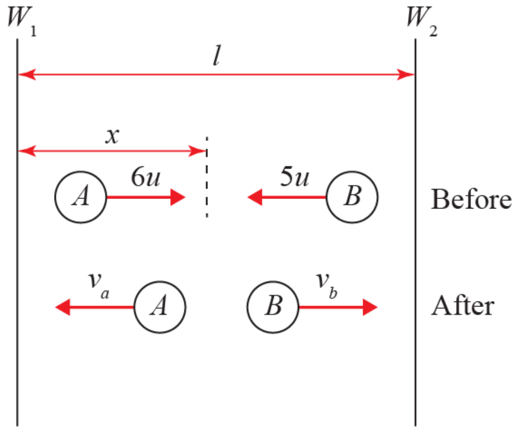
The loss in kinetic energy is

$$\frac{1}{2}mu^2 - \frac{15}{32}mu^2 = \frac{1}{32}mu^2$$

Before the collision only A is moving and it has speed u, so the initial kinetic energy of the system is $\frac{1}{2}mu^2$

Challenge

1



Let the balls collide a distance x from W_1 at time T_1, and the speeds of the balls after collision be v_A and v_B

Using $v = \frac{s}{t}$ for each ball:

$$T_1 = \frac{x}{6u} = \frac{l-x}{5u}$$

$$5x = 6l - 6x$$

$$11x = 6l$$

$$x = \tfrac{6}{11}l$$

$$\Rightarrow T_1 = \frac{\frac{6}{11}l}{6u} = \frac{l}{11u}$$

Using conservation of momentum:

$$6mu - 5mu = mv_B - mv_A$$

$$u = v_B - v_A \qquad \textbf{(1)}$$

Using Newton's law of restitution:

relative speed after collision = e × relative speed prior to collision

$$v_B - (-v_A) = e(6u - (-5u))$$

$$v_B + v_A = 11eu \qquad \textbf{(2)}$$

(2) − (1)

$$v_B + v_A - (v_B - v_A) = 11eu - u$$

$$2v_A = (11e - 1)u$$

Therefore time, T_2, taken for A to return to W_1 is:

$$T_2 = \frac{x}{v_A}$$

$$T_2 = \frac{\frac{6}{11}l}{\frac{u}{2}(11e-1)}$$

$$T_2 = \frac{12l}{11u(11e-1)}$$

Challenge
1 continued

Total time, $T = T_1 + T_2$:

$$T = \frac{l}{11u} + \frac{12l}{11u(11e-1)}$$

$$T = \frac{l(11e-1)+12l}{11u(11e-1)}$$

$$T = \frac{l(11e-1+12)}{11u(11e-1)}$$

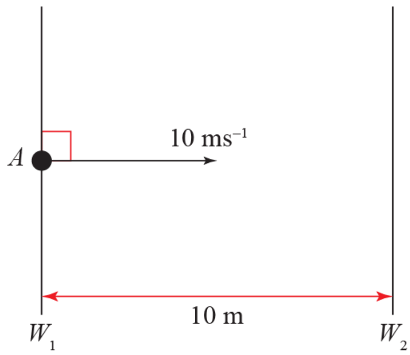
$$T = \frac{11l(e+1)}{11u(11e-1)}$$

$$T = \frac{l(e+1)}{u(11e-1)} \quad \text{as required.}$$

Challenge

2 For all journeys, $s = 10$ m and $a = 0$ ms^{-2} so $s = vt = 10$ throughout.
Let v_n be the speed after the nth collision, t_n the time taken to travel between the walls at this speed and Tn the total time to reach the opposite wall after the nth collision.

a i $v_0 = 10$ ms^{-1} → $t_0 = 1$ s by inspection.
Using Newton's law of restitution for the first collision:



$$v_1 = ev_0$$

$$v_1 = 0.8 \times 10 = 8$$

$$\Rightarrow t_1 = \frac{10}{8} = 1.25$$

$$\Rightarrow T_1 = 1 + 1.25 = 2.25$$

The sphere first returns to A after 2.25 s.

Challenge

2 a ii Using the same method:

$v_2 = ev_1$

$v_2 = 0.8\times(0.8\times10) = 6.4$

$\Rightarrow t_2 = \dfrac{10}{6.4} = 1.5625$

$v_3 = ev_2$

$v_3 = 0.8\times(0.8\times0.8\times10) = 5.12$

$\Rightarrow t_3 = \dfrac{10}{5.12} = 1.9531...$

$\Rightarrow t_2 + t_3 = 1.5625 + 1.9531... = 3.5156...$

The sphere returns to A again after a further 3.52 s (3s.f.)

