Centres of mass of plane figures 2C

1 a G is $\left(\frac{1+2+3}{3}, \frac{2+6+1}{3}\right)$ — Find the mean of the vertices of the triangle.

i.e. (2, 3)

b G is $\left(\frac{-1+3+7}{3}, \frac{4+5+3}{3}\right)$ — Find the mean of the vertices of the triangle.

i.e. (3, 4)

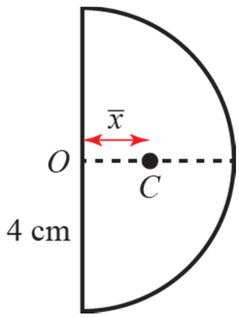
c G is $\left(\frac{-3+4+0}{3}, \frac{2+0+1}{3}\right)$ — Find the mean of the vertices of the triangle.

i.e. $\left(\frac{1}{3}, 1\right)$

d G is $\left(\frac{a+3a+4a}{3}, \frac{a+2a+6a}{3}\right)$ — Find the mean of the vertices of the triangle.

i.e. $\left(\frac{8a}{3}, 3a\right)$

2 For a semicircle,



$$2\alpha = \pi$$

$$\alpha = \frac{\pi}{2}$$

$$\bar{x} = \frac{2r\sin\alpha}{3\alpha} = \frac{8\sin\frac{\pi}{2}}{3\frac{\pi}{2}}$$ — This result is in the formula booklet provided by Edexcel.

$$= \frac{16}{3\pi}$$

Centre of mass is on the axis of symmetry at a distance $\frac{16}{3\pi}$ cm from the centre.

3 $(2, a) = \left(\frac{4+b-1}{3}, \frac{3+1+5}{3}\right)$ — Use the face that the centre of mass is at the 'mean point'.

i.e. $(2, a) = \left(\frac{3+b}{3}, 3\right)$ — Simplify.

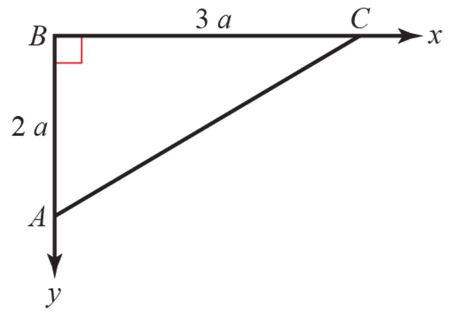
So,

$2 = \frac{3+b}{3}$ and $a = 3$ — Compare x and y coordinates.

$6 = 3 + b$

$3 = b$ — Solve for b.

4 a



We need to set up our own axes here.

Using the axes shown, B is (0, 0)
C is $(3a, 0)$ and A is $(0, 2a)$.

Centre of mass G is $\left(\frac{0+0+3a}{3}, \frac{2a+0+0}{3}\right)$

i.e. $\left(a, \frac{2a}{3}\right)$

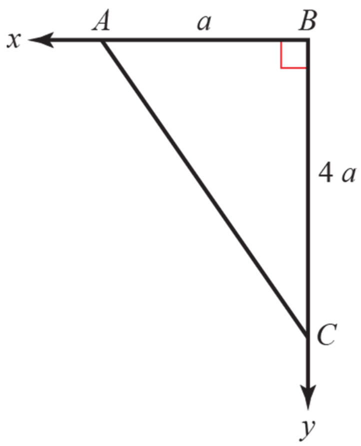
Use the fact that the centre of mass is at the 'mean point'.

Centre of the mass is a distance a from AB and a distance $\frac{2a}{3}$ from BC.

State your answer carefully.

B is (0, 0)
A is $(a, 0)$
C is $(0, 4a)$

b



Use the axes chosen (see the diagram).

Centre of mass G is $\left(\frac{0+a+0}{3}, \frac{0+0+4a}{3}\right)$

i.e. $\left(\frac{a}{3}, \frac{4a}{3}\right)$

Centre of mass is a distance $\frac{a}{3}$ from BC and a distance $\frac{4a}{3}$ from AB.

Note that $\bar{x}$ gives the distance from the y-axis and $\bar{y}$ gives the distance from the x-axis.

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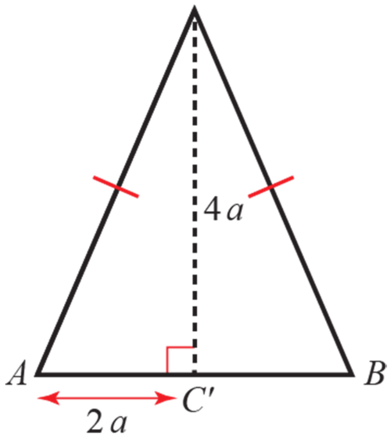
4 c



Since $AC = BC$ i.e. the Δ is isosceles so $AB = 4a$.

We need to set up our own axes here.

Take the mean of the vertices.

Taking A as the origin with AB as the x-axis, the coordinates of A, B and C are $(0, 0)$, $(4a, 0)$ and $(2a, 4a)$ respectively.