In general:

$v_n = e^n v_0$

$\Rightarrow v_n = 10\times0.8^n$

$t_n = \dfrac{s}{v_n} = \dfrac{s}{v_0}\dfrac{1}{e^n}$

$\Rightarrow t_n = \dfrac{10}{10}\times\dfrac{1}{0.8^n} = 1.25^n$

$T_n = \dfrac{s}{v_0} + \sum_{n=1}^{n} t_n$

$\Rightarrow T_n = 1 + \sum_{n=1}^{n} 1.25^n$

The second term in the expression for T_n is the sum of geometric series with first term, $a = 1.25$, and ratio $r = 1.25$. Since for a finite geometrical series:

$S = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow T_n = 1 + \dfrac{1.25(1-1.25^n)}{1-1.25} = 1-5(1-1.25^n)$

$\therefore T_n = (5\times1.25^n) - 4$

b The sphere returns to A for the mth time after $2m - 1$ collisions, i.e. at time Tn where $n = 2m - 1$

So the sphere returns to A for the 20th time after 40 – 1 = 39 collisions.

Using the identities above:

$v_{39} = 10\times0.8^{39} = 1.6615...\times10^{-3}$

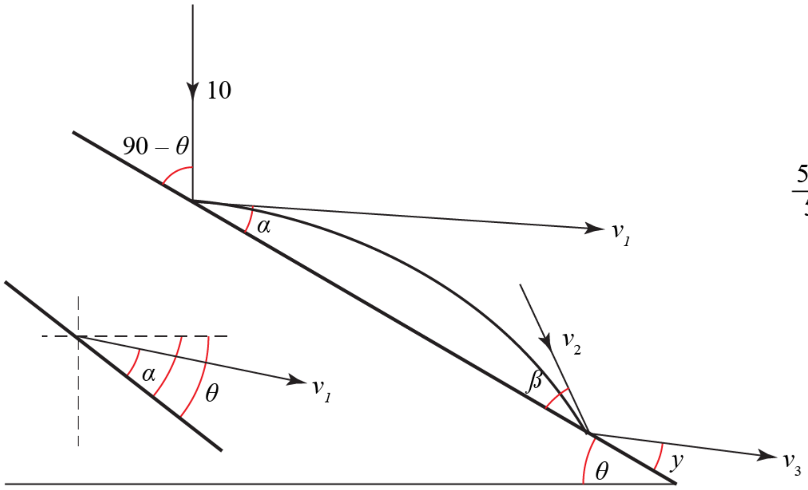
$T_{39} = (5\times1.25^{39}) - 4 = 30088.65...$

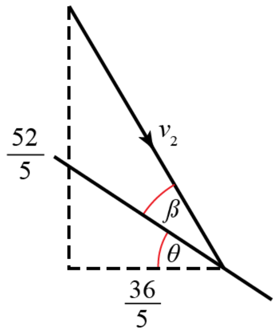
The ball returns to A for the 20th time after 30 100 s at which time it is travelling at 1.66×10^{-3} ms^{-1} (both to 3s.f.)

Challenge

2 c In practice, for real pairs of objects/surfaces the coefficient of restitution would vary with speed and tends to be smaller at smaller speeds. This means that the ball is likely to have stopped moving before it makes this number of collisions.

3 $\sin\theta = \frac{3}{5} \Rightarrow \cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$





a For the initial fall, down positive: $u = 0\ \text{ms}^{-1}$, $a = g$, $s = 5$ m, $v = ?$

$v^2 = u^2 + 2as$

$v^2 = 0 + (2\times10\times2) = 100$

Momentum is conserved in the direction parallel to the plane so:

$10m\cos(90-\theta) = mv_1\cos\alpha$

$10\sin\theta = v_1\cos\alpha$ **(1)**

Considering the components of movement perpendicular to the plane and using Newton's law of restitution:

$v_1\sin\alpha = 10e\sin(90-\theta)$

$v_1\sin\alpha = 10e\cos\theta$ **(2)**

(2) ÷ **(1)**

$\tan\alpha = \frac{e}{\tan\theta}$

$\tan\alpha = \frac{0.5}{0.75} = \frac{2}{3}$

$\Rightarrow \sin\alpha = \frac{2}{\sqrt{13}}$ and $\cos\alpha = \frac{3}{\sqrt{13}}$

Since $\tan\alpha < \tan\theta$, $\alpha < \theta$ and initial motion of ball after the bounce is below the horizontal.

Substituting into **(1)**

$\frac{3}{5}\times10 = \frac{3}{\sqrt{13}}v_1$

$v_1 = 2\sqrt{13}$

Challenge

3 a continued

The vertical component of this velocity is $v_1 \sin(\theta - \alpha)$ and the horizontal component is $v_1 \cos(\theta - \alpha)$

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$\tan(\theta - \alpha) = \frac{\frac{3}{4} - \frac{2}{3}}{1 + \frac{3 \times 2}{4 \times 3}}$$

$$\tan(\theta - \alpha) = \frac{1}{18}$$

$$\Rightarrow \cos(\theta - \alpha) = \frac{18}{5\sqrt{13}} \text{ and } \sin(\theta - \alpha) = \frac{1}{5\sqrt{13}}$$

In time t, the ball travels $4k$ horizontally, $3k$ vertically and $5k$ along the slope where k is a constant.
Considering the horizontal motion and using $s = vt$

$$4k = v_1 \cos(\theta - \alpha)t$$

$$t = \frac{4k}{v_1 \cos(\theta - \alpha)}$$

Vertically, taking down as positive and using $s = ut + \frac{1}{2}at^2$

$$s = ut + \tfrac{1}{2}at^2$$

$$3k = \frac{4kv_1 \sin(\theta - \alpha)}{v_1 \cos(\theta - \alpha)} + \frac{10}{2}\left(\frac{4k}{v_1 \cos(\theta - \alpha)}\right)^2$$

$$3k = 4k\tan(\theta - \alpha) + \frac{5 \times 16k^2}{v_1^2 \cos^2(\theta - \alpha)}$$

$$3k = \frac{4k}{18} + \frac{5 \times 16k^2}{4 \times 13 \times \frac{18^2}{25 \times 13}}$$

$$3k = \frac{2k}{9} + \frac{125k^2}{81}$$

$243k = 18k + 125k^2$ since k is not zero:

$$243 - 18 = 125k$$

$$k = \frac{9}{5}$$

Therefore, the distance travelled along the slope = $5k = \frac{9}{5} \times 5 = 9$ m

Challenge

3 b Kinetic energy $E_k = \frac{1}{2}mv^2$ so fraction of kinetic energy lost is given by:

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{u^2 - v^2}{u^2}$$

For the first bounce, $u = 10$ ms^{-1} and $v = v_1 = 2\sqrt{13}$ ms^{-1} so proportion of kinetic energy lost

$$= \frac{100 - (4 \times 13)}{100} = 0.48$$

i.e. 48%, as required.

The ball hits the slope for a second time at a velocity v_2 and at angle β to the slope.
Since there is no horizontal acceleration, the horizontal component of this velocity is:

$$v_1 \cos(\theta - \alpha) = 2\sqrt{13} \times \frac{18}{5\sqrt{13}} = \frac{36}{5}$$

and the vertical component can be found using $v^2 = u^2 + 2as$ with $u = v_1 \sin(\theta - \alpha)$, $s = 3k = \frac{9}{5} \times 3$

$$v_{2\downarrow}{}^2 = \left(\frac{2\sqrt{13}}{5\sqrt{13}}\right)^2 + \left(2 \times 10 \times \frac{27}{5}\right)$$

$$v_{2\downarrow}{}^2 = \frac{4}{25} + 108$$

$$v_{2\downarrow} = \frac{52}{5}$$

The value of v_2 is therefore given by:

$$v_2{}^2 = \left(\frac{52}{5}\right)^2 + \left(\frac{36}{5}\right)^2 = 160$$

$$v_2 = 4\sqrt{10}$$

From the diagram, it can be seen that $\tan(\theta + \beta) = \frac{52}{36}$

$$\tan((\theta + \beta) - \theta) = \frac{\tan(\theta + \beta) - \tan\theta}{1 + (\theta + \beta)\tan\theta}$$

$$\tan\beta = \frac{\frac{52}{36} - \frac{3}{4}}{1 + \frac{52 \times 3}{30 \times 4}} = \frac{\frac{25}{36}}{\frac{25}{13}}$$

$$\tan\beta = \frac{1}{3}$$

$$\Rightarrow \cos\beta = \frac{3}{\sqrt{10}} \text{ and } \sin\beta = \frac{1}{\sqrt{10}}$$

On the second bounce:
Momentum is conserved in the direction parallel to the plane so:

$v_2 \cos\beta = v_3 \cos\gamma$ **(3)**

Challenge

3 b continued

Considering the components of movement perpendicular to the plane and using Newton's law of restitution:

$ev_2 \sin\beta = v_3 \sin\gamma$ **(4)**

(4) ÷ **(3)**

$\frac{1}{2}\tan\beta = \tan\gamma$

$\tan\gamma = \frac{1}{6}$

$\Rightarrow \sin\gamma = \frac{1}{\sqrt{37}}$ and $\cos\gamma = \frac{6}{\sqrt{37}}$

Substituting into **(3)**

$4\sqrt{10} \times \frac{3}{\sqrt{10}} = v_3 \times \frac{6}{\sqrt{37}}$

$v_3 = 2\sqrt{37}$

For the second bounce, $u = v_2 = 4\sqrt{10}\text{ ms}^{-1}$ and $v = v_3 = 2\sqrt{37}\text{ ms}^{-1}$,

so the proportion of kinetic energy lost $= \frac{160 - (4 \times 37)}{160} = 0.075$

i.e. 7.5%, as required.

On the second bounce, the angle at which the ball strikes the plane is shallow.

This means the component parallel to the plane is larger than that perpendicular to the plane.

Since the former is unaffected by the collision, most of the kinetic energy is retained.

On the first bounce, when the angle is higher, the larger component is the one affected by the collision and there is therefore a greater change in the amount of kinetic energy.