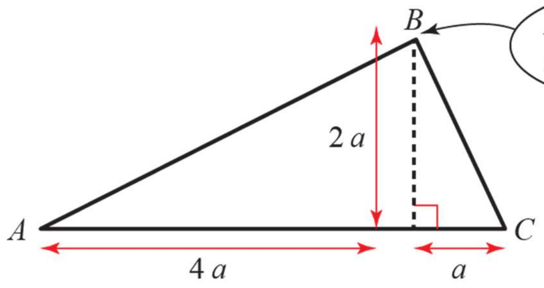
G is $\left(\frac{0+4a+2a}{3}, \frac{0+0+4a}{3}\right)$

i.e. $\left(2a, \frac{4a}{3}\right)$

Note that we could have found G by using the *symmetry* of the Δ. G must lie on the axis of symmetry and since this line is also a median, G divides CC' in the ratio 2:1, i.e. it is $\frac{2}{3}$ of the way down the median from C.

This type of argument is perfectly acceptable when answering examination questions.

d



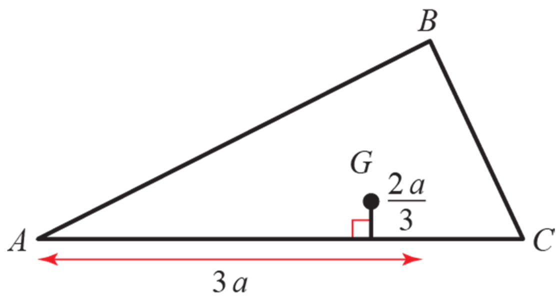
Again we need to set up some axes.

A is $(0, 0)$; B is $(4a, 2a)$; C is $(5a, 0)$.

Then G is $\left(\frac{0+4a+5a}{3}, \frac{0+2a+0}{3}\right)$

i.e. $\left(3a, \frac{2a}{3}\right)$

Take the mean of the vertices.



The diagram shows the position of G.

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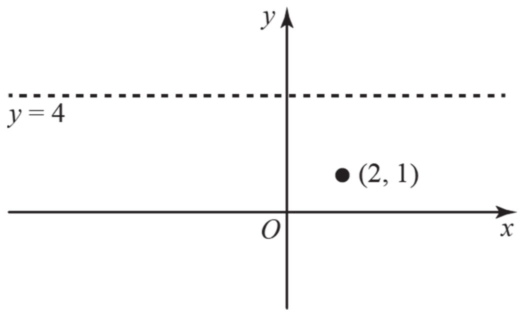

Axis of symmetry $y = 4$

Draw a diagram showing all the information.

Let (2, 1) be the point A. Then since $y = 4$ is the line of symmetry, the point (2, 7) must be another vertex.

A is 3 units below the axis of symmetry, so the other after vertex must be 3 units above the axis of symmetry.

The third vertex must be on the line of symmetry, $(x, 4)$ say. Then,

$$-3 = \frac{2+2+x}{3}$$
$$-13 = x$$

The mean of the x-coordinates must be –3.

The other two vertices, are (2, 7) and (–13, 4).

State the answer.

6 a We have that B lies on the line $y = 2x+1$ now the line BC is perpendicular to this line and hence has the form $y = -\frac{1}{2}x + K$ now considering the point $C = (6,7)$ lies on this line we have $K = 10$ hence B is the intersection of these two lines given by

$$2x+1 = -\tfrac{1}{2}x+10$$

So

$$\frac{5}{2}x = 9$$

i.e. $x = \frac{18}{5}$ and so $y = \frac{41}{5}$ so the coordinates of B are $\left(\frac{18}{5}, \frac{41}{5}\right)$

Likewise to find the coordinates of D we start by noting that the line CD has the form $y = 2x + K$ and by considering the coordinates of C we have $K = -5$ and the line AD has the form $y = -\frac{1}{2}x + K$ and considering the coordinates of A we have $K = 1$ hence the coordinates of D are the intersection of these lines given by $2x - 5 = -\frac{1}{2}x + 1$ i.e. $x = \frac{12}{5}$ and $y = -\frac{1}{5}$ so the coordinates of D are $\left(\frac{12}{5}, -\frac{1}{5}\right)$

b The coordinates for the centre of mass of the rectangle is just given by the average of the coordinates of each vertex, hence let the centre of mass have coordinates (x, y) then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 6 \\ 7 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} \frac{18}{5} \\ \frac{41}{5} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} \frac{12}{5} \\ -\frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{4} + \frac{18}{20} + \frac{12}{20} \\ \frac{1}{4} + \frac{7}{4} + \frac{41}{20} + -\frac{1}{20} \end{pmatrix} = \begin{pmatrix} \frac{60}{20} \\ \frac{80}{20} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

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7 a Since the centre of mass lies on the line $x = 3$ by symmetry we must have $x = 3$ hence the coordinates of C are $(3, y)$ the area of the triangle is then

$$\frac{1}{2}\times \text{base}\times \text{height}=\frac{1}{2}\times 2\times |y-1|$$

Hence we have $|y-1| = 4$ so $y = 5$ or $y = -3$ and the coordinates are either

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

b The coordinates of the centre of mass are just the average of the coordinates of the vertices hence in the case $y = 5$ the coordinates are

$$\frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{7}{3} \end{pmatrix}$$

In the case $y = -3$ the coordinates are

$$\frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{1}{3} \end{pmatrix}$$

8 Here we use the characterisation of the centre of mass as the intersection of the lines connecting a vertex to the midpoint of the opposite side. Consider the line connecting A to the midpoint of BC, the angle this line makes with AB is $\frac{\pi}{6}$ hence the distance from the centre of mass to AC is

$$2\tan\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$

So the distance from the centre of mass to B is

$$4\sin\frac{\pi}{3} - \frac{2\sqrt{3}}{3} = 2\sqrt{3} - \frac{2\sqrt{3}}{3} = \frac{4\sqrt{3}}{3}$$

